Nonlinear Systems
Third Edition

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To my mentor,
Petar V. Kokotović

and my family
Amina, Mohammad, Omar, Yousuf, and Suzanne

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Preface

This text is intended for a first-year graduate-level course on nonlinear systems or control. It may also be used for self study or reference by engineers and applied mathematicians. It is an outgrowth of my experience teaching the nonlinear systems course at Michigan State University, East Lansing. Students taking this course have had background in electrical engineering, mechanical engineering, or applied mathematics. The prerequisite for the course is a graduate-level course in linear systems, taught at the level of the texts by Antsaklis and Michel [9], Chen [35], Kailath [94], or Rugh [158]. The linear systems prerequisite allowed me not to worry about introducing the concept of "state" and to refer freely to "transfer functions," "state transition matrices," and other linear system concepts. The mathematical background is the usual level of calculus, differential equations, and matrix theory that any graduate student in engineering or mathematics would have. In the Appendix, I have collected a few mathematical facts that are used throughout the book.

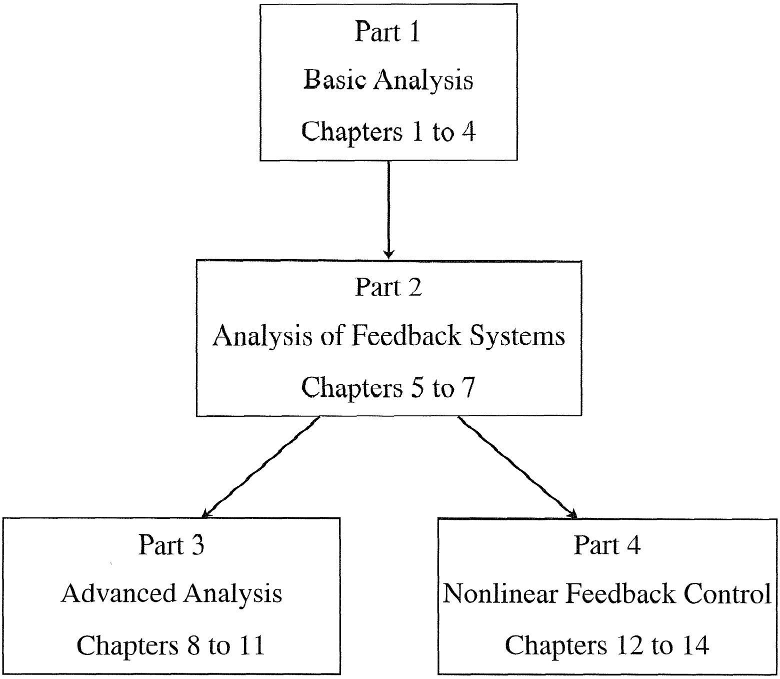
I have written the text in such a way that the level of mathematical sophistication increases as we advance from chapter to chapter. This is why the second chapter is written in an elementary context. Actually, this chapter could be taught at senior, or even junior, level courses without difficulty. This is also the reason I have split the treatment of Lyapunov stability into two parts. In Sections 4.1 through 4.3, I introduce the essence of Lyapunov stability for autonomous systems where I do not have to worry about technicalities such as uniformity, class $\mathcal{K}$ functions, etc. In Sections 4.4 through 4.6, I present Lyapunov stability in a more general setup that accommodates nonautonomous systems and allows for a deeper look into advanced aspects of the stability theory. The level of mathematical sophistication at the end of Chapter 4 is the level to which I like to bring the students, so that they can comfortably read the rest of the text.

There is yet a higher level of mathematical sophistication that is assumed in writing the proofs in the Appendix. These proofs are not intended for classroom use. They are included to make the text on one hand, self contained, and, on the other, to respond to the need or desire of some students to read such proofs, such as students continuing on to conduct Ph.D. research into nonlinear systems or control theory. Those students can continue to read the Appendix in a self-study manner.

This third edition has been written with the following goals in mind:

1. To make the book (especially the early chapters) more accessible to first-year graduate students. As an example of the changes made toward that end, note the change in Chapter 3: All the material on mathematical background, the contraction mapping theorem, and the proof of the existence and uniqueness theorem have been moved to the Appendix. Several parts of the books have been rewritten to improve readability.

2. To reorganize the book in such a way that makes it easier to structure nonlinear systems or control courses around it. In the new organization, the book has four parts, as shown in the flow chart. A course on nonlinear systems analysis will cover material from Parts 1, 2, and 3, while a course on nonlinear control will cover material from Parts 1, 2, and 4.


Part 1
Basic Analysis
Chapters 1 to 4
Part 2
Analysis of Feedback Systems
Chapters 5 to 7
Part 3
Advanced Analysis
Chapters 8 to 11
Part 4
Nonlinear Feedback Control
Chapters 12 to 14

3. To update the material of the book to include topics or results that have proven to be useful in nonlinear control design in recent years. New to the third addition are the: expanded treatment of passivity and passivity-based control, integral control, sliding mode control, and high-gain observers. Moreover, bifurcation is introduced in the context of second-order systems. On the technical side, the reader will find Kurzweil's converse Lyapunov theorem, nonlocal results in Chapters 10 and 11, and new results on integral control and gain scheduling.

4. To update the exercises. More than 170 new exercises have been included.

I am indebted to many colleagues, students, and readers, who helped me in writing the book, through discussions, suggestions, corrections, constructive comments, and feedback on the first two editions. There are, probably, more than 100 names that I would like to acknowledge, but my fear of inadvertently omitting some names, leads to me settle for a big *thank you* to each one of you.

I am grateful to Michigan State University for providing an environment that allowed me to write this book, and to the National Science Foundation for supporting my research on nonlinear feedback control.

The book was typeset using LaTeX. All computations, including numerical solution of differential equations, were done using MATLAB and SIMULINK. The figures were generated using MATLAB or the graphics tool of LaTeX.

As much as I wish the book to be free of errors, I know this will not be the case. Therefore, reports of errors, sent electronically to

khalil@msu.edu

will be greatly appreciated. An up-to-date errata list will be available at the homepage of the book:

www.egr.msu.edu/~khalil/NonlinearSystems

The homepage also will contain a list of changes from the second edition, additional exercises, and other useful material.

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East Lansing, Michigan

Nonlinear Systems
Third Edition

Chapter 1

Introduction

When engineers analyze and design nonlinear dynamical systems in electrical circuits, mechanical systems, control systems, and other engineering disciplines, they need to absorb and digest a wide range of nonlinear analysis tools. In this book, we introduce some of the these tools. In particular, we present tools for the stability analysis of nonlinear systems, with emphasis on Lyapunov's method. We give special attention to the stability of feedback systems from input–output and passivity perspectives. We present tools for the detection and analysis of "free" oscillations, including the describing function method. We introduce the asymptotic tools of perturbation theory, including averaging and singular perturbations. Finally, we introduce nonlinear feedback control tools, including linearization, gain scheduling, integral control, feedback linearization, sliding mode control, Lyapunov redesign, backstepping, passivity-based control, and high-gain observers.

1.1 Nonlinear Models and Nonlinear Phenomena

We will deal with dynamical systems that are modeled by a finite number of coupled first-order ordinary differential equations

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t and u_1, u_2, …, u_p are specified input variables. We call the variables x_1, x_2, …, x_n the state variables. They represent the memory that the dynamical system has of its past.

We usually use vector notation to write these equations in a compact form. Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

and rewrite the n first-order differential equations as one n-dimensional first-order vector differential equation

$$\dot{x} = f(t,x,u) \tag{1.1}$$

We call (1.1) the state equation and refer to x as the *state* and u as the *input*. Sometimes, another equation

$$y = h(t,x,u) \tag{1.2}$$

is associated with (1.1), thereby defining a q-dimensional *output* vector y that comprises variables of particular interest in the analysis of the dynamical system, (e.g., variables that can be physically measured or variables that are required to behave in a specified manner). We call (1.2) the output equation and refer to equations (1.1) and (1.2) together as the state-space model, or simply the state model. Mathematical models of finite-dimensional physical systems do not always come in the form of a state model. However, more often than not, we can model physical systems in this form by carefully choosing the state variables. Examples and exercises that will appear later in the chapter will demonstrate the versatility of the state model.

A good part of our analysis in this book will deal with the state equation, many times without explicit presence of an input u, that is, the so-called unforced state equation

$$\dot{x} = f(t,x) \tag{1.3}$$

Working with an unforced state equation does not necessarily mean that the input to the system is zero. It could be that the input has been specified as a given function of time, $u = \gamma(t)$, a given feedback function of the state, $u = \gamma(x)$, or both, $u = \gamma(t,x)$. Substituting $u = \gamma$ in (1.1) eliminates u and yields an unforced state equation.

A special case of (1.3) arises when the function f does not depend explicitly on t; that is,

$$\dot{x} = f(x) \tag{1.4}$$

in which case the system is said to be *autonomous* or *time invariant*. The behavior of an autonomous system is invariant to shifts in the time origin, since changing the time variable from t to $\tau = t - a$ does not change the right-hand side of the state equation. If the system is not autonomous, then it is called *nonautonomous* or *time varying.*

An important concept in dealing with the state equation is the concept of an equilibrium point. A point $x = x^*$ in the state space is said to be an equilibrium point of (1.3) if it has the property that whenever the state of the system starts at x^*, it will remain at x^* for all future time. For the autonomous system (1.4), the equilibrium points are the real roots of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points.

For linear systems, the state model (1.1)–(1.2) takes the special form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

We assume that the reader is familiar with the powerful analysis tools for linear systems, founded on the basis of the *superposition principle.* As we move from linear to nonlinear systems, we are faced with a more difficult situation. The superposition principle does not hold any longer, and analysis tools involve more advanced mathematics. Because of the powerful tools we know for linear systems, the first step in analyzing a nonlinear system is usually to linearize it, if possible, about some nominal operating point and analyze the resulting linear model. This is a common practice in engineering, and it is a useful one. There is no question that, whenever possible, we should make use of linearization to learn as much as we can about the behavior of a nonlinear system. However, linearization alone will not be sufficient; we must develop tools for the analysis of nonlinear systems. There are two basic limitations of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" behavior far from the operating point and certainly not the "global" behavior throughout the state space. Second, the dynamics of a nonlinear system are much richer than the dynamics of a linear system. There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity; hence, they cannot be described or predicted by linear models. The following are examples of essentially nonlinear phenomena:

- *Finite escape time.* The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

- *Multiple isolated equilibria.* A linear system can have only one isolated equilibrium point; thus, it can have only one steady-state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

- *Limit cycles.* For a linear time-invariant system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a nonrobust condition that is almost impossible to maintain in the presence of perturbations. Even if we do, the amplitude of oscillation will be dependent on the initial state. In real life, stable oscillation must be produced by nonlinear systems. There are nonlinear systems that can go into an oscillation of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as a limit cycle.

- *Subharmonic, harmonic, or almost-periodic oscillations.* A stable linear system under a periodic input produces an output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, an example is the sum of periodic oscillations with frequencies that are not multiples of each other.

- *Chaos.* A nonlinear system can have a more complicated steady-state behavior that is not equilibrium, periodic oscillation, or almost-periodic oscillation. Such behavior is usually referred to as chaos. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

- *Multiple modes of behavior.* It is not unusual for two or more modes of behavior to be exhibited by the same nonlinear system. An unforced system may have more than one limit cycle. A forced system with periodic excitation may exhibit harmonic, subharmonic, or more complicated steady-state behavior, depending upon the amplitude and frequency of the input. It may even exhibit a discontinuous jump in the mode of behavior as the amplitude or frequency of the excitation is smoothly changed.

In this book, we will encounter only the first three of these phenomena.[1] Multiple equilibria and limit cycles will be introduced in the next chapter, as we examine second-order autonomous systems, while the phenomenon of finite escape time will be introduced in Chapter 3.

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[1] To read about forced oscillation, chaos, bifurcation, and other important topics, the reader may consult [70], [74], [187], and [207].

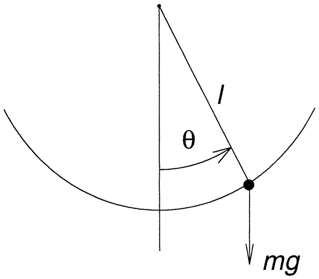

Figure 1.1: Pendulum.

1.2 Examples

1.2.1 Pendulum Equation

Consider the simple pendulum shown in Figure 1.1, where l denotes the length of the rod and m denotes the mass of the bob. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum is free to swing in the vertical plane. The bob of the pendulum moves in a circle of radius l. To write the equation of motion of the pendulum, let us identify the forces acting on the bob. There is a downward gravitational force equal to mg, where g is the acceleration due to gravity. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. Using Newton's second law of motion, we can write the equation of motion in the tangential direction as

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

Writing the equation of motion in the tangential direction has the advantage that the rod tension, which is in the normal direction, does not appear in the equation. We could have arrived at the same equation by writing the moment equation about the pivot point. To obtain a state model for the pendulum, let us take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then, the state equations are

$$\dot{x}_1 = x_2 \tag{1.5}$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \tag{1.6}$$

To find the equilibrium points, we set $\dot{x}_1 = \dot{x}_2 = 0$ and solve for x_1 and x_2:

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

The equilibrium points are located at $(n\pi, 0)$, for $n = 0, \pm 1, \pm 2, \ldots$. From the physical description of the pendulum, it is clear that the pendulum has only two equilibrium positions corresponding to the equilibrium points $(0, 0)$ and $(\pi, 0)$. Other equilibrium points are repetitions of these two positions, which correspond to the number of full swings the pendulum would make before it rests at one of the two equilibrium positions. For example, if the pendulum makes m complete 360° revolutions before it rests at the downward vertical position, then, mathematically, we say that the pendulum approaches the equilibrium point $(2m\pi, 0)$. In our investigation of the pendulum, we will limit our attention to the two "nontrivial" equilibrium points at $(0, 0)$ and $(\pi, 0)$. Physically, we can see that these two equilibrium positions are quite distinct from each other. While the pendulum can indeed rest at the $(0, 0)$ equilibrium point, it can hardly maintain rest at the $(\pi, 0)$ equilibrium point because infinitesimally small disturbance from that equilibrium will take the pendulum away. The difference between the two equilibrium points is in their stability properties, a topic we will study in some depth.

Sometimes it is instructive to consider a version of the pendulum equation where the frictional resistance is neglected by setting $k = 0$. The resulting system

$$\dot{x}_1 = x_2 \tag{1.7}$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 \tag{1.8}$$

is conservative in the sense that if the pendulum is given an initial push, it will keep oscillating forever with a nondissipative energy exchange between kinetic and potential energies. This, of course, is not realistic, but gives insight into the behavior of the pendulum. It may also help in finding approximate solutions of the pendulum equation when the friction coefficient k is small. Another version of the pendulum equation arises if we can apply a torque T to the pendulum. This torque may be viewed as a control input in the equation

$$\dot{x}_1 = x_2 \tag{1.9}$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T \tag{1.10}$$

Interestingly enough, several unrelated physical systems are modeled by equations similar to the pendulum equation. Such examples are the model of a synchronous generator connected to an infinite bus (Exercise 1.8), the model of a Josephson junction circuit (Exercise 1.9), and the model of a phase-locked loop (Exercise 1.11). Consequently, the pendulum equation is of great practical importance.

1.2.2 Tunnel-Diode Circuit

Consider the tunnel-diode circuit shown in Figure 1.2,[2] where the tunnel diode is characterized by $i_R = h(v_R)$. The energy-storing elements in this circuit are the

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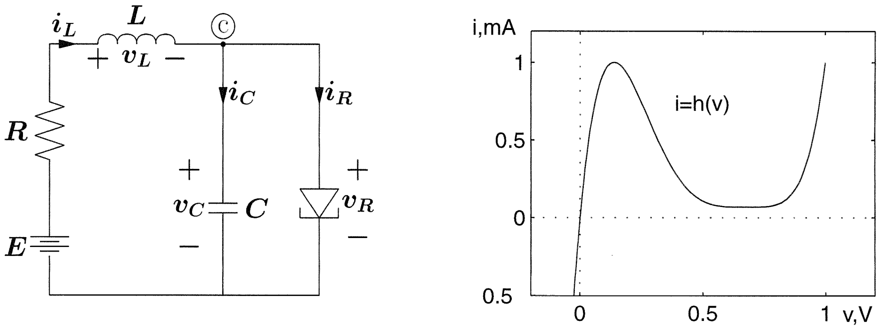
[2] This figure, as well as Figures 1.3 and 1.7, are taken from [39].



(a) (b)

Figure 1.2: (a) Tunnel-diode circuit; (b) Tunnel-diode v_R–i_R characteristic.

capacitor C and the inductor L. Assuming they are linear and time invariant, we can model them by the equations

$$i_C = C\frac{dv_C}{dt} \quad \text{and} \quad v_L = L\frac{di_L}{dt}$$

where i and v are the current through and the voltage across an element, with the subscript specifying the element. To write a state model for the system, let us take $x_1 = v_C$ and $x_2 = i_L$ as the state variables and $u = E$ as a constant input. To write the state equation for x_1, we need to express i_C as a function of the state variables x_1, x_2 and the input u. Using Kirchhoff's current law, we can write an equation that the algebraic sum of all currents leaving node ⓒ is equal to zero:

$$i_C + i_R - i_L = 0$$

Therefore,

$$i_C = -h(x_1) + x_2$$

Similarly, we need to express v_L as a function of the state variables x_1, x_2 and the input u. Using Kirchhoff's voltage law, we can write an equation that the algebraic sum of all voltages across elements in the left loop is equal to zero:

$$v_C - E + Ri_L + v_L = 0$$

Hence,

$$v_L = -x_1 - Rx_2 + u$$

We can now write the state model for the circuit as

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right] \tag{1.11}$$

$$\dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + u\right] \tag{1.12}$$

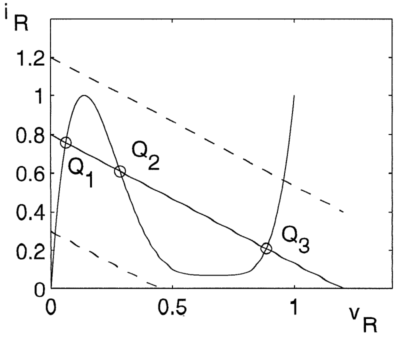


Figure 1.3: Equilibrium points of the tunnel-diode circuit.

The equilibrium points of the system are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$ and solving for x_1 and x_2:

$$\begin{aligned} 0 &= -h(x_1) + x_2 \\ 0 &= -x_1 - Rx_2 + u \end{aligned}$$

Therefore, the equilibrium points correspond to the roots of the equation

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$

Figure 1.3 shows graphically that, for certain values of E and R, this equation has three isolated roots which correspond to three isolated equilibrium points of the system. The number of equilibrium points might change as the values of E and R change. For example, if we increase E for the same value of R, we will reach a point beyond which only the point Q_3 will exist. On the other hand, if we decrease E for the same value of R, we will end up with the point Q_1 as the only equilibrium. Suppose that we are in the multiple equilibria situation, which of these equilibrium points can we observe in an experimental setup of this circuit? The answer depends on the stability properties of the equilibrium points. We will come back to this example in Chapter 2 and answer the question.

1.2.3 Mass–Spring System

In the mass–spring mechanical system, shown in Figure 1.4, we consider a mass m sliding on a horizontal surface and attached to a vertical surface through a spring. The mass is subjected to an external force F. We define y as the displacement from a reference position and write Newton's law of motion

$$m\ddot{y} + F_f + F_{sp} = F$$

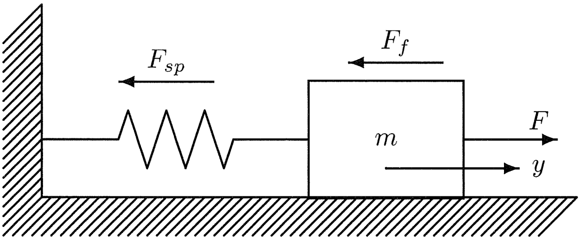


Figure 1.4: Mass-spring mechanical system.

where F_f is a resistive force due to friction and F_{sp} is the restoring force of the spring. We assume that F_{sp} is a function only of the displacement y and write it as $F_{sp} = g(y)$. We assume also that the reference position has been chosen such that $g(0) = 0$. The external force F is at our disposal. Depending upon F, F_f, and g, several interesting autonomous and nonautonomous second-order models arise.

For a relatively small displacement, the restoring force of the spring can be modeled as a linear function $g(y) = ky$, where k is the spring constant. For a large displacement, however, the restoring force may depend nonlinearly on y. For example, the function

$$g(y) = k(1 - a^2y^2)y, \quad |ay| < 1$$

models the so-called *softening spring*, where, beyond a certain displacement, a large displacement increment produces a small force increment. On the other hand, the function

$$g(y) = k(1 + a^2y^2)y$$

models the so-called *hardening spring*, where, beyond a certain displacement, a small displacement increment produces a large force increment.

The resistive force F_f may have components due to static, Coulomb, and viscous friction. When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm\mu_s mg$, where $0 < \mu_s < 1$ is the static friction coefficient. This force takes whatever value, between its limits, to keep the mass at rest. For motion to begin, there must be a force acting on the mass to overcome the resistance to motion caused by static friction. In the absence of an external force, $F = 0$, the static friction force will balance the restoring force of the spring and maintain equilibrium for $|g(y)| \le \mu_s mg$. Once motion has started, the resistive force F_f, which acts in the direction opposite to motion, is modeled as a function of the sliding velocity $v = \dot{y}$. The resistive force due to *Coulomb friction* F_c has a constant magnitude $\mu_k mg$, where μ_k is the kinetic friction coefficient, that is,

$$F_c = \begin{cases} -\mu_k mg, & \text{for} \quad v < 0 \\ \mu_k mg, & \text{for} \quad v > 0 \end{cases}$$

As the mass moves in a viscous medium, such as air or lubricant, there will be a frictional force due to viscosity. This force is usually modeled as a nonlinear

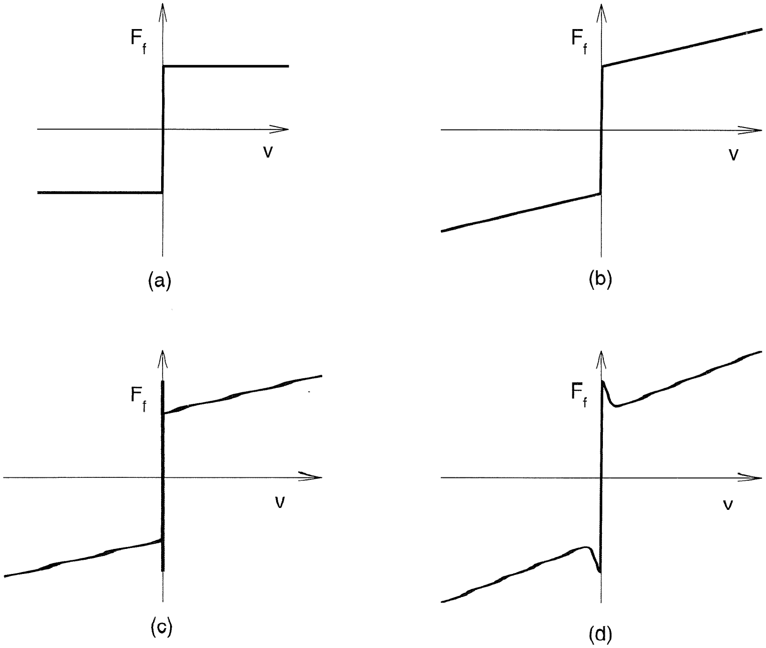


Figure 1.5: Examples of friction models. (a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect.

function of the velocity; that is, $F_v = h(v)$, where $h(0) = 0$. For small velocity, we can assume that $F_v = cv$. Figures 1.5(a) and (b) show examples of friction models for Coulomb friction and Coulombs plus linear viscous friction, respectively. Figure 1.5(c) shows an example where the static friction is higher than the level of Coulomb friction, while Figure 1.5(d) shows a similar situation, but with the force decreasing continuously with increasing velocity, the so-called *Stribeck effect.*

The combination of a hardening spring, linear viscous friction, and a periodic external force $F = A\cos\omega t$ results in the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t \tag{1.13}$$

which is a classical example in the study of periodic excitation of nonlinear systems.

The combination of a linear spring, static friction, Coulomb friction, linear viscous friction, and zero external force results in

$$m\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = 0$$

where

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg \operatorname{sign}(\dot{y}), & \text{for} \quad |\dot{y}| > 0 \\ -ky, & \text{for} \quad \dot{y} = 0 \quad \text{and} \quad |y| \le \mu_s mg/k \\ -\mu_s mg \operatorname{sign}(y), & \text{for} \quad \dot{y} = 0 \quad \text{and} \quad |y| > \mu_s mg/k \end{cases}$$

The value of $\eta(y, \dot{y})$ for $\dot{y} = 0$ and $|y| \le \mu_s mg/k$ is obtained from the equilibrium condition $\ddot{y} = \dot{y} = 0$. With $x_1 = y$ and $x_2 = \dot{y}$, the state model is

$$\dot{x}_1 = x_2 \tag{1.14}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{1}{m}\eta(x_1, x_2) \tag{1.15}$$

Let us note two features of this state model. First, it has an equilibrium set, rather than isolated equilibrium points. Second, the right-hand side function is a discontinuous function of the state. The discontinuity is a consequence of the idealization we adopted in modeling friction. One would expect the physical friction to change from its static friction mode into its sliding friction mode in a smooth way, not abruptly as our idealization suggests.[3] The discontinuous idealization, however, simplifies the analysis. For example, when $x_2 > 0$, we can model the system by the linear model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \mu_k g \end{aligned}$$

Similarly, when $x_2 < 0$, we can model it by the linear model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \mu_k g \end{aligned}$$

Thus, in each region, we can predict the behavior of the system via linear analysis. This is an example of the so-called *piecewise linear analysis*, where a system is represented by linear models in various regions of the state space, certain coefficients changing from region to region.

1.2.4 Negative-Resistance Oscillator

Figure 1.6 shows the basic circuit structure of an important class of electronic oscillators. The inductor and capacitor are assumed to be linear, time invariant and passive, that is, $L > 0$ and $C > 0$. The resistive element is an active circuit characterized by the v–i characteristic $i = h(v)$, shown in the figure. The function

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[3] The smooth transition from static to sliding friction can be captured by dynamic friction models; see, for example, [12] and [144].

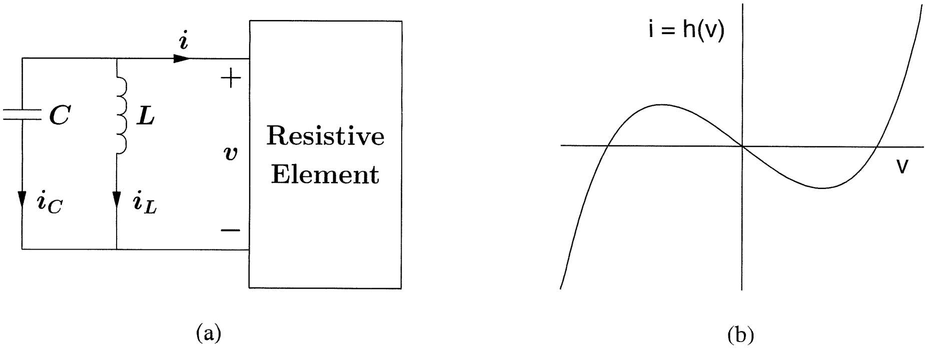

Figure 1.6: (a) Basic oscillator circuit; (b) Typical driving-point characteristic.

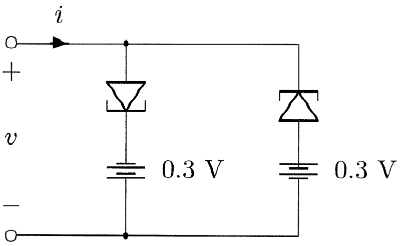

Figure 1.7: A negative-resistance twin-tunnel-diode circuit.

$h(\cdot)$ satisfies the conditions

$$h(0) = 0, \qquad h'(0) < 0$$

$$h(v) \to \infty \text{ as } v \to \infty, \quad \text{and} \quad h(v) \to -\infty \text{ as } v \to -\infty$$

where $h'(v)$ is the first derivative of $h(v)$ with respect to v. Such v–i characteristic can be realized, for example, by the twin-tunnel-diode circuit of Figure 1.7, with the tunnel-diode characteristic shown in Figure 1.2. Using Kirchhoff's current law, we can write the equation

$$i_C + i_L + i = 0$$

Hence,

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)\ ds + h(v) = 0$$

Differentiating once with respect to t and multiplying through by L, we obtain

$$CL\frac{d^2v}{dt^2} + v + Lh'(v)\frac{dv}{dt} = 0$$

The foregoing equation can be written in a form that coincides with some well-known equations in nonlinear systems theory. To do that, let us change the time variable from t to $\tau = t/\sqrt{CL}$. The derivatives of v with respect to t and τ are related by

$$\frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} \quad \text{and} \quad \frac{d^2v}{d\tau^2} = CL\frac{d^2v}{dt^2}$$

Denoting the derivative of v with respect to τ by $\dot{v}$, we can rewrite the circuit equation as

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0$$

where $\varepsilon = \sqrt{L/C}$. This equation is a special case of *Liénard's equation*

$$\ddot{v} + f(v)\dot{v} + g(v) = 0 \tag{1.16}$$

When

$$h(v) = -v + \tfrac{1}{3}v^3$$

the circuit equation takes the form

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0 \tag{1.17}$$

which is known as the *Van der Pol equation.* This equation, which was used by Van der Pol to study oscillations in vacuum tube circuits, is a fundamental example in nonlinear oscillation theory. It possesses a periodic solution that attracts every other solution except the zero solution at the unique equilibrium point $v = \dot{v} = 0$. To write a state model for the circuit, let us take $x_1 = v$ and $x_2 = \dot{v}$ to obtain

$$\dot{x}_1 = x_2 \tag{1.18}$$

$$\dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2 \tag{1.19}$$

Note that an alternate state model could have been obtained by choosing the state variables as the voltage across the capacitor and the current through the inductor. Denoting the state variables by $z_1 = i_L$ and $z_2 = v_C$, the state model is given by

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{1}{L}z_2 \\ \frac{dz_2}{dt} &= -\frac{1}{C}[z_1 + h(z_2)] \end{aligned}$$

Since the first state model has been written with respect to the time variable $\tau = t/\sqrt{CL}$, let us write this model with respect to τ.

$$\dot{z}_1 = \frac{1}{\varepsilon}z_2 \tag{1.20}$$

$$\dot{z}_2 = -\varepsilon[z_1 + h(z_2)] \tag{1.21}$$

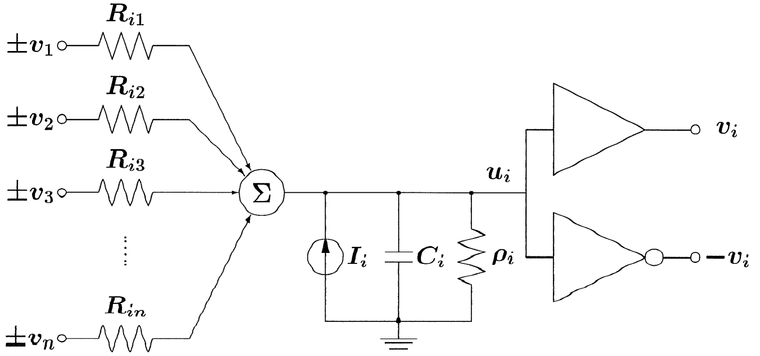

Figure 1.8: Hopfield neural network model.

The state models in x and z look different, but they are equivalent representations of the system. This equivalence can be seen by noting that these models can be obtained from each other by a change of coordinates

$$z = T(x)$$

Since we have chosen both x and z in terms of the physical variables of the circuit, it is not hard to find the map $T(\cdot)$. We have

$$\begin{aligned} x_1 &= v = z_2 \\ x_2 &= \frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} = \sqrt{\frac{L}{C}}[-i_L - h(v_C)] = \varepsilon[-z_1 - h(z_2)] \end{aligned}$$

Thus,

$$z = T(x) = \begin{bmatrix} -h(x_1) - (1/\varepsilon)x_2 \\ x_1 \end{bmatrix}$$

and the inverse mapping is

$$x = T^{-1}(z) = \begin{bmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{bmatrix}$$

1.2.5 Artificial Neural Network

Artificial neural networks, in analogy to biological structures, take advantage of distributed information processing and their inherent potential for parallel computation. Figure 1.8 shows an electric circuit that implements one model of neural networks, known as the *Hopfield model.* The circuit is based on an RC network con-

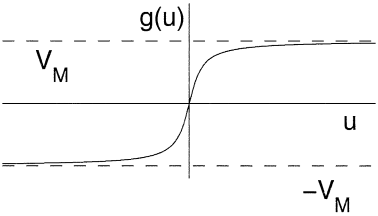

Figure 1.9: A typical input–output characteristic for the amplifiers in Hopfield network.

necting amplifiers. The input–output characteristics of the amplifiers are given by $v_i = g_i(u_i)$, where u_i and v_i are the input and output voltages of the ith amplifier. The function $g_i(\cdot) : R \to (-V_M, V_M)$ is a sigmoid function with asymptotes $-V_M$ and V_M, as shown in Figure 1.9. It is continuously differentiable, odd, monotonically increasing, and $g_i(u_i) = 0$ if and only if $u_i = 0$. Examples of possible $g_i(\cdot)$ are

$$g_i(u_i) = \frac{2V_M}{\pi} \tan^{-1}\left(\frac{\lambda \pi u_i}{2V_M}\right), \quad \lambda > 0$$

and

$$g_i(u_i) = V_M \frac{e^{\lambda u_i} - e^{-\lambda u_i}}{e^{\lambda u_i} + e^{-\lambda u_i}} = V_M \tanh(\lambda u_i), \quad \lambda > 0$$

where λ determines the slope of $g_i(u_i)$ at $u_i = 0$. Such sigmoid input–output characteristics can be realized by using operational amplifiers. For each amplifier, the circuit contains an inverting amplifier whose output is $-v_i$, which permits a choice of the sign of the amplifier output that is connected to a given input line. The outputs v_i and $-v_i$ are usually provided by two output terminals of the same operational amplifier circuit. The pair of nonlinear amplifiers is referred to as a "neuron." The circuit also contains an RC section at the input of each amplifier. The capacitance $C_i > 0$ and the resistance $\rho_i > 0$ represent the total shunt capacitance and shunt resistance at the ith amplifier input. Writing Kirchhoff's current law at the input node of the ith amplifier, we obtain

$$C_i \frac{du_i}{dt} = \sum_j \frac{1}{R_{ij}}(\pm v_j - u_i) - \frac{1}{\rho_i} u_i + I_i = \sum_j T_{ij} v_j - \frac{1}{R_i} u_i + I_i$$

where

$$\frac{1}{R_i} = \frac{1}{\rho_i} + \sum_j \frac{1}{R_{ij}}$$

T_{ij} is a signed conductance whose magnitude is $1/R_{ij}$, and whose sign is determined by the choice of the positive or negative output of the jth amplifier, and I_i is a constant input current. For a circuit containing n amplifiers, the motion is described by n first-order differential equations. To write a state model for the circuit, let us choose the state variables as $x_i = v_i$ for $i = 1, 2, \ldots, n$. Then

$$\dot{x}_i = \frac{dg_i}{du_i}(u_i) \times \dot{u}_i = \frac{dg_i}{du_i}(u_i) \times \frac{1}{C_i}\left(\sum_j T_{ij}x_j - \frac{1}{R_i}u_i + I_i\right)$$

By defining

$$h_i(x_i) = \left.\frac{dg_i}{du_i}(u_i)\right|_{u_i = g_i^{-1}(x_i)}$$

we can write the state equation as

$$\dot{x}_i = \frac{1}{C_i}h_i(x_i)\left[\sum_j T_{ij}x_j - \frac{1}{R_i}g_i^{-1}(x_i) + I_i\right] \tag{1.22}$$

for $i = 1, 2, \ldots, n$. Note that, due to the sigmoid characteristic of $g_i(\cdot)$, the function $h_i(\cdot)$ satisfies

$$h_i(x_i) > 0, \quad \forall\ x_i \in (-V_M, V_M)$$

The equilibrium points of the system are the roots of the n simultaneous equations

$$0 = \sum_j T_{ij}x_j - \frac{1}{R_i}g_i^{-1}(x_i) + I_i, \quad 1 \le i \le n$$

They are determined by the sigmoid characteristics, the linear resistive connection, and the input currents. We can obtain an equivalent state model by choosing the state variables as u_i for $i = 1, 2, \ldots, n$.

Stability analysis of this neural network depends critically on whether the symmetry condition $T_{ij} = T_{ji}$ is satisfied. An example of the analysis when $T_{ij} = T_{ji}$ is given in Section 4.2, while an example when $T_{ij} \neq T_{ji}$ is given in Section 9.5.

1.2.6 Adaptive Control

Consider a first-order linear system described by the model

$$\dot{y}_p = a_p y_p + k_p u$$

where u is the control input and y_p is the measured output. We refer to this system as the plant. Suppose that it is desirable to obtain a closed-loop system whose input–output behavior is described by the reference model

$$\dot{y}_m = a_m y_m + k_m r$$

where r is the reference input and the model has been chosen such that $y_m(t)$ represents the desired output of the closed-loop system. This goal can be achieved by the linear feedback control

$$u(t) = \theta_1^* r(t) + \theta_2^* y_p(t)$$

provided that the plant parameters a_p and k_p are known, $k_p \neq 0$, and the controller parameters θ_1^* and θ_2^* are chosen as

$$\theta_1^* = \frac{k_m}{k_p} \quad \text{and} \quad \theta_2^* = \frac{a_m - a_p}{k_p}$$

When a_p and k_p are unknown, we may consider the controller

$$u(t) = \theta_1(t) r(t) + \theta_2(t) y_p(t)$$

where the time-varying gains $\theta_1(t)$ and $\theta_2(t)$ are adjusted on-line by using the available data, namely, $r(\tau)$, $y_m(\tau)$, $y_p(\tau)$, and $u(\tau)$ for $\tau < t$. The adaptation should be such that $\theta_1(t)$ and $\theta_2(t)$ evolve to their nominal values θ_1^* and θ_2^*. The adaptation rule is chosen based on stability considerations. One such rule, known as the gradient algorithm,[4] is to use

$$\begin{aligned} \dot{\theta}_1 &= -\gamma(y_p - y_m)r \\ \dot{\theta}_2 &= -\gamma(y_p - y_m)y_p \end{aligned}$$

where γ is a positive constant that determines the speed of adaptation. This adaptive control law assumes that the sign of k_p is known and, without loss of generality, takes it to be positive. To write a state model that describes the closed-loop system under the adaptive control law, it is more convenient to define the output error e_o and the parameter errors ϕ_1 and ϕ_2 as

$$e_o = y_p - y_m, \quad \phi_1 = \theta_1 - \theta_1^*, \quad \text{and} \quad \phi_2 = \theta_2 - \theta_2^*$$

By using the definition of θ_1^* and θ_2^*, the reference model can be rewritten as

$$\dot{y}_m = a_p y_m + k_p(\theta_1^* r + \theta_2^* y_m)$$

On the other hand, the plant output y_p satisfies the equation

$$\dot{y}_p = a_p y_p + k_p(\theta_1 r + \theta_2 y_p)$$

Subtracting the above two equations, we obtain the error equation

$$\begin{aligned} \dot{e}_o &= a_p e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 y_p - \theta_2^* y_m) \\ &= a_p e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 y_p - \theta_2^* y_m + \theta_2^* y_p - \theta_2^* y_p) \\ &= (a_p + k_p\theta_2^*)e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 - \theta_2^*)y_p \end{aligned}$$

Thus, the closed-loop system is described by the nonlinear, nonautonomous, third-order state model

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p\phi_1 r(t) + k_p\phi_2[e_o + y_m(t)] && (1.23) \\ \dot{\phi}_1 &= -\gamma e_o r(t) && (1.24) \\ \dot{\phi}_2 &= -\gamma e_o[e_o + y_m(t)] && (1.25) \end{aligned}$$

---

[4] This adaptation rule will be justified in Section 8.3.

where we used $\dot{\phi}_i(t) = \dot{\theta}_i(t)$ and wrote $r(t)$ and $y_m(t)$ as explicit functions of time to emphasize the nonautonomous nature of the system. The signals $r(t)$ and $y_m(t)$ are the external driving inputs of the closed-loop system.

A simpler version of this model arises if we know k_p. In this case, we can take $\theta_1 = \theta_1^*$ and only θ_2 needs to be adjusted on-line. The closed-loop model reduces to

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p \phi[e_o + y_m(t)] && (1.26) \\ \dot{\phi} &= -\gamma e_o [e_o + y_m(t)] && (1.27) \end{aligned}$$

where we dropped the subscript from ϕ_2. If the goal of the control design is to regulate the plant output y_p to zero, we take $r(t) \equiv 0$ (hence, $y_m(t) \equiv 0$) and the closed-loop model simplifies to the autonomous second-order model

$$\begin{aligned} \dot{e}_o &= (a_m + k_p \phi) e_o \\ \dot{\phi} &= -\gamma e_o^2 \end{aligned}$$

The equilibrium points of this system are determined by setting $\dot{e}_o = \dot{\phi} = 0$ to obtain the algebraic equations

$$\begin{aligned} 0 &= (a_m + k_p \phi) e_o \\ 0 &= -\gamma e_o^2 \end{aligned}$$

The system has equilibrium at $e_o = 0$ for all values of ϕ; that is, it has an equilibrium set $e_o = 0$. There are no isolated equilibrium points.

The particular adaptive control scheme described here is called *direct model reference adaptive control.* The term "model reference" stems from the fact that the controller's task is to match a given closed-loop reference model, while the term "direct" is used to indicate that the controller parameters are adapted directly as opposed, for example, to an adaptive control scheme that would estimate the plant parameters a_p and k_p on-line and use their estimates to calculate the controller parameters.[5] The adaptive control problem generates some interesting nonlinear models that will be used to illustrate some of the stability and perturbation techniques of this book.

1.2.7 Common Nonlinearities

In the foregoing examples, we saw some typical nonlinearities that arise in modeling physical systems, such as nonlinear resistance, nonlinear friction, and sigmoid nonlinearities. In this section, we cover some other typical nonlinearities. Figure 1.10 shows four typical memoryless nonlinearities. They are called memoryless, zero memory, or static because the output of the nonlinearity at any instant of time is

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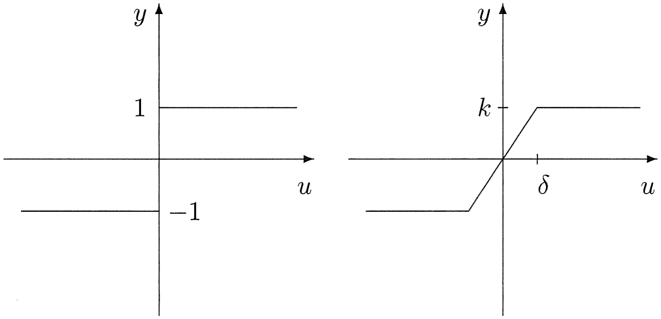
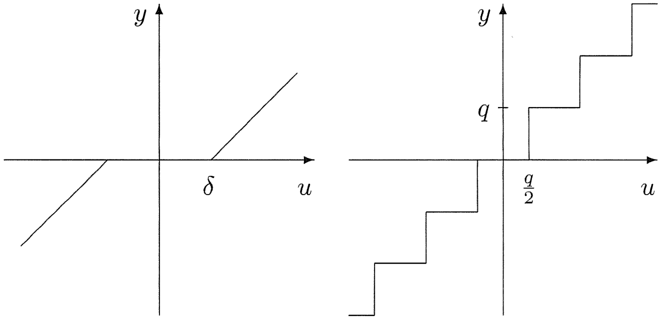
[5] For a comprehensive treatment of adaptive control, the reader may consult [5], [15], [87], [139], or [168].



(a) Relay (b) Saturation



(c) Dead zone (d) Quantization

Figure 1.10: Typical memoryless nonlinearities.

determined uniquely by its input at that instant; it does not depend on the history of the input.

Figure 1.10(a) shows an ideal relay described by the signum function

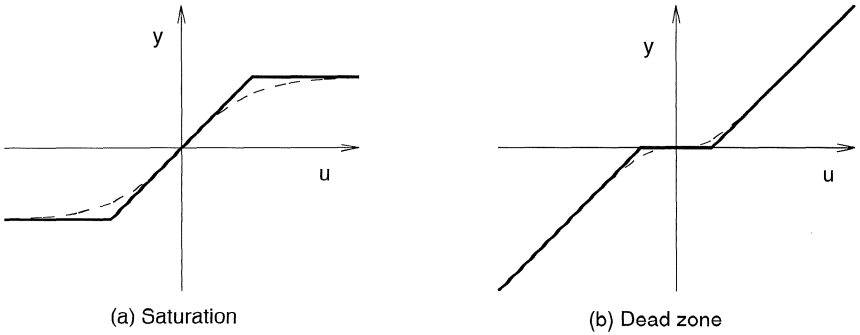
$$\text{sgn}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0 \end{cases} \tag{1.28}$$

Such nonlinear characteristic can model electromechanical relays, thyristor circuits, and other switching devices.

Figure 1.10(b) shows an ideal saturation nonlinearity. Saturation characteristics are common in all practical amplifiers (electronic, magnetic, pneumatic, or hydraulic), motors, and other devices. They are also used, intentionally, as limiters to restrict the range of a variable. We define the saturation function

$$\text{sat}(u) = \begin{cases} u, & \text{if } |u| \le 1 \\ \text{sgn}(u), & \text{if } |u| > 1 \end{cases} \tag{1.29}$$

to represent a normalized saturation nonlinearity and generate the graph of Fig-



(a) Saturation (b) Dead zone

Figure 1.11: Practical characteristics (dashed) of saturation and dead-zone nonlinearities are approximated by piecewise linear characteristics (solid).

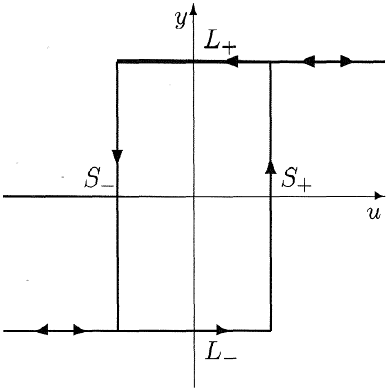


Figure 1.12: Relay with hysteresis.

ure 1.10(b) as $k\ \mathrm{sat}(u/\delta)$.

Figure 1.10(c) shows an ideal dead-zone nonlinearity. Such characteristic is typical of valves and some amplifiers at low input signals. The piecewise linear functions used in Figure 1.10(b) and (c) to represent saturation and dead-zone characteristics are approximations of more realistic smooth functions, as shown in Figure 1.11

Figure 1.10(d) shows a quantization nonlinearity, which is typical in analog-to-digital conversion of signals.

Quite frequently, we encounter nonlinear elements whose input–output characteristics have memory; that is, the output at any instant of time may depend on the whole history of the input. Figures 1.12, 1.15(b), and 1.16 show three such characteristics of the hysteresis type. The first of the three elements, Figure 1.12, is a relay with hysteresis. For highly negative values of the input, the output will

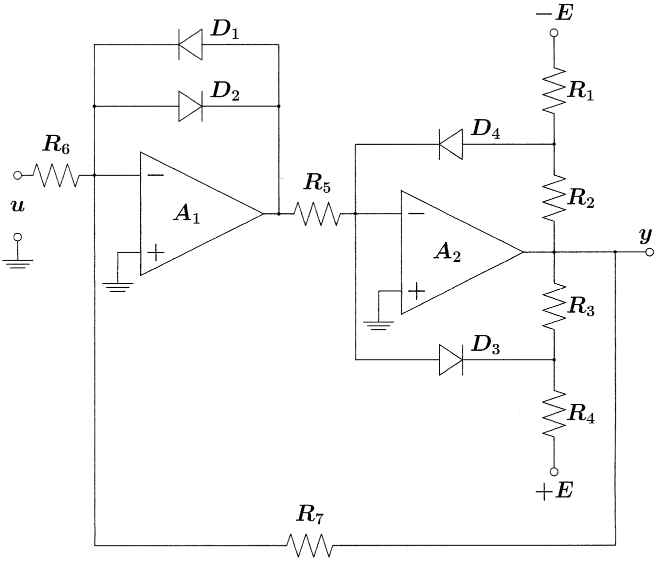


Figure 1.13: An operational amplifier circuit that realizes the relay with hysteresis characteristic of Figure 1.12.

be at the lower level L_-. As the input is increased, the output stays at L_- until the input reaches S_+. Increasing the input beyond S_+, the output switches to the higher level L_+ and stays there for higher values of the input. Now, if we decrease the input, the output stays at the higher level L_+ until the input crosses the value S_- at which point the output switches to the lower level L_- and stays there for lower values of the input. Such input–output characteristic can be generated, for example, by the operational amplifier circuit of Figure 1.13.[6] The circuit features ideal operational amplifiers and ideal diodes. An ideal operational amplifier has the voltage at its inverting (-) input equal to the voltage at its noninverting (+) input and has zero input currents at both inputs. An ideal diode has the v–i characteristic shown in Figure 1.14. When the input voltage u is highly negative, the diodes D_1 and D_3 will be *on* while D_2 and D_4 will be *off*.[7] Because the inverting inputs of both amplifiers are at virtual ground, the currents in R_5 and D_3 will be zero and the output of D_3 will be at virtual ground. Therefore, the output voltage y will be given by $y = -(R_3/R_4)E$. This situation will remain as long as the current in D_1

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[6] This circuit is taken from [204].

[7] To see why D_3 is *on* when D_1 is *on*, notice that when D_1 is *on*, the voltage at the output of A_1 will be V_d, the offset voltage of the diode. This will cause a current V_d/R_5 to flow in R_5 heading towards A_2. Since the input current to A_2 is zero, the current in R_5 must flow through D_3. In modeling the diodes, we neglect the offset voltage V_d; therefore, the currents in R_5 and D_3 are neglected.

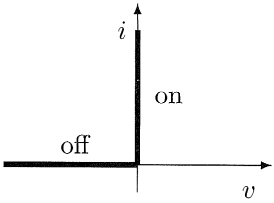


Figure 1.14: v–i characteristic of an ideal diode.

is positive; that is,

$$i_{D1} = \frac{R_3E}{R_4R_7} - \frac{u}{R_6} > 0 \Leftrightarrow u < \frac{R_3R_6E}{R_4R_7}$$

As we increase the input u, the output y will stay at $-(R_3/R_4)E$ until the input reaches the value R_3R_6E/R_4R_7. Beyond this value, the diodes D_1 and D_3 will be *off* while D_2 and D_4 will be *on*. Once again, because the inverting inputs of both amplifiers are at virtual ground, the currents in R_5 and D_4 will be zero, and the input of D_4 will be at virtual ground. Therefore, the output y will be given by $y = (R_2/R_1)E$. This situation will remain as long as the current in D_2 is positive; that is,

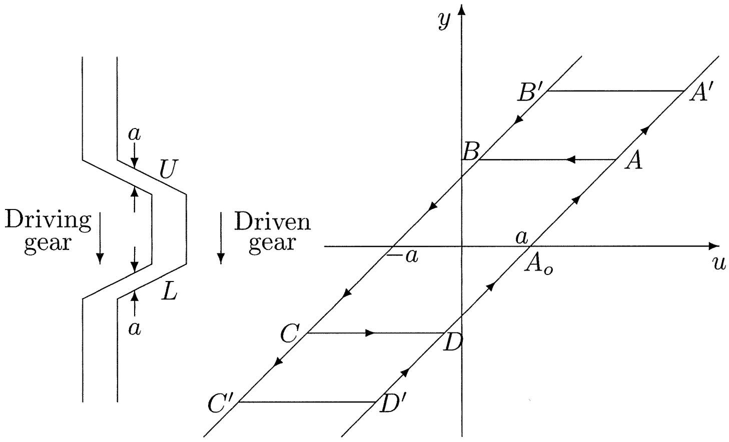
$$i_{D2} = \frac{u}{R_6} + \frac{R_2E}{R_1R_7} > 0 \Leftrightarrow u > -\frac{R_2R_6E}{R_1R_7}$$

Thus, we obtain the input–output characteristic of Figure 1.12 with

$$L_- = -\frac{R_3E}{R_4}, \quad L_+ = \frac{R_2E}{R_1}, \quad S_- = -\frac{R_2R_6E}{R_1R_7}, \quad S_+ = \frac{R_3R_6E}{R_4R_7}$$

We will see in Example 2.1 that the tunnel-diode circuit of Section 1.2.2 produces a similar characteristic when its input voltage is much slower than the dynamics of the circuit.

Another type of hysteresis nonlinearity is the backlash characteristic shown in Figure 1.15(b), which is common in gears. To illustrate backlash, the sketch of Figure 1.15(a) shows a small gap between a pair of mating gears. Suppose that the driven gear has a high friction to inertia ratio so that when the driving gear starts to decelerate, the surfaces will remain in contact at L. The input–output characteristic shown in Figure 1.15(b) depicts the angle of the driven gear y versus the angle of the driving gear u. Starting from the position shown in Figure 1.15(a), when the driving gear rotates an angle smaller than a, the driven gear does not move. For rotation larger than a, a contact is established at L and the driven gear follows the driving one, corresponding to the A_oA piece of the input–output characteristic. When the driving gear reverses direction, it rotates an angle $2a$ before a contact is established at U. During this motion, the angle y remains constant, producing the AB piece of



(a) (b)

Figure 1.15: Backlash nonlinearity.

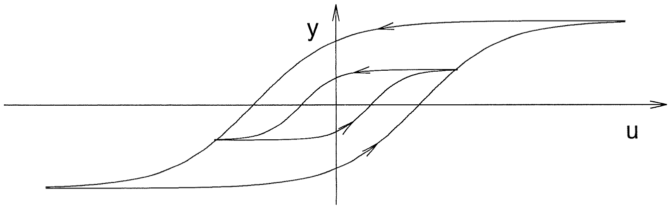


Figure 1.16: Hysteresis nonlinearity.

the characteristic. After a contact is established at U, the driven gear follows the driving one, producing the BC piece, until another reversal of direction produces the CDA piece. Thus, a periodic input of amplitude higher than a produces the $ABCD$ hysteresis loop of Figure 1.15(b). Notice that for a larger amplitude, the hysteresis loop will be $A'B'C'D'$—an important difference between this type of hysteresis characteristic and the relay with hysteresis characteristic of Figure 1.12, where the hysteresis loop is independent of the amplitude of the input.

Similar to backlash, the hysteresis characteristic of Figure 1.16, which is typical in magnetic material, has a hysteresis loop that is dependent on the amplitude of the input.[8]

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[8] Modeling the hysteresis characteristics of Figures 1.15(b) and 1.16 is quite complex. Various modeling approaches are given in [106], [126] and [203].

1.3 Exercises

✓**1.1** A mathematical model that describes a wide variety of physical nonlinear systems is the nth-order differential equation

$$y^{(n)} = g\left(t, y, \dot{y}, \dots, y^{(n-1)}, u\right)$$

where u and y are scalar variables. With u as input and y as output, find a state model.

1.2 Consider a single-input–single-output system described by the nth-order differential equation

$$y^{(n)} = g_1\left(t, y, \dot{y}, \dots, y^{(n-1)}, u\right) + g_2\left(t, y, \dot{y}, \dots, y^{(n-2)}\right)\dot{u}$$

where g_2 is a differentiable function of its arguments. With u as input and y as output, find a state model.
Hint: Take $x_n = y^{(n-1)} - g_2\left(t, y, \dot{y}, \dots, y^{(n-2)}\right) u$.

1.3 Consider a single-input–single-output system described by the nth-order differential equation

$$y^{(n)} = g\left(y, \dots, y^{(n-1)}, z, \dots, z^{(m)}\right), \quad m < n$$

where z is the input and y is the output. Extend the dynamics of the system by adding a series of m integrators at the input side and define $u = z^{(m)}$ as the input to the extended system; see Figure 1.17. Using $y, \dots, y^{(n-1)}$ and $z, \dots, z^{(m-1)}$ as state variables, find a state model of the extended system.

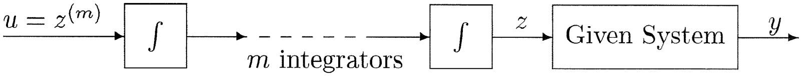

Figure 1.17: Exercise 1.3.

1.4 The nonlinear dynamic equations for an m-link robot [171, 185] take the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + +D\dot{q} + g(q) = u$$

where q is an m-dimensional vector of generalized coordinates representing joint positions, u is an m-dimensional control (torque) input, and $M(q)$ is a symmetric inertia matrix, which is positive definite for all $q \in R^m$. The term $C(q, \dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces. The matrix C has the property that $\dot{M} - 2C$ is a skew-symmetric matrix for all $q, \dot{q} \in R^m$, where $\dot{M}$ is the total derivative of $M(q)$ with respect to t. The term $D\dot{q}$ account for viscous damping, where D is a positive semidefinite symmetric matrix. The term $g(q)$, which accounts for gravity forces, is given by $g(q) = [\partial P(q)/\partial q]^T$, where $P(q)$ is the total potential energy of the links due to gravity. Choose appropriate state variables and find the state equation.

1.5 The nonlinear dynamic equations for a single-link manipulator with flexible joints [185], damping ignored, is given by

$$\begin{aligned} I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - k(q_1 - q_2) &= u \end{aligned}$$

where q_1 and q_2 are angular positions, I and J are moments of inertia, k is a spring constant, M is the total mass, L is a distance, and u is a torque input. Choose state variables for this system and write down the state equation.

1.6 The nonlinear dynamic equations for an m-link robot with flexible joints [185] take the form

$$\begin{aligned} M(q_1)\ddot{q}_1 + h(q_1, \dot{q}_1) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - K(q_1 - q_2) &= u \end{aligned}$$

where q_1 and q_2 are m-dimensional vectors of generalized coordinates, $M(q_1)$ and J are symmetric nonsingular inertia matrices, and u is an m-dimensional control input. The term $h(q, \dot{q})$ accounts for centrifugal, Coriolis, and gravity forces, and K is a diagonal matrix of joint spring constants. Choose state variables for this system and write down the state equation.

1.7 Figure 1.18 shows a feedback connection of a linear time-invariant system represented by the transfer function $G(s)$ and a nonlinear time-varying element defined by $z = \psi(t, y)$. The variables r, u, y, and z are vectors of the same dimension, and $\psi(t, y)$ is a vector-valued function. With r as input and y as output, find a state model.

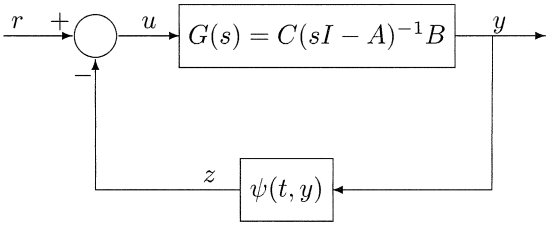

Figure 1.18: Exercise 1.7.

1.8 A synchronous generator connected to an infinite bus can be represented [148] by

$$\begin{aligned} M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin\delta \\ \tau\dot{E}_q &= -\eta_2 E_q + \eta_3 \cos\delta + E_{FD} \end{aligned}$$

where δ is an angle in radians, E_q is voltage, P is mechanical input power, E_{FD} is field voltage (input), D is damping coefficient, M is inertial coefficient, τ is time constant, and η_1, η_2, and η_3 are constant parameters.

(a) Using δ, $\dot{\delta}$, and E_q as state variables, find the state equation.

(b) Let $P = 0.815$, $E_{FD} = 1.22$, $\eta_1 = 2.0$, $\eta_2 = 2.7$, $\eta_3 = 1.7$, $\tau = 6.6$, $M = 0.0147$, and $D/M = 4$. Find all equilibrium points.

(c) Suppose that τ is relatively large so that $\dot{E}_q \approx 0$. Show that assuming E_q to be constant reduces the model to a pendulum equation.

1.9 The circuit shown in Figure 1.19 contains a nonlinear inductor and is driven by a time-dependent current source. Suppose that the nonlinear inductor is a Josephson junction [39], described by $i_L = I_0 \sin k\phi_L$, where ϕ_L is the magnetic flux of the inductor and I_0 and k are constants.

(a) Using ϕ_L and v_C as state variables, find the state equation.

(b) Is it easier to choose i_L and v_C as state variables?

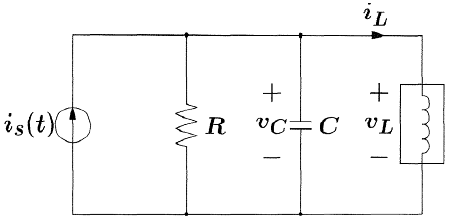

Figure 1.19: Exercises 1.9 and 1.10.

1.10 The circuit shown in Figure 1.19 contains a nonlinear inductor and is driven by a time-dependent current source. Suppose that the nonlinear inductor is described by $i_L = L\phi_L + \mu\phi_L^3$, where ϕ_L is the magnetic flux of the inductor and L and μ are positive constants.

(a) Using ϕ_L and v_C as state variables, find the state equation.

(b) Find all equilibrium points when $i_s = 0$.

1.11 A phase-locked loop [64] can be represented by the block diagram of Figure 1.20. Let $\{A, B, C\}$ be a minimal realization of the scalar, strictly proper transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$, and $\theta_i =$ constant. Let z be the state of the realization $\{A, B, C\}$.

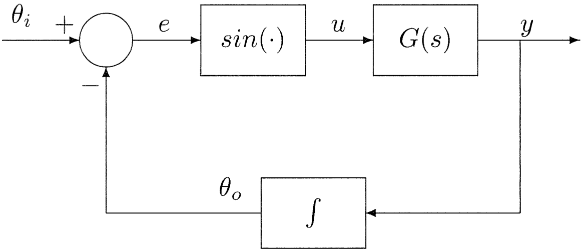

Figure 1.20: Exercise 1.11.

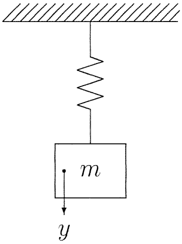

Figure 1.21: Exercise 1.12.

(a) Show that the closed-loop system can be represented by the state equations

$$\dot{z} = Az + B\sin e, \qquad \dot{e} = -Cz$$

(b) Find all equilibrium points of the system.

(c) Show that when $G(s) = 1/(\tau s + 1)$, the closed-loop model coincides with the model of a pendulum equation.

1.12 Consider the mass–spring system shown in Figure 1.21. Assuming a linear spring and nonlinear viscous damping described by $c_1\dot{y}+c_2\dot{y}|\dot{y}|$, find a state equation that describes the motion of the system.

1.13 An example of a mechanical system in which friction can be negative in a certain region is the structure shown in Figure 1.22 [7]. On a belt moving uniformly with velocity v_0, there lies a mass m fixed by linear springs, with spring constants k_1 and k_2. The friction force $h(v)$ exerted by the belt on the mass is a function of the relative velocity $v = v_0 - \dot{y}$. We assume that $h(v)$ is a smooth function for $|v| > 0$. In addition to this friction, assume that there is a linear viscous friction proportional to $\dot{y}$.

(a) Write down the equation of motion of the mass m.

(b) By restricting our analysis to the region $|\dot{y}| \ll v_0$, we can use a Taylor series to approximate $h(v)$ by $h(v_0) - \dot{y}h'(v_0)$. Using this approximation, simplify the model of the system.

(c) In view of the friction models discussed in Section 1.2.3, describe what kind of friction characteristic $h(v)$ would result in a system with negative friction.

1.14 Figure 1.23 shows a vehicle moving on a road with grade angle θ, where v is the vehicle's velocity, M is its mass, and F is the tractive force generated by the engine. Assume that the friction is due to Coulomb friction, linear viscous friction, and a drag force proportional to v^2. Viewing F as the control input and θ as a disturbance input, find a state model of the system.

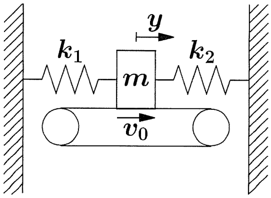

Figure 1.22: Exercise 1.13.

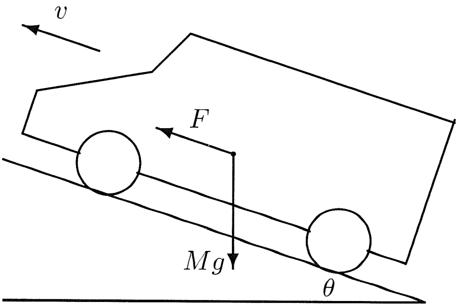

Figure 1.23: Exercise 1.14.

1.15 Consider the inverted pendulum of Figure 1.24 [110]. The pivot of the pendulum is mounted on a cart that can move in a horizontal direction. The cart is driven by a motor that exerts a horizontal force F on the cart. The figure shows also the forces acting on the pendulum, which are the force mg at the center of gravity, a horizontal reaction force H, and a vertical reaction force V at the pivot. Writing horizontal and vertical Newton's laws at the center of gravity of the pendulum yields

$$m\,\frac{d^2}{dt^2}(y + L\sin\theta) = H \quad \text{and} \quad m\,\frac{d^2}{dt^2}(L\cos\theta) = V - mg$$

Taking moments about the center of gravity yields the torque equation

$$I\ddot{\theta} = VL\sin\theta - HL\cos\theta$$

while a horizontal Newton's law for the cart yields

$$M\ddot{y} = F - H - k\dot{y}$$

Here m is the mass of the pendulum, M is the mass of the cart, L is the distance from the center of gravity to the pivot, I is the moment of inertia of the pendulum with respect to the center of gravity, k is a friction coefficient, y is the displacement of the pivot, θ is the angular rotation of the pendulum (measured clockwise), and g is the acceleration due to gravity.

(a) Carrying out the indicated differentiation and eliminating V and H, show that the equations of motion reduce to

$$I\ddot{\theta} = mgL\sin\theta - mL^2\ddot{\theta} - mL\ddot{y}\cos\theta$$

$$M\ddot{y} = F - m\left(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right) - k\dot{y}$$

(b) Solving the foregoing equations for $\ddot{\theta}$ and $\ddot{y}$, show that

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix} \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

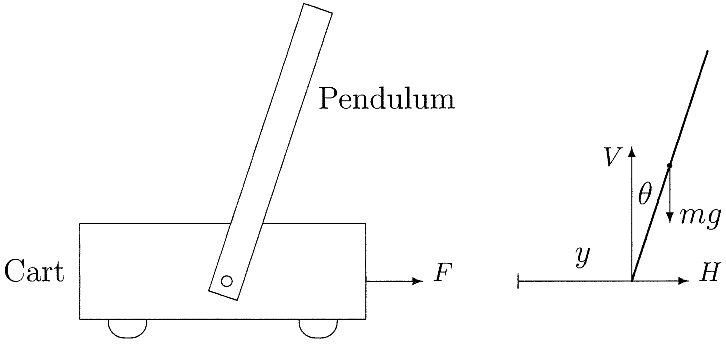

Figure 1.24: Inverted pendulum of Exercise 1.15.

where

$$\Delta(\theta) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta \geq (I + mL^2)M + mI > 0$$

(c) Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as the state variables and $u = F$ as the control input, write down the state equation.

1.16 Figure 1.25 shows a schematic diagram of a Translational Oscillator with Rotating Actuator (TORA) system [205]. The system consists of a platform of mass M connected to a fixed frame of reference by a linear spring, with spring constant k. The platform can only move in the horizontal plane, parallel to the spring axis. On the platform, a rotating proof mass is actuated by a DC motor. It has mass m and moment of inertial I around its center of mass, located at a distance L from its rotational axis. The control torque applied to the proof mass is denoted by u. The rotating proof mass creates a force which can be controlled to dampen the translational motion of the platform. We will derive a model for the system, neglecting friction. Figure 1.25 shows that the proof mass is subject to forces F_x and F_y and a torque u. Writing Newton's law at the center of mass and taking moments about the center of mass yield the equations

$$m\,\frac{d^2}{dt^2}(x_c + L\sin\theta) = F_x, \quad m\,\frac{d^2}{dt^2}(L\cos\theta) = F_y, \quad \text{and} \quad I\ddot{\theta} = u + F_yL\sin\theta - F_xL\cos\theta$$

where θ is the angular position of the proof mass (measured counter clockwise). The platform is subject to the forces F_x and F_y, in the opposite directions, as well as the restoring force of the spring. Newton's law for the platform yields

$$M\ddot{x}_c = -F_x - kx_c$$

where x_c is the translational position of the platform.

(a) Carrying out the indicated differentiation and eliminating F_x and F_y, show that the equations of motion reduce to

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2 \sin\theta - kx_c \end{bmatrix}, \text{ where } D(\theta) = \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & M + m \end{bmatrix}$$

(b) Solving the foregoing equation for $\ddot{\theta}$ and $\ddot{x}_c$, show that

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)}\begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix}\begin{bmatrix} u \\ mL\dot{\theta}^2 \sin\theta - kx_c \end{bmatrix}$$

where

$$\Delta(\theta) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta \geq (I + mL^2)M + mI > 0$$

(c) Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x_c$, and $x_4 = \dot{x}_c$ as the state variables and u as the control input, write down the state equation.

(d) Find all equilibrium points of the system.

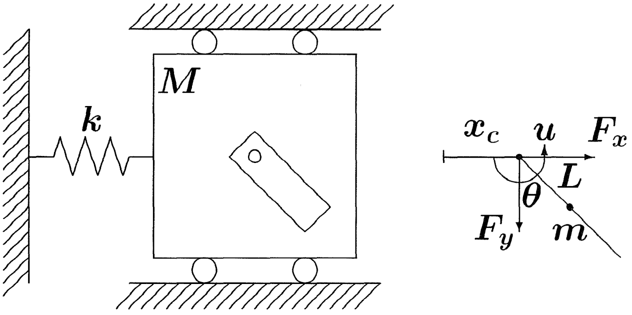

Figure 1.25: Translational Oscillator with Rotating Actuator (TORA) system.

1.17 The dynamics of a DC motor [178] can be described by

$$\begin{aligned} v_f &= R_f i_f + L_f \frac{di_f}{dt} \\ v_a &= c_1 i_f \omega + L_a \frac{di_a}{dt} + R_a i_a \\ J\frac{d\omega}{dt} &= c_2 i_f i_a - c_3 \omega \end{aligned}$$

The first equation is for the field circuit with v_f, i_f, R_f, and L_f being its voltage, current, resistance, and inductance. The variables v_a, i_a, R_a, and L_a are the corresponding variables for the armature circuit described by the second equation. The third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as a damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f. induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

(a) For a separately excited DC motor, the voltages v_a and v_f are independent control inputs. Choose appropriate state variables and find the state equation.

(b) Specialize the state equation of part(a) to the field controlled DC motor, where v_f is the control input, while v_a is held constant.

(c) Specialize the state equation of part(a) to the armature controlled DC motor, where v_a is the control input, while v_f is held constant. Can you reduce the order of the model in this case?

(d) In a shunt wound DC motor, the field and armature windings are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux; that is, $v = v_a = v_f + R_x i_f$. With v as the control input, write down the state equation.

1.18 Figure 1.26 shows a schematic diagram of a magnetic suspension system, where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the, optically measured, ball position [211, pp. 192–200]. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers, and fast trains. The equation of motion of the ball is

$$m\ddot{y} = -k\dot{y} + mg + F(y, i)$$

where m is the mass of the ball, $y \geq 0$ is the vertical (downward) position of the ball measured from a reference point ($y = 0$ when the ball is next to the coil), k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y, i)$ is the force generated by the electromagnet, and i is its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as

$$L(y) = L_1 + \frac{L_0}{1 + y/a}$$

where L_1, L_0, and a are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y, i) = \frac{1}{2}L(y)i^2$ as the energy stored in the electromagnet, the force $F(y, i)$ is given by

$$F(y, i) = \frac{\partial E}{\partial y} = -\frac{L_0 i^2}{2a(1 + y/a)^2}$$

When the electric circuit of the coil is driven by a voltage source with voltage v, Kirchhoff's voltage law gives the relationship $v = \dot{\phi} + Ri$, where R is the series resistance of the circuit and $\phi = L(y)i$ is the magnetic flux linkage.

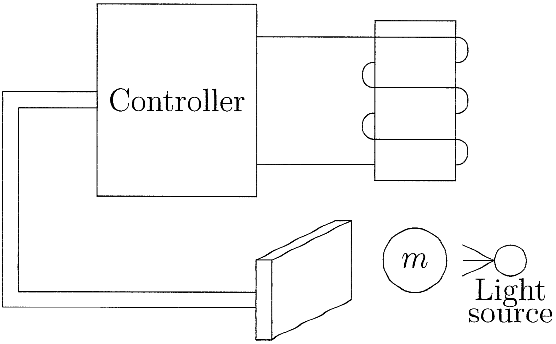

Figure 1.26: Magnetic suspension system of Exercise 1.18.

(a) Using $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = i$ as state variables and $u = v$ as control input, find the state equation.

(b) Suppose it is desired to balance the ball at a certain position $r > 0$. Find the steady-state values I_{ss} and V_{ss} of i and v, respectively, which are necessary to maintain such balance.

The next three exercises give examples of hydraulic systems [41].

1.19 Figure 1.27 shows a hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)\, d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve that obeys the flow-pressure relationship $w_o = k\sqrt{\Delta p}$. In the current case, $\Delta p = p - p_a$. Take $u = w_i$ to be the control input and $y = h$ to be the output.

(a) Using h as the state variable, determine the state model.

(b) Using $p - p_a$ as the state variable, determine the state model.

(c) Find u_{ss} that is needed to maintain the output at a constant value r.

1.20 The hydraulic system shown in Figure 1.28 consists of a constant speed centrifugal pump feeding a tank from which liquid flows through a pipe and a valve that obeys the relationship $w_o = k\sqrt{p - p_a}$. The pump characteristic for the specified pump speed is shown in Figure 1.29. Let us denote this relationship by $\Delta p = \phi(w_i)$ and denote its inverse, whenever defined, by $w_i = \phi^{-1}(\Delta p)$. For the current pump, $\Delta p = p - p_a$. The cross-sectional area of the tank is uniform; therefore, $v = Ah$ and $p = p_a + \rho g v / A$, where the variables are defined in the previous exercise.

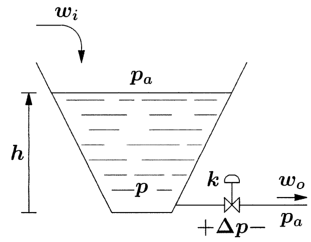

Figure 1.27: Exercise 1.19.

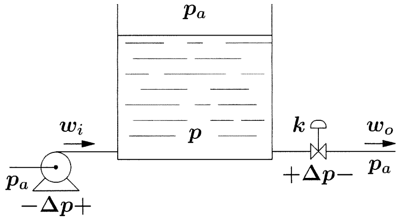

Figure 1.28: Exercise 1.20.

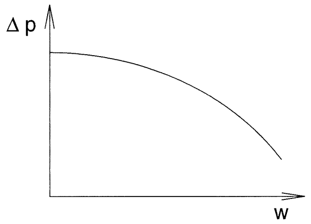

Figure 1.29: Typical centrifugal pump characteristic.

(a) Using $(p - p_a)$ as the state variable, find the state model.

(b) Find all equilibrium points of the system.

1.21 The valves in the hydraulic system of Figure 1.30 obey the flow relationships $w_1 = k_1\sqrt{p_1 - p_2}$ and $w_2 = k_2\sqrt{p_2 - p_a}$. The pump has the characteristic shown in Figure 1.29 for $(p_1 - p_a)$ versus w_p. The various components and variables are defined in the previous two exercises.

(a) Using $(p_1 - p_a)$ and $(p_2 - p_a)$ as the state variables, find the state equation.

(b) Find all equilibrium points of the system.

1.22 Consider a biochemical reactor with two components—biomass and substrate—where the biomass cells consume the substrate [23]; a schematic is shown in Figure 1.31. Assume that the reactor is perfectly mixed and the volume V is constant. Let x_1 and x_2 be the concentrations (mass/volume) of the biomass cells and substrate, respectively, and x_{1f} and x_{2f} be the corresponding concentrations in the feed stream. Let r_1 be the rate of biomass cell generation (mass/volume/time), r_2 be the rate of the substrate consumption, and F be the flow rate (volume/time). The dynamic model is developed by writing material balances on the biomass and substrate; that is,

$$\text{rate of biomass accumulation} = \text{in by flow} - \text{out by flow} + \text{generation}$$

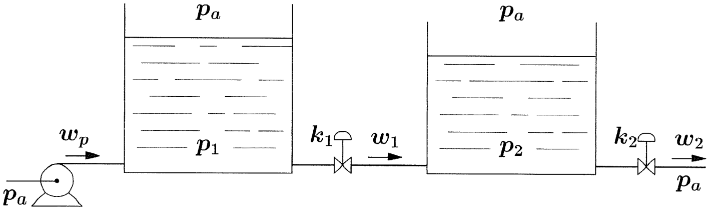

Figure 1.30: The hydraulic system of Exercise 1.21.

$$\text{rate of substrate accumulation} = \text{in by flow} - \text{out by flow} - \text{consumption}$$

The generation rate r_1 is modeled as $r_1 = \mu x_1$, where the specific growth coefficient μ is a function of x_2. We assume that there is no biomass in the feed stream, so $x_{1f} = 0$, the dilution rate $d = F/V$ is constant, and the yield $Y = r_1/r_2$ is constant.

(a) Using x_1 and x_2 as state variables, find the state model.

(b) Find all equilibrium points when $\mu = \mu_m x_2/(k_m + x_2)$ for some positive constants μ_m and k_m. Assume that $d < \mu_m$.

(c) Find all equilibrium points when $\mu = \mu_m x_2/(k_m + x_2 + k_1 x_2^2)$ for some positive constants μ_m, k_m, and k_1. Assume that $d < \max_{x_2 \geq 0}\{\mu(x_2)\}$.

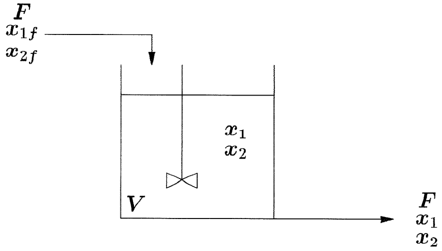

Figure 1.31: Biochemical reactor of Exercise 1.22.

Chapter 2

Second-Order Systems

Second-order autonomous systems occupy an important place in the study of nonlinear systems because solution trajectories can be represented by curves in the plane. This allows for easy visualization of the qualitative behavior of the system. The purpose of this chapter is to use second-order systems to introduce, in an elementary context, some of the basic ideas of nonlinear systems. In particular, we will look at the behavior of a nonlinear system near equilibrium points, the phenomenon of nonlinear oscillation, and bifurcation.

A second-order autonomous system is represented by two scalar differential equations

$$\dot{x}_1 = f_1(x_1, x_2) \tag{2.1}$$

$$\dot{x}_2 = f_2(x_1, x_2) \tag{2.2}$$

Let $x(t) = (x_1(t), x_2(t))$ be the solution[1] of (2.1)–(2.2) that starts at a certain initial state $x_0 = (x_{10}, x_{20})$; that is, $x(0) = x_0$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit* of (2.1)–(2.2) from x_0. The x_1–x_2 plane is usually called the *state plane* or *phase plane*. The right-hand side of (2.1)–(2.2) expresses the tangent vector $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to the curve. Using the vector notation

$$\dot{x} = f(x)$$

where $f(x)$ is the vector $(f_1(x), f_2(x))$, we consider $f(x)$ as a *vector field* on the state plane, which means that to each point x in the plane, we assign a vector $f(x)$. For easy visualization, we represent $f(x)$ as a vector based at x; that is, we assign to x the directed line segment from x to $x + f(x)$. For example, if $f(x) = (2x_1^2, x_2)$, then at $x = (1, 1)$, we draw an arrow pointing from $(1, 1)$ to $(1, 1) + (2, 1) = (3, 2)$. (See Figure 2.1.) Repeating this at every point in a grid covering the plane, we

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[1] It is assumed that there is a unique solution.

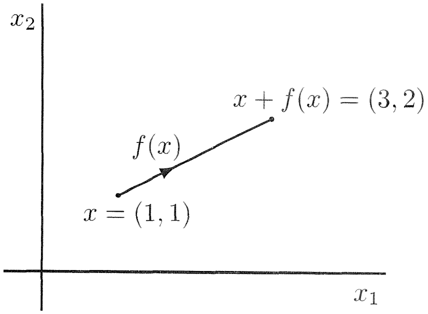

Figure 2.1: Vector field representation.

obtain a *vector field diagram*, such as the one shown in Figure 2.2 for the pendulum equation without friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin x_1 \end{aligned}$$

In the figure, the length of the arrow at a given point x is proportional to the length of $f(x)$, that is, $\sqrt{f_1^2(x) + f_2^2(x)}$. Sometimes, for convenience, we draw arrows of equal length at all points. Since the vector field at a point is tangent to the trajectory through that point, we can, in essence, construct trajectories from the vector field diagram. Starting at a given initial point x_0, we can construct the trajectory from x_0 by moving along the vector field at x_0. This motion takes us to a new point x_a, where we continue the trajectory along the vector field at x_a. If the process is repeated carefully and the consecutive points are chosen close enough to each other, we can obtain a reasonable approximation of the trajectory through x_0. In the case of Figure 2.2, a careful implementation of the foregoing process would show that the trajectory through $(2, 0)$ is a closed curve.

The family of all trajectories or solution curves is called the *phase portrait* of (2.1)–(2.2). An (approximate) picture of the phase portrait can be constructed by plotting trajectories from a large number of initial states spread all over the x_1–x_2 plane. Since numerical subroutines for solving general nonlinear differential equations are widely available, we can easily construct the phase portrait by using computer simulations. (Some hints are given in Section 2.5.) Note that since the time t is suppressed in a trajectory, it is not possible to recover the solution $(x_1(t), x_2(t))$ associated with a given trajectory. Hence, a trajectory gives only the *qualitative*, but not *quantitative*, behavior of the associated solution. For example, a closed trajectory shows that there is a periodic solution; that is, the system has a sustained oscillation, whereas a shrinking spiral shows a decaying oscillation. In the rest of this chapter, we will qualitatively analyze the behavior of second-order systems by using their phase portraits.

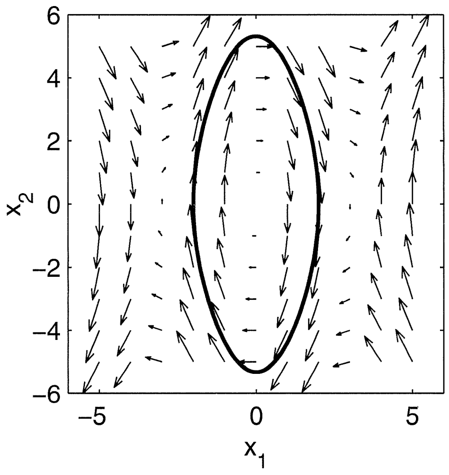

Figure 2.2: Vector field diagram of the pendulum equation without friction.

2.1 Qualitative Behavior of Linear Systems

Consider the linear time-invariant system

$$\dot{x} = Ax \tag{2.3}$$

where A is a 2×2 real matrix. The solution of (2.3) for a given initial state x_0 is given by

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

where J_r is the real Jordan form of A and M is a real nonsingular matrix such that $M^{-1}AM = J_r$. Depending on the eigenvalues of A, the real Jordan form may take one of three forms

real, distinct real, equal complex

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where k is either 0 or 1. The first form corresponds to the case when the eigenvalues λ_1 and λ_2 are real and distinct, the second form corresponds to the case when the eigenvalues are real and equal, and the third form corresponds to the case of complex eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$. In our analysis, we have to distinguish between these three cases. Moreover, with real eigenvalues, we have to isolate the case when at least one of the eigenvalues is zero. In that situation, the origin is not an isolated equilibrium point and the qualitative behavior is quite different from the behavior in the other cases.

Case 1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$.

In this case, $M = [v_1, v_2]$, where v_1 and v_2 are the real eigenvectors associated with λ_1 and λ_2. The change of coordinates $z = M^{-1}x$ transforms the system into two decoupled first-order differential equations,

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

whose solution, for a given initial state (z_{10}, z_{20}), is given by

$$z_1(t) = z_{10}e^{\lambda_1 t}, \qquad z_2(t) = z_{20}e^{\lambda_2 t}$$

Eliminating t between the two equations, we obtain

$$z_2 = c z_1^{\lambda_2/\lambda_1} \tag{2.4}$$

where $c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$. The phase portrait of the system is given by the family of curves generated from (2.4) by allowing the constant c to take arbitrary values in R. The shape of the phase portrait depends on the signs of λ_1 and λ_2.

Consider first the case when both eigenvalues are negative. Without loss of generality, let $\lambda_2 < \lambda_1 < 0$. Here, both exponential terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$. Moreover, since $\lambda_2 < \lambda_1 < 0$, the term $e^{\lambda_2 t}$ tends to zero faster than the term $e^{\lambda_1 t}$. Hence, we call λ_2 the fast eigenvalue and λ_1 the slow eigenvalue. For later reference, we call v_2 the fast eigenvector and v_1 the slow eigenvector. The trajectory tends to the origin of the z_1–z_2 plane along the curve of (2.4), which now has a ratio λ_2/λ_1 that is greater than one. The slope of the curve is given by

$$\frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

+ve

Since $[(\lambda_2/\lambda_1) - 1]$ is positive, the slope of the curve approaches zero as $|z_1| \to 0$ and approaches ∞ as $|z_1| \to \infty$. Therefore, as the trajectory approaches the origin, it becomes tangent to the z_1-axis; as it approaches ∞, it becomes parallel to the z_2-axis. These observations allow us to sketch the typical family of trajectories shown in Figure 2.3. When transformed back into the x-coordinates, the family of trajectories will have the typical portrait shown in Figure 2.4(a). Note that in the x_1–x_2 plane, the trajectories become tangent to the slow eigenvector v_1 as they approach the origin and parallel to the fast eigenvector v_2 far from the origin. In this situation, the equilibrium point $x = 0$ is called a *stable node.*

When λ_1 and λ_2 are positive, the phase portrait will retain the character of Figure 2.4(a), but with the trajectory directions reversed, since the exponential terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially as t increases. Figure 2.4(b) shows the phase portrait for the case $\lambda_2 > \lambda_1 > 0$. The equilibrium point $x = 0$ is referred to in this instance as an *unstable node.*

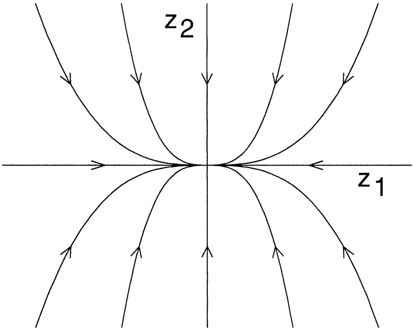

Figure 2.3: Phase portrait of a stable node in modal coordinates.

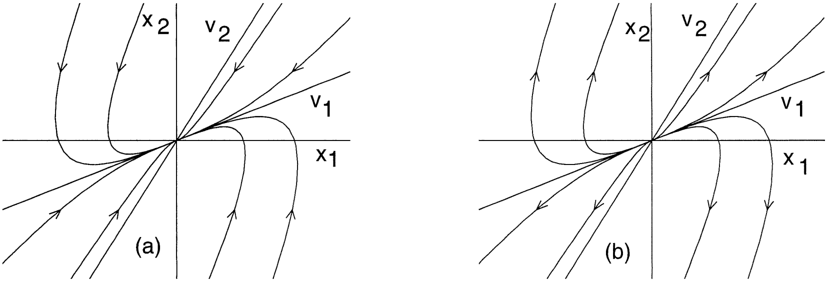

Figure 2.4: Phase portraits for (a) a stable node; (b) an unstable node.

Suppose now that the eigenvalues have opposite signs. In particular, let $\lambda_2 < 0 < \lambda_1$. In this case, $e^{\lambda_1 t} \to \infty$, while $e^{\lambda_2 t} \to 0$ as $t \to \infty$. Hence, we call λ_2 the stable eigenvalue and λ_1 the unstable eigenvalue. Correspondingly, v_2 and v_1 are called the stable and unstable eigenvectors, respectively. Equation (2.4) will have a negative exponent (λ_2/λ_1). Thus, the family of trajectories in the z_1–z_2 plane will take the typical form shown in Figure 2.5(a). Trajectories have hyperbolic shapes. They become tangent to the z_1-axis as $|z_1| \to \infty$ and tangent to the z_2-axis as $|z_1| \to 0$. The only exception to these hyperbolic shapes are the four trajectories along the axes. The two trajectories along the z_2-axis are called the stable trajectories since they approach the origin as $t \to \infty$, while the two trajectories along the z_1-axis are called the unstable trajectories since they approach infinity as $t \to \infty$. The phase portrait in the x_1–x_2 plane is shown in Figure 2.5(b). Here the stable trajectories are along the stable eigenvector v_2 and the unstable trajectories are along the unstable eigenvector v_1. In this case, the equilibrium point is called a *saddle*.

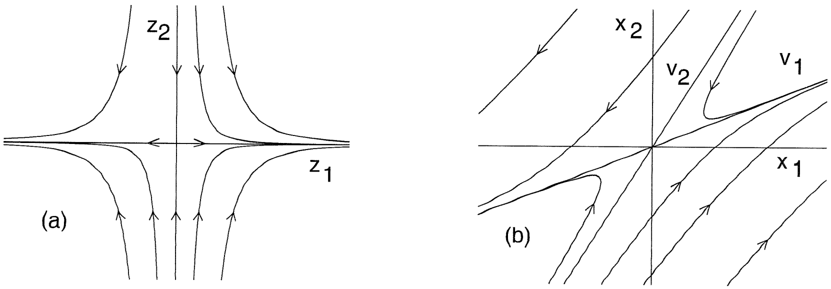

Figure 2.5: Phase portrait of a saddle point (a) in modal coordinates; (b) in original coordinates.

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$.

The change of coordinates $z = M^{-1}x$ transforms the system (2.3) into the form

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

The solution of these equations is oscillatory and can be expressed more conveniently in the polar coordinates

$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

where we have two uncoupled first-order differential equations:

$$\dot{r} = \alpha r \quad \text{and} \quad \dot{\theta} = \beta$$

The solution for a given initial state (r_0, θ_0) is given by

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

which define a logarithmic spiral in the z_1–z_2 plane. Depending on the value of α, the trajectory will take one of the three forms shown in Figure 2.6. When $\alpha < 0$, the spiral converges to the origin; when $\alpha > 0$, it diverges away from the origin. When $\alpha = 0$, the trajectory is a circle of radius r_0. Figure 2.7 shows the trajectories in the x_1–x_2 plane. The equilibrium point $x = 0$ is referred to as a *stable focus* if $\alpha < 0$, *unstable focus* if $\alpha > 0$, and *center* if $\alpha = 0$.

$\alpha < c$ $\alpha > c$

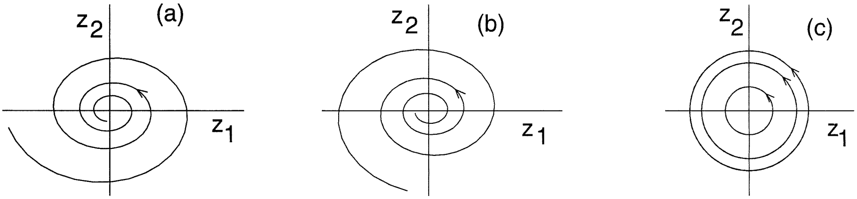

Figure 2.6: Typical trajectories in the case of complex eigenvalues.
(a) $\alpha < 0$; (b) $\alpha > 0$; (c) $\alpha = 0$.

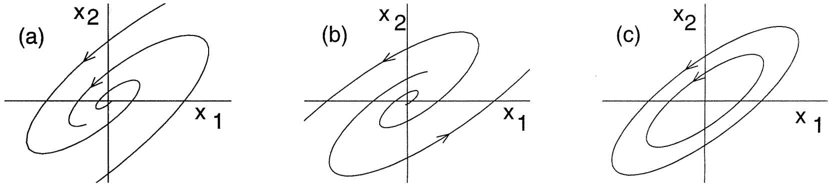

Figure 2.7: Phase portraits for (a) a stable focus; (b) an unstable focus; (c) a center.

Case 3. Nonzero multiple eigenvalues: $\lambda_1 = \lambda_2 = \lambda \neq 0$.

The change of coordinates $z = M^{-1}x$ transforms the system (2.3) into the form

$$\dot{z}_1 = \lambda z_1 + k z_2, \qquad \dot{z}_2 = \lambda z_2$$

whose solution, for a given initial state (z_{10}, z_{20}), is given by

$$z_1(t) = e^{\lambda t}(z_{10} + k z_{20} t), \qquad z_2(t) = e^{\lambda t} z_{20}$$

Eliminating t, we obtain the trajectory equation

$$z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln \left(\frac{z_2}{z_{20}} \right) \right]$$

Figure 2.8 shows the form of the trajectories when $k = 0$, while Figure 2.9 shows their form when $k = 1$. The phase portrait has some similarity with the portrait of a node. Therefore, the equilibrium point $x = 0$ is usually referred to as a stable node if $\lambda < 0$ and unstable node if $\lambda > 0$. Note, however, that the phase portraits of Figures 2.8 and 2.9 do not have the asymptotic slow–fast behavior that we saw in Figures 2.3 and 2.4.

Before we discuss the degenerate case when one or both of the eigenvalues are zero, let us summarize our findings about the qualitative behavior of the system when the equilibrium point $x = 0$ is isolated. We have seen that the system can display six qualitatively different phase portraits, which are associated with different

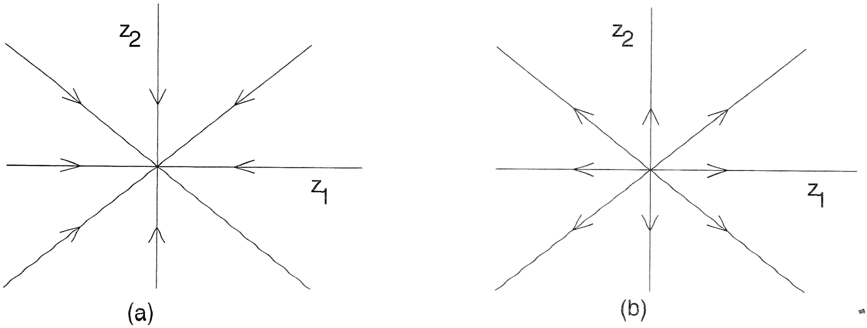

Figure 2.8: Phase portraits for the case of nonzero multiple eigenvalues when $k = 0$: (a) $\lambda < 0$; (b) $\lambda > 0$.

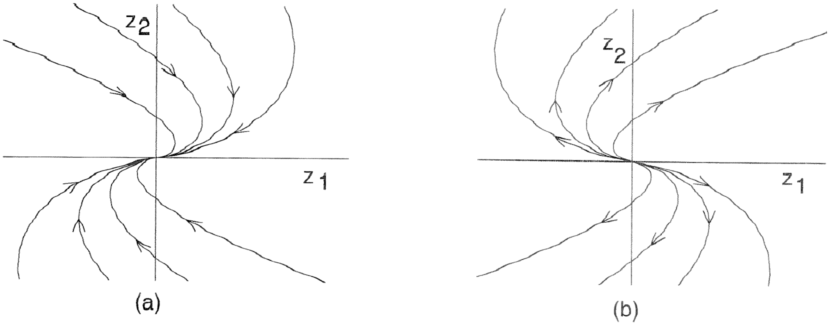

Figure 2.9: Phase portraits for the case of nonzero multiple eigenvalues when $k = 1$: (a) $\lambda < 0$; (b) $\lambda > 0$.

types of equilibria: stable node, unstable node, saddle point, stable focus, unstable focus, and center. The type of equilibrium point is completely specified by the location of the eigenvalues of A. Note that the global (throughout the phase plane) qualitative behavior of the system is determined by the type of equilibrium point. This is a characteristic of linear systems. When we study the qualitative behavior of nonlinear systems in the next section, we shall see that the type of equilibrium point can only determine the qualitative behavior of the trajectories in the vicinity of that point.

Case 4. One or both eigenvalues are zero.

When one or both eigenvalues of A are zero, the phase portrait is in some sense degenerate. Here, the matrix A has a nontrivial null space. Any vector in the null space of A is an equilibrium point for the system; that is, the system has an

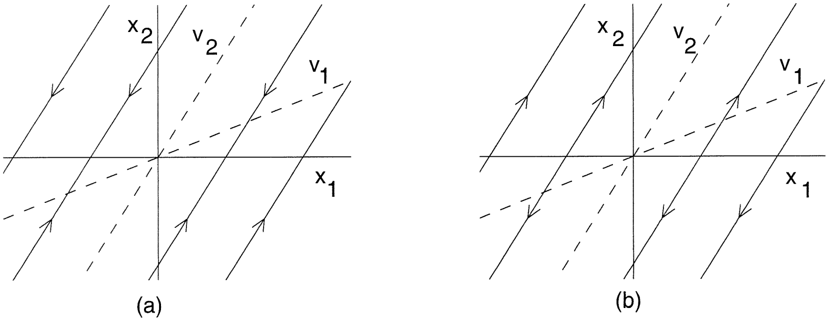

Figure 2.10: Phase portraits for (a) $\lambda_1 = 0,\ \lambda_2 < 0$; (b) $\lambda_1 = 0,\ \lambda_2 > 0$.

equilibrium subspace, rather than an equilibrium point. The dimension of the null space could be one or two; if it is two, the matrix A will be the zero matrix. This is a trivial case where every point in the plane is an equilibrium point. When the dimension of the null space is one, the shape of the Jordan form of A will depend on the multiplicity of the zero eigenvalue. When $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the matrix M is given by $M = [v_1, v_2]$ where v_1 and v_2 are the associated eigenvectors. Note that v_1 spans the null space of A. The change of variables $z = M^{-1}x$ results in

$$\dot{z}_1 = 0, \qquad \dot{z}_2 = \lambda_2 z_2$$

whose solution is

$$z_1(t) = z_{10}, \qquad z_2(t) = z_{20}e^{\lambda_2 t}$$

The exponential term will grow or decay, depending on the sign of λ_2. Figure 2.10 shows the phase portrait in the x_1–x_2 plane. All trajectories converge to the equilibrium subspace when $\lambda_2 < 0$, and diverge away from it when $\lambda_2 > 0$.

When both eigenvalues are at the origin, the change of variables $z = M^{-1}x$ results in

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = 0$$

whose solution is

$$z_1(t) = z_{10} + z_{20}t, \qquad z_2(t) = z_{20}$$

The term $z_{20}t$ will increase or decrease, depending on the sign of z_{20}. The z_1-axis is the equilibrium subspace. Figure 2.11 shows the phase portrait in the x_1–x_2 plane; the dashed line is the equilibrium subspace. The phase portrait in Figure 2.11 is quite different from that in Figure 2.10. Trajectories starting off the equilibrium subspace move parallel to it.

The study of the behavior of linear systems about the equilibrium point $x = 0$ is important because, in many cases, the local behavior of a nonlinear system near an equilibrium point can be deduced by linearizing the system about that point and

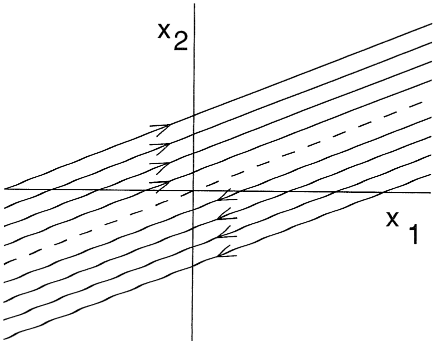

Figure 2.11: Phase portrait when $\lambda_1 = \lambda_2 = 0$.

studying the behavior of the resultant linear system. How conclusive the linearization approach is depends to a great extent on how the various qualitative phase portraits of a linear system persist under perturbations. We can gain insight into the behavior of a linear system under perturbations by examining the special case of linear perturbations. Suppose A has distinct eigenvalues and consider $A + \Delta A$, where ΔA is a 2×2 real matrix whose elements have arbitrarily small magnitudes. From the perturbation theory of matrices,[2] we know that the eigenvalues of a matrix depend continuously on its parameters. This means that, given any positive number ε, there is a corresponding positive number δ such that if the magnitude of the perturbation in each element of A is less than δ, the eigenvalues of the perturbed matrix $A + \Delta A$ will lie in open discs of radius ε centered at the eigenvalues of A. Consequently, any eigenvalue of A that lies in the open right-half plane (positive real part) or in the open left-half plane (negative real part) will remain in its respective half of the plane after arbitrarily small perturbations. On the other hand, eigenvalues on the imaginary axis, when perturbed, might go into either the right-half or the left-half of the plane, since a disc centered on the imaginary axis will extend in both halves no matter how small ε is. Consequently, we can conclude that if the equilibrium point $x = 0$ of $\dot{x} = Ax$ is a node, focus, or saddle point, then the equilibrium point $x = 0$ of $\dot{x} = (A + \Delta A)x$ will be of the same type for sufficiently small perturbations. The situation is quite different if the equilibrium point is a center. Consider the perturbation of the real Jordan form in the case of a center

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

where μ is a perturbation parameter. When μ is positive, the equilibrium point of the perturbed system is an unstable focus; when μ is negative, it is a stable focus.

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[2] See [67, Chapter 7].

This is true no matter how small μ is, as long as it is different from zero. Because the phase portraits of a stable focus and unstable focus are qualitatively different from the phase portrait of a center, we see that a center equilibrium point will not persist under perturbations. The node, focus, and saddle equilibrium points are said to be *structurally stable* because they maintain their qualitative behavior under infinitesimally small perturbations,[3] while the center equilibrium point is not structurally stable. The distinction between the two cases is due to the location of the eigenvalues of A, with the eigenvalues on the imaginary axis being vulnerable to perturbations. This brings in the definition of a *hyperbolic equilibrium point.* The origin $x = 0$ is said to be a hyperbolic equilibrium point of $\dot{x} = Ax$ if A has no eigenvalues with zero real part.[4]

When A has multiple nonzero real eigenvalues, infinitesimally small perturbations could result in a pair of complex eigenvalues. Hence, a stable (respectively, unstable) node would either remain a stable (respectively, unstable) node or become a stable (respectively, unstable) focus.

When A has eigenvalues at zero, one would expect perturbations to move these eigenvalues away from zero, resulting in a major change in the phase portrait. It turns out, however, that there is an important difference between the case when there is only one eigenvalue at zero and the case when both eigenvalues are at zero ($A \neq 0$). In the first case, perturbation of the zero eigenvalue results in a real eigenvalue $\lambda_1 = \mu$, where μ could be positive or negative. Since the other eigenvalue λ_2 is different from zero, its perturbation will keep it away from zero. Moreover, since we are talking about arbitrarily small perturbations, $|\lambda_1| = |\mu|$ will be much smaller than $|\lambda_2|$. Thus, we end up with two real distinct eigenvalues, which means that the equilibrium point of the perturbed system will be a node or a saddle point, depending on the signs of λ_2 and μ. This is already an important change in the phase portrait. However, a careful examination of the phase portrait gives more insight into the qualitative behavior of the system. Since $|\lambda_1| \ll |\lambda_2|$, the exponential term $e^{\lambda_2 t}$ will change with t much faster than the exponential term $e^{\lambda_1 t}$, resulting in the typical phase portraits of a node and a saddle shown in Figure 2.12, for the case $\lambda_2 < 0$. Comparing these phase portraits with Figure 2.10(a) shows some similarity. In particular, similar to Figure 2.10, trajectories starting off the eigenvector v_1 converge to that vector along lines (almost) parallel to the eigenvector v_2. As they approach the vector v_1, they become tangent to it and move along it. When $\mu < 0$, the motion along v_1 converges to the origin (stable node), while when $\mu > 0$ the motion along v_1 tends to infinity (saddle point). This qualitative behavior is characteristic of singularly perturbed systems, which will be studied in Chapter 11.

When both eigenvalues of A are zeros, the effect of perturbations is more dra-

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[3] See [81, Chapter 16] for a rigorous and more general definition of structural stability.

[4] This definition of a hyperbolic equilibrium point extends to higher-dimensional systems. It also carries over to equilibria of nonlinear systems by applying it to the eigenvalues of the linearized system.

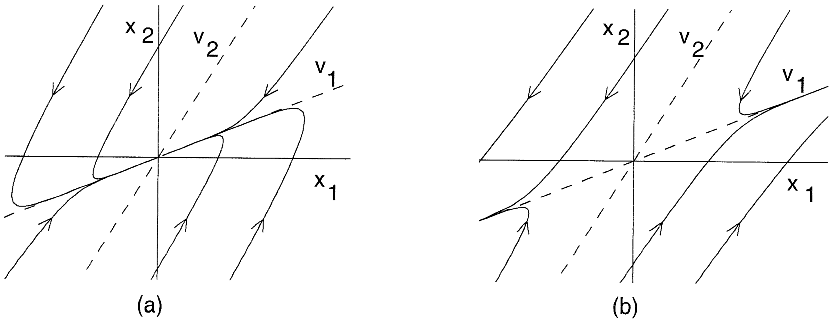

Figure 2.12: Phase portraits of a perturbed system when $\lambda_1 = 0$ and $\lambda_2 < 0$: (a) $\mu < 0$; (b) $\mu > 0$.

matic. Consider the four possible perturbations of the Jordan form

$$\begin{bmatrix} 0 & 1 \\ -\mu^2 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mu & 1 \\ -\mu^2 & \mu \end{bmatrix}, \quad \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \mu & 1 \\ 0 & -\mu \end{bmatrix}$$

where μ is a perturbation parameter that could be positive or negative. It can easily be seen that the equilibrium points in these four cases are a center, a focus, a node, and a saddle point, respectively. In other words, all the possible phase portraits of an isolated equilibrium point could result from perturbations.

2.2 Multiple Equilibria

The linear system $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$ if A has no zero eigenvalues, that is, if $\det A \neq 0$. When $\det A = 0$, the system has a continuum of equilibrium points. These are the only possible equilibria patterns that a linear system may have. A nonlinear system can have multiple isolated equilibrium points. In the following two examples, we explore the qualitative behavior of the tunnel-diode circuit of Section 1.2.2 and the pendulum equation of Section 1.2.1. Both systems exhibit multiple isolated equilibria.

Example 2.1 The state model of a tunnel-diode circuit is given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u] \end{aligned}$$

Assume that the circuit parameters are[5] $u = 1.2\ V$, $R = 1.5\ k\Omega = 1.5 \times 10^3\ \Omega$, $C = 2\ pF = 2 \times 10^{-12}\ F$, and $L = 5\ \mu H = 5 \times 10^{-6}\ H$. Measuring time in

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[5] The numerical data are taken from [39].

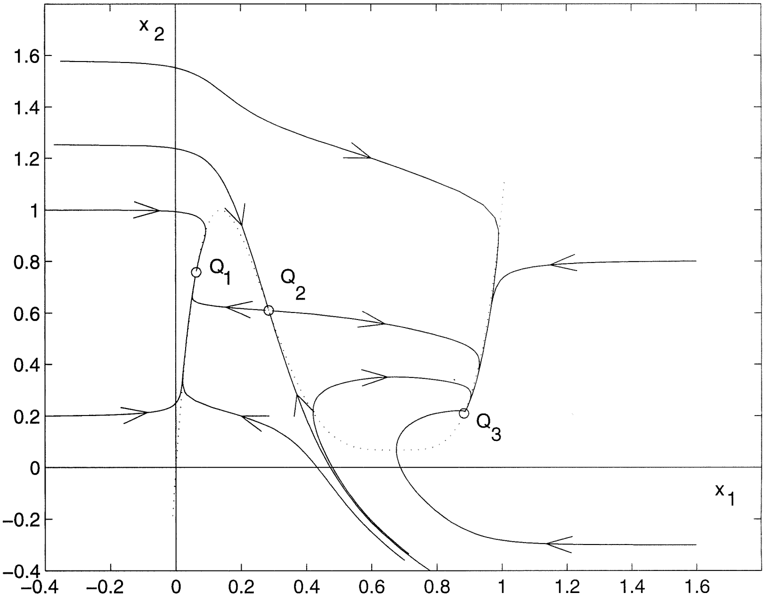

Figure 2.13: Phase portrait of the tunnel-diode circuit of Example 2.1.

nanoseconds and the currents x_2 and $h(x_1)$ in mA, the state model is given by

$$\begin{aligned} \dot{x}_1 &= 0.5[-h(x_1) + x_2] \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2) \end{aligned}$$

Suppose that $h(\cdot)$ is given by

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$

By setting $\dot{x}_1 = \dot{x}_2 = 0$ and solving for the equilibrium points, we can verify that there are three equilibrium points at $(0.063, 0.758)$, $(0.285, 0.61)$, and $(0.884, 0.21)$. The phase portrait of the system, generated by a computer program, is shown in Figure 2.13. The three equilibrium points are denoted in the portrait by Q_1, Q_2, and Q_3, respectively. Examination of the phase portrait shows that, except for two special trajectories, which approach Q_2, all trajectories eventually approach either Q_1 or Q_3. Near the equilibrium points, the trajectories take the form of a saddle for Q_2 and stable nodes for Q_1 and Q_3. The two special trajectories, which approach Q_2, are the stable trajectories of the saddle. They form a curve that divides the plane into two halves. All trajectories originating from the left side of the curve will approach Q_1, while all trajectories originating from the right side will approach Q_3. This special curve is called a *separatrix*, because it partitions the

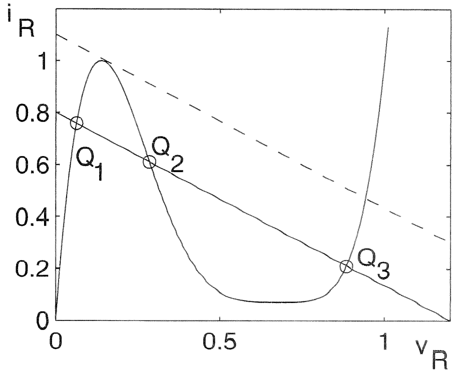

Figure 2.14: Adjustment of the load line of the tunnel-diode circuit during triggering.

plane into two regions of different qualitative behavior.[6] In an experimental setup, we shall observe one of the two steady-state operating points Q_1 or Q_3, depending on the initial capacitor voltage and inductor current. The equilibrium point at Q_2 is never observed in practice because the ever-present physical noise would cause the trajectory to diverge from Q_2 even if it were possible to set up the exact initial conditions corresponding to Q_2.

The phase portrait in Figure 2.13 tells us the global qualitative behavior of the tunnel-diode circuit. The range of x_1 and x_2 was chosen so that all essential qualitative features are displayed. The portrait outside this range does not contain any new qualitative features.

The tunnel-diode circuit with multiple equilibria is referred to as a *bistable* circuit, because it has two steady-state operating points. It has been used as a computer memory, where the equilibrium point Q_1 is associated with the binary state "0" and the equilibrium point Q_3 is associated with the binary state "1." Triggering from Q_1 to Q_3 or vice versa is achieved by a triggering signal of sufficient amplitude and duration that allows the trajectory to move to the other side of the separatrix. For example, if the circuit is initially at Q_1, then a positive pulse added to the supply voltage u will carry the trajectory to the right side of the separatrix. The pulse must be adequate in amplitude to raise the load line beyond the dashed line in Figure 2.14 and long enough to allow the trajectory to reach the right side of the separatrix.

Another feature of this circuit can be revealed if we view it as a system with input $u = E$ and output $y = v_R$. Suppose we start with a small value of u such that the only equilibrium point is Q_1. After a transient period, the system settles at Q_1. Let us now increase u gradually, allowing the circuit to settle at an equilibrium point

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[6] In general, the state plane decomposes into a number of regions, within each of which the trajectories may show a different type of behavior. The curves separating these regions are called separatrices.

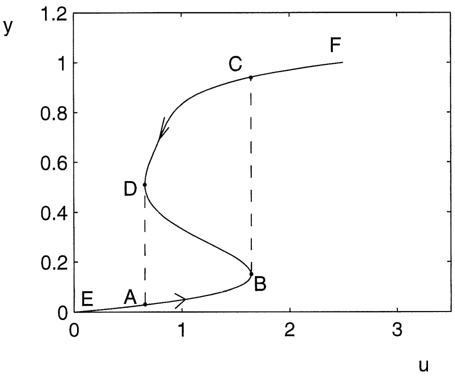

Figure 2.15: Hysteresis characteristics of the tunnel-diode circuit.

after each increment of u. For a range of values of u, Q_1 will be the only equilibrium point. On the input–output characteristic of the system, shown in Figure 2.15, this range corresponds to the segment EA. As the input is increased beyond the point A, the circuit will have two steady-state operating points at Q_1, on the segment AB, and Q_3, on the segment CD. Since we are increasing u gradually, the initial conditions will be near Q_1 and the circuit will settle there. Hence, the output will be on the segment AB. With further increase of u, we will reach a point where the circuit will have only one equilibrium point at Q_3. Therefore, after a transient period the circuit will settle at Q_3. On the input–output characteristic, it will appear as a jump from B to C. For higher values of u, the output will remain on the segment CF. Suppose now that we start decreasing u gradually. First, there will be only one equilibrium point Q_3; that is, the output will move along the segment FC. Beyond a certain value of u, corresponding to the point C, the circuit will have two steady-state operating points at Q_1 and Q_3, but will settle at Q_3 because its initial conditions will be closer to it. Hence, the output will be on the segment CD. Eventually, as we decrease u beyond the value corresponding to D, the circuit will have only one equilibrium point at Q_1 and the characteristic will exhibit another jump from D to A. Thus, the input–output characteristic of the system features a hysteresis behavior. Notice that by drawing the input–output characteristic of Figure 2.15, we ignore the dynamics of the system. Such viewpoint will be reasonable when the input is slowly varying relative to the dynamics of the system so that the transient time between different steady-state operating points can be neglected.[7] $\triangle$

Example 2.2 Consider the following pendulum equation with friction:

$$\dot{x}_1 = x_2$$

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[7] This statement can be justified by the singular perturbation theory presented in Chapter 11.

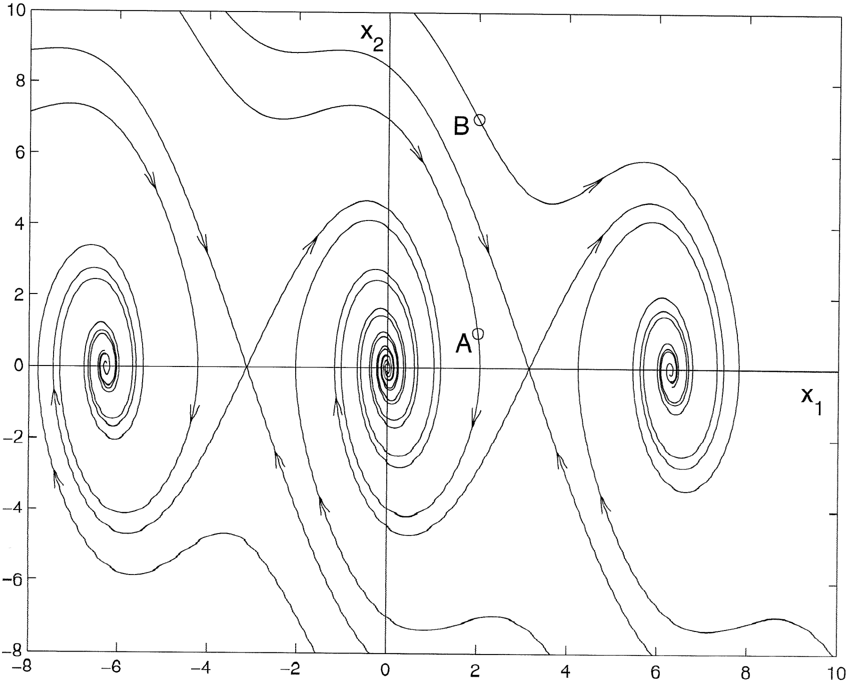

Figure 2.16: Phase portrait of the pendulum equation of Example 2.2.

$$\dot{x}_2 \quad = \quad -10 \sin x_1 - x_2$$

A computer-generated phase portrait is shown in Figure 2.16. The phase portrait is periodic in x_1 with period 2π. Consequently, all distinct features of the system's qualitative behavior can be captured by drawing the portrait in the vertical strip $-\pi \leq x_1 \leq \pi$. As we noted earlier, the equilibrium points $(0,0)$, $(2\pi, 0)$, $(-2\pi, 0)$, etc., correspond to the downward equilibrium position $(0,0)$. Trajectories near these equilibrium points have the pattern of a stable focus. On the other hand, the equilibrium points at $(\pi, 0)$, $(-\pi, 0)$, etc., correspond to the upward equilibrium position $(\pi, 0)$. Trajectories near these equilibrium points have the pattern of a saddle. The stable trajectories of the saddles at $(\pi, 0)$ and $(-\pi, 0)$ form separatrices which contain a region with the property that all trajectories in its interior approach the equilibrium point $(0,0)$. This picture is repeated periodically. The fact that trajectories could approach different equilibrium points correspond to the number of full swings a trajectory would take before it settles at the downward equilibrium position. For example, the trajectories starting at points A and B have the same initial position, but different speeds. The trajectory starting at A oscillates with decaying amplitude until it settles down at equilibrium. The trajectory starting at B, on the other hand, has more initial kinetic energy. It makes a full swing before it starts to oscillate with decaying amplitude. Once again, notice that the "unstable" equilibrium position $(\pi, 0)$ cannot be maintained in practice, because noise would cause trajectories to diverge away from that position. △

2.3 Qualitative Behavior Near Equilibrium Points

Examination of the phase portraits in Examples 2.1 and 2.2 shows that the qualitative behavior in the vicinity of each equilibrium point looks just like those we saw in Section 2.1 for linear systems. In particular, in Figure 2.13 the trajectories near Q_1, Q_2, and Q_3 are similar to those associated with a stable node, saddle point, and stable node, respectively. Similarly, in Figure 2.16 the trajectories near $(0,0)$ and $(\pi, 0)$ are similar to those associated with a stable focus and saddle point, respectively. In this section, we will see that we could have seen this behavior near the equilibrium points without drawing the phase portrait. It will follow from the general property that, except for some special cases, the qualitative behavior of a nonlinear system near an equilibrium point can be determined via *linearization* with respect to that point.

Let $p = (p_1, p_2)$ be an equilibrium point of the nonlinear system (2.1)–(2.2) and suppose that the functions f_1 and f_2 are continuously differentiable. Expanding the right-hand side of (2.1)–(2.2) into its Taylor series about the point (p_1, p_2), we obtain

$$\begin{aligned}\dot{x}_1 &= f_1(p_1,p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.}\\ \dot{x}_2 &= f_2(p_1,p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}\end{aligned}$$

where

$$\begin{aligned}a_{11} &= \left.\frac{\partial f_1(x_1,x_2)}{\partial x_1}\right|_{x_1=p_1,x_2=p_2}, & a_{12} &= \left.\frac{\partial f_1(x_1,x_2)}{\partial x_2}\right|_{x_1=p_1,x_2=p_2}\\ a_{21} &= \left.\frac{\partial f_2(x_1,x_2)}{\partial x_1}\right|_{x_1=p_1,x_2=p_2}, & a_{22} &= \left.\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|_{x_1=p_1,x_2=p_2}\end{aligned}$$

and H.O.T. denotes higher order terms of the expansion, that is, terms of the form $(x_1-p_1)^2$, $(x_2-p_2)^2$, $(x_1-p_1)\times(x_2-p_2)$, and so on. Since (p_1, p_2) is an equilibrium point, we have

$$f_1(p_1,p_2) = f_2(p_1,p_2) = 0$$

Moreover, since we are interested in the trajectories near (p_1, p_2), we define

$$y_1 = x_1 - p_1 \quad \text{and} \quad y_2 = x_2 - p_2$$

and rewrite the state equations as

$$\begin{aligned}\dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.}\\ \dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.}\end{aligned}$$

If we restrict attention to a sufficiently small neighborhood of the equilibrium point such that the higher-order terms are negligible, then we may drop these terms and approximate the nonlinear state equations by the linear state equations

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 \end{aligned}$$

Rewriting the equations in a vector form, we obtain

$$\dot{y} = Ay$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

The matrix $[\partial f/\partial x]$ is called the Jacobian matrix of $f(x)$, and A is the Jacobian matrix evaluated at the equilibrium point p.

It is reasonable to expect the trajectories of the nonlinear system in a small neighborhood of an equilibrium point to be "close" to the trajectories of its linearization about that point. Indeed, it is true that[8] *if the origin of the linearized state equation is a stable (respectively, unstable) node with distinct eigenvalues, a stable (respectively, unstable) focus, or a saddle point, then, in a small neighborhood of the equilibrium point, the trajectories of the nonlinear state equation will behave like a stable (respectively, unstable) node, a stable (respectively, unstable) focus, or a saddle point.* Consequently, we call an equilibrium point of the nonlinear state equation (2.1)–(2.2) a stable (respectively, unstable) node, a stable (respectively, unstable) focus, or a saddle point if the linearized state equation about the equilibrium point has the same behavior. The type of equilibrium points in Examples 2.1 and 2.2 could have been determined by linearization without the need to construct the global phase portrait of the system.

Example 2.3 The Jacobian matrix of the function $f(x)$ of the tunnel-diode circuit in Example 2.1 is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

where

$$h'(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4$$

---

[8] The proof of this linearization property can be found in [76]. It is valid under the assumption that $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ have continuous first partial derivatives in a neighborhood of the equilibrium point (p_1, p_2). A related, but different, linearization result will be proved in Chapter 3 for higher-dimensional systems. (See Theorem 4.7.)

Evaluating the Jacobian matrix at the equilibrium points $Q_1 = (0.063, 0.758)$, $Q_2 = (0.285, 0.61)$, and $Q_3 = (0.884, 0.21)$, respectively, yields the three matrices

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: \; -3.57, \; -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: \; 1.77, \; -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: \; -1.33, \; -0.4$$

Thus, Q_1 is a stable node, Q_2 is a saddle point, and Q_3 is a stable node. △

Example 2.4 The Jacobian matrix of the function $f(x)$ of the pendulum equation in Example 2.2 is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}$$

Evaluating the Jacobian matrix at the equilibrium points $(0, 0)$ and $(\pi, 0)$ yields, respectively, the two matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad \text{Eigenvalues}: \; -0.5 \pm j3.12$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \quad \text{Eigenvalues}: \; -3.7, \; 2.7$$

Thus, the equilibrium point $(0, 0)$ is a stable focus and the equilibrium point $(\pi, 0)$ is a saddle point. △

Note that the foregoing linearization property dealt only with cases when the linearized state equation has no eigenvalues on the imaginary axis, that is, when the origin is a hyperbolic equilibrium point of the linear system. We extend this definition to nonlinear systems and say that an equilibrium point is hyperbolic if the Jacobian matrix, evaluated at that point, has no eigenvalues on the imaginary axis. If the Jacobian matrix has eigenvalues on the imaginary axis, then the qualitative behavior of the nonlinear state equation near the equilibrium point could be quite distinct from that of the linearized state equation. This should come as no surprise in view of our earlier discussion on the effect of linear perturbations on the qualitative behavior of a linear system when the origin is not a hyperbolic equilibrium point. The example that follows considers a case when the origin of the linearized state equation is a center.

Example 2.5 The system

$$\begin{aligned} \dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2) \end{aligned}$$

has an equilibrium point at the origin. The linearized state equation at the origin has eigenvalues $\pm j$. Thus, the origin is a center equilibrium point for the linearized system. We can determine the qualitative behavior of the nonlinear system by representing it in the polar coordinates:

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta$$

which yield

$$\dot{r} = -\mu r^3 \quad \text{and} \quad \dot{\theta} = 1$$

From these equations, it can be easily seen that the trajectories of the nonlinear system will resemble a stable focus when $\mu > 0$ and an unstable focus when $\mu < 0$. △

The preceding example shows that the qualitative behavior describing a center in the linearized state equation is not preserved in the nonlinear state equation.

The foregoing discussion excludes the case when the linearized state equation has a node with multiple eigenvalues. Exercise 2.5 shows a case where the linearization has a stable node, while the trajectories of the nonlinear state equation behave like a stable focus. It should be mentioned, however, that a smoother function $f(x)$ will not allow this to happen. In particular, if $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are analytic functions[9] in a neighborhood of the equilibrium point, then it is true that[10] *if the origin of the linearized state equation is a stable (respectively, unstable) node, then, in a small neighborhood of the equilibrium point, the trajectories of the nonlinear state equation will behave like a stable (respectively, unstable) node whether or not the eigenvalues of the linearization are distinct.*

Determining the type of equilibrium points via linearization provides useful information that should be used when we construct a global phase portrait of a second-order system, whether we do that graphically or numerically. In fact, the first step in constructing a phase portrait should be the calculation of all equilibrium points and determining the type of isolated ones via linearization, which will give us a clear idea about the expected portrait in the neighborhood of the equilibrium points.

2.4 Limit Cycles

Oscillation is one of the most important phenomena that occur in dynamical systems. A system oscillates when it has a *nontrivial periodic solution*

$$x(t+T) = x(t), \quad \forall\, t \geq 0$$

for some $T > 0$. The word "nontrivial" is used to exclude constant solutions corresponding to equilibrium points. A constant solution satisfies the preceding equation,

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[9] That is, f_1 and f_2 have convergent Taylor series representations.

[10] See [115, Theorem 3.4, page 188].

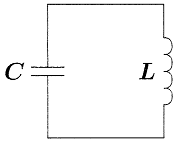


Figure 2.17: A linear LC circuit for the harmonic oscillator.

but it is not what we have in mind when we talk of oscillation or periodic solutions. Unless otherwise specified, from this point on whenever we refer to a periodic solution, we will mean a nontrivial one. The image of a periodic solution in the phase portrait is a closed trajectory, which is usually called a *periodic orbit* or a *closed orbit.*

We have already seen an example of oscillation in Section 2.1: the second-order linear system with eigenvalues $\pm j\beta$. The origin of that system is a center and the trajectories are closed orbits. When the system is transformed into its real Jordan form, the solution is given by

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

where

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

Therefore, the system has a sustained oscillation of amplitude r_0. It is usually referred to as the *harmonic oscillator.* If we think of the harmonic oscillator as a model for the linear LC circuit of Figure 2.17, then we can see that the physical mechanism leading to these oscillations is a periodic exchange (without dissipation) of the energy stored in the capacitor's electric field with the energy stored in the inductor's magnetic field. There are, however, two fundamental problems with this linear oscillator. The first problem is one of robustness. We have seen that infinitesimally small right-hand side (linear or nonlinear) perturbations will destroy the oscillation. That is, *the linear oscillator is not structurally stable.* In fact, it is impossible to build an LC circuit that realizes the harmonic oscillator, for the resistance in the electric wires alone will eventually consume whatever energy was initially stored in the capacitor and inductor. Even if we succeeded in building the linear oscillator, we would face the second problem: *the amplitude of oscillation is dependent on the initial conditions.*

The two fundamental problems of the linear oscillator can be eliminated in nonlinear oscillators. It is possible to build physical nonlinear oscillators such that

- The nonlinear oscillator is structurally stable.
- The amplitude of oscillation (at steady state) is independent of initial conditions.

The negative-resistance oscillator of Section 1.2.4 is an example of such nonlinear oscillators. The state equations of the system are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2 \end{aligned}$$

where the function h satisfies certain properties, stated in Section 1.2.4. The system has only one equilibrium point at $x_1 = x_2 = 0$. The Jacobian matrix at this point is given by

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

Since $h'(0) < 0$, the origin is either an unstable node or unstable focus, depending on the value of $\varepsilon h'(0)$. In either case, all trajectories starting near the origin would diverge away from it and head toward infinity. The repelling feature of the origin is due to the negative resistance of the resistive element near the origin, which means that the resistive element is "active" and supplies energy. This point can be seen analytically by writing an expression for the rate of change of energy. The total energy stored in the capacitor and inductor at any time t is given by

$$E = \tfrac{1}{2}Cv_C^2 + \tfrac{1}{2}Li_L^2$$

We have seen in Section 1.2.4 that

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

Thus, recalling that $\varepsilon = \sqrt{L/C}$, we can rewrite the energy expression as

$$E = \tfrac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

The rate of change of energy is given by

$$\begin{aligned} \dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\ &= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\ &= C[x_1x_2 - \varepsilon x_1h(x_1) - x_1x_2] \\ &= -\varepsilon Cx_1h(x_1) \end{aligned}$$

The preceding expression confirms that, near the origin, the trajectory gains energy since for small $|x_1|$ the term $x_1h(x_1)$ is negative. It also shows that there is a strip $-a \le x_1 \le b$ such that the trajectory gains energy within the strip and loses energy outside the strip. The strip boundaries $-a$ and b are roots of $h(x_1) = 0$, as shown in Figure 2.18. As a trajectory moves in and out of the strip, there is an exchange of energy with the trajectory gaining energy inside the strip and losing it outside. A stationary oscillation will occur if, along a trajectory, the net exchange of energy

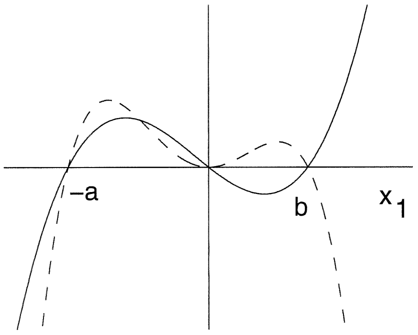
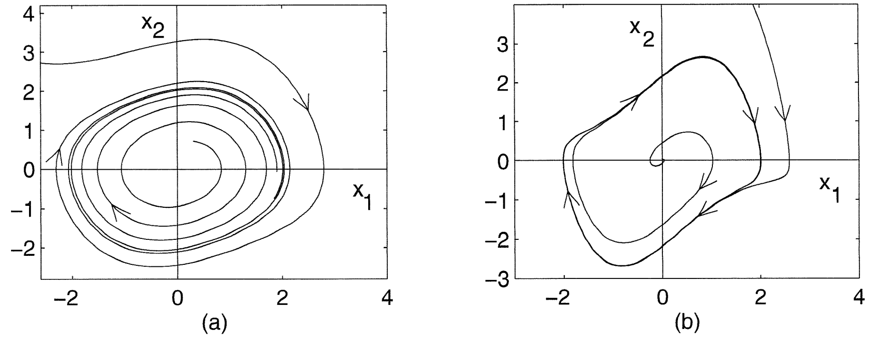


Figure 2.18: A sketch of $h(x_1)$ (solid) and $-x_1 h(x_1)$ (dashed), which shows that $\dot{E}$ is positive for $-a \le x_1 \le b$.



(a) (b)

Figure 2.19: Phase portraits of the Van der Pol oscillator: (a) $\varepsilon = 0.2$; (b) $\varepsilon = 1.0$.

over one cycle is zero. Such a trajectory will be a closed orbit. It turns out that the negative-resistance oscillator has an isolated closed orbit, which is illustrated in the next example for the Van der Pol oscillator.

Example 2.6 Figures 2.19(a), 2.19(b), and 2.20(a) show the phase portraits of the Van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 & (2.5)\\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 & (2.6) \end{aligned}$$

for three different values of the parameter ε: a small value of 0.2, a medium value of 1.0, and a large value of 5.0. In all three cases, the phase portraits show that there is a unique closed orbit that attracts all trajectories starting off the orbit. For $\varepsilon = 0.2$, the closed orbit is a smooth orbit that is close to a circle of radius 2. This

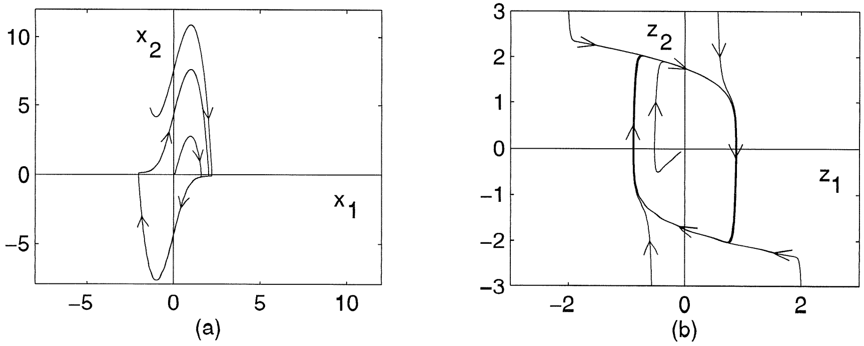


Figure 2.20: Phase portrait of the Van der Pol oscillator with $\varepsilon = 5.0$: (a) in x_1–x_2 plane; (b) in z_1–z_2 plane.

is typical for small ε (say, $\varepsilon < 0.3$). For the medium value of $\varepsilon = 1.0$, the circular shape of the closed orbit is distorted as shown in Figure 2.19(b). For the large value of $\varepsilon = 5.0$, the closed orbit is severely distorted as shown in Figure 2.20(a). A more revealing phase portrait in this case can be obtained when the state variables are chosen as $z_1 = i_L$ and $z_2 = v_C$, resulting in the state equations

$$\begin{aligned} \dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\ \dot{z}_2 &= -\varepsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3) \end{aligned}$$

The phase portrait in the z_1–z_2 plane for $\varepsilon = 5.0$ is shown in Figure 2.20(b). The closed orbit is very close to the curve $z_1 = z_2 - (1/3)z_2^3$, except at the corners, where it becomes nearly vertical. The vertical portion of the closed orbit can be viewed as if the closed orbit jumps from one branch of the curve to the other as it reaches the corner. Oscillations where the *jump phenomenon* takes place are usually referred to as *relaxation oscillations*. This phase portrait is typical for large values of ε (say, $\varepsilon > 3.0$). $\triangle$

The closed orbit we have seen in Example 2.6 is different from what we have seen in the harmonic oscillator. In the case of the harmonic oscillator, there is a continuum of closed orbits, while in the Van der Pol example, there is only one isolated periodic orbit. An isolated periodic orbit is called a *limit cycle*. The limit cycle of the Van der Pol oscillator has the property that all trajectories in the vicinity of the limit cycle ultimately tend toward the limit cycle as $t \to \infty$. A limit cycle with this property is classically known as a stable limit cycle. We shall also encounter *unstable limit cycles*, which have the property that all trajectories starting from points arbitrarily close to the limit cycle will tend away from it as $t \to \infty$. (See Figure 2.21.) To see an example of an unstable limit cycle, consider

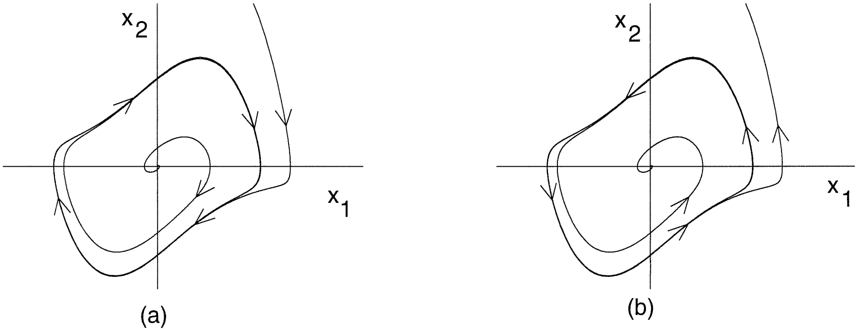


Figure 2.21: (a) A stable limit cycle; (b) an unstable limit cycle.

the Van der Pol equation in reverse time; that is,

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - \varepsilon(1 - x_1^2)x_2 \end{aligned}$$

The phase portrait of this system is identical to that of the Van der Pol oscillator, except that the arrowheads are reversed. Consequently, the limit cycle is unstable.

The limit cycle of the Van der Pol oscillator of Example 2.6 takes special forms in the limiting cases when ε is very small and very large. These special forms can be predicted analytically by using asymptotic methods. In Chapter 10, we will use the *averaging method* to derive the special form of the limit cycle as $\varepsilon \to 0$; while in Chapter 11, we will use the *singular perturbation method* to derive the special form of the limit cycle as $\varepsilon \to \infty$.

2.5 Numerical Construction of Phase Portraits

Computer programs for numerical solution of ordinary differential equations are widely available. They can be effectively used to construct phase portraits for second-order systems. In this section, we give some hints[11] that might be useful for beginners.

The first step in constructing the phase portrait is to find all equilibrium points and determine the type of isolated ones via linearization.

Drawing trajectories involves three tasks:[12]

- Selection of a bounding box in the state plane where trajectories are to be drawn. The box takes the form

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[11] These hints are taken from [149, Chapter 10], which contains more instructions on how to generate informative phase portraits.

[12] A fourth task that we left out is placing arrowheads on the trajectory. For the purpose of this textbook, it can be conveniently done manually.

$$x_{1min} \leq x_1 \leq x_{1max}, \qquad x_{2min} \leq x_2 \leq x_{2max}$$

- Selection of initial points (conditions) inside the bounding box.
- Calculation of trajectories.

Let us talk first about calculating trajectories. To find the trajectory passing through a point x_0, solve the equation

$$\dot{x} = f(x), \quad x(0) = x_0$$

in forward time (with positive t) and in reverse time (with negative t). Solution in reverse time is equivalent to solution in forward time of the equation

$$\dot{x} = -f(x), \quad x(0) = x_0$$

since the change of time variable $\tau = -t$ reverses the sign of the right-hand side. The arrowhead on the forward trajectory is placed heading away from x_0, while the one on the reverse trajectory is placed heading into x_0. Note that solution in reverse time is the only way we can get a good portrait in the neighborhood of unstable focus, unstable node, or unstable limit cycle. Trajectories are continued until they get out of the bounding box. If processing time is a concern, you may want to add a stopping criterion when trajectories converge to an equilibrium point.

The bounding box should be selected so that all essential qualitative features are displayed. Since some of these features will not be known *a priori*, we may have to adjust the bounding box interactively. However, our initial choice should make use of all prior information. For example, the box should include all equilibrium points. Care should be exercised when a trajectory travels out of bounds, for such a trajectory is either unbounded or is attracted to a stable limit cycle.

The simplest approach to select initial points is to place them uniformly on a grid throughout the bounding box. However, an evenly spaced set of initial conditions rarely yields an evenly spaced set of trajectories. A better approach is to select the initial points interactively after plotting the already calculated trajectories. Since most computer programs have sophisticated plotting tools, this approach should be quite feasible.

For a saddle point, we can use linearization to generate the stable and unstable trajectories. This is useful because, as we saw in Examples 2.1 and 2.2, the stable trajectories of a saddle define a separatrix. Let the eigenvalues of the linearization be $\lambda_1 > 0 > \lambda_2$ and the corresponding eigenvectors be v_1 and v_2. The stable and unstable trajectories of the nonlinear saddle will be tangent to the stable eigenvector v_2 and the unstable eigenvector v_1, respectively, as they approach the equilibrium point p. Therefore, the two unstable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_1$, where α is a small positive number. Similarly, the two stable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_2$. The major parts of the unstable trajectories will be generated by solution in forward time, while the major parts of the stable ones will be generated by solution in reverse time.

2.6 Existence of Periodic Orbits

Periodic orbits in the plane are special in that they divide the plane into a region inside the orbit and a region outside it. This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems. The most celebrated of these criteria are the Poincaré–Bendixson theorem, the Bendixson criterion, and the index method.

We consider the second-order autonomous system

$$\dot{x} = f(x) \tag{2.7}$$

where $f(x)$ is continuously differentiable. Poincaré–Bendixson theorem gives a condition for the existence of periodic orbits of (2.7). We will not give the formal statement of the theorem,[13] but will give a corollary of the theorem which summarizes how the theorem is actually applied. We refer to this corollary as the Poincaré–Bendixson criterion.

Lemma 2.1 (Poincaré–Bendixson Criterion) *Consider the system* (2.7) *and let M be a closed bounded subset of the plane such that*

- *M contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix $[\partial f/\partial x]$ at this point has eigenvalues with positive real parts.* (*Hence, the equilibrium point is unstable focus or unstable node.*)
- *Every trajectory starting in M stays in M for all future time.*

Then, M contains a periodic orbit of (2.7).

The intuition behind the criterion is that bounded trajectories in the plane will have to approach periodic orbits or equilibrium points as time tends to infinity. If M contains no equilibrium points, then it must contain a periodic orbit. If M contains only one equilibrium point that satisfies the stated conditions, then in the vicinity of that point all trajectories will be moving away from it. Therefore, we can choose a simple closed curve[14] around the equilibrium point such that the vector field on the curve points outward.[15] By redefining the set M to exclude the region enclosed by this curve (see Figure 2.22), we end up with a set that is free of equilibrium points, and all trajectories are trapped in it.

As a tool for investigating whether trajectories are trapped inside a set M, consider a simple closed curve defined by the equation $V(x) = c$, where $V(x)$ is continuously differentiable. The vector field $f(x)$ at a point x on the curve points

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[13] For the statement and proof of the Poincaré–Bendixson theorem, see, for example, [135] or the second edition of this book.

[14] A simple closed curve divides the plane into a bounded region inside the curve and an unbounded region outside it (examples are circles, ellipses, and polygons).

[15] See Exercise 4.33.

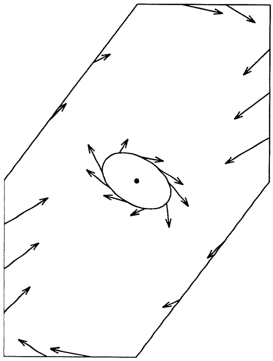

Figure 2.22: Redefinition of the set M to exclude the vicinity of an unstable focus or node.

inward if the inner product of $f(x)$ and the gradient vector $\nabla V(x)$ is negative; that is,

$$f(x) \cdot \nabla V(x) = \frac{\partial V}{\partial x_1}(x) f_1(x) + \frac{\partial V}{\partial x_2}(x) f_2(x) < 0$$

The vector field $f(x)$ points outward if $f(x) \cdot \nabla V(x) > 0$, and it is tangent to the curve if $f(x) \cdot \nabla V(x) = 0$. Trajectories can leave a set only if the vector field points outward at some point on its boundary. Therefore, for a set of the form $M = \{V(x) \leq c\}$, for some $c > 0$, trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \leq 0$ on the boundary $V(x) = c$. For an annular region of the form $M = \{W(x) \geq c_1 \text{ and } V(x) \leq c_2\}$, for some $c_1 > 0$ and $c_2 > 0$, trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \leq 0$ on $V(x) = c_2$ and $f(x) \cdot \nabla W(x) \geq 0$ on $W(x) = c_1$.

We illustrate the application of the Poincaré–Bendixson criterion in the next two examples, while the third example is a nontrivial application to the negative-resistance oscillator of Section 1.2.4.

Example 2.7 Consider the harmonic oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

and the annular region $M = \{\, c_1 \leq V(x) \leq c_2\}$, where $V(x) = x_1^2 + x_2^2$ and $c_2 > c_1 > 0$. The set M is closed, bounded, and free of equilibrium points, since the only equilibrium point is at the origin $(0,0)$. Trajectories are trapped inside M since $f(x) \cdot \nabla V(x) = 0$ everywhere. Hence, by the Poincaré–Bendixson criterion, we conclude that there is a periodic orbit in M. △

The preceding example emphasizes the fact that the Poincaré–Bendixson criterion assures the existence of a periodic orbit, but not its uniqueness. From our earlier study of the harmonic oscillator, we know that it has a continuum of periodic orbits in M.

Example 2.8 The system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

has a unique equilibrium point at the origin, and the Jacobian matrix

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left[\begin{array}{cc} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1x_2 \\ -2 - 2x_1x_2 & 1 - x_1^2 - 3x_2^2 \end{array}\right]_{x=0} = \left[\begin{array}{cc} 1 & 1 \\ -2 & 1 \end{array}\right]$$

has eigenvalues $1 \pm j\sqrt{2}$. Let $M = \{V(x) \leq c\}$, where $V(x) = x_1^2 + x_2^2$ and $c > 0$. It is clear that M is closed, bounded, and contains only one equilibrium point at which the Jacobian matrix has eigenvalues with positive real parts. On the surface $V(x) = c$, we have

$$\begin{aligned}\frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 &= 2x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)] + 2x_2[-2x_1 + x_2 - x_2(x_1^2 + x_2^2)]\\ &= 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1x_2\\ &\leq 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2)\\ &= 3c - 2c^2\end{aligned}$$

where we used the fact that $|2x_1x_2| \leq x_1^2 + x_2^2$. By choosing $c \geq 1.5$, we can ensure that all trajectories are trapped inside M. Hence, by the Poincaré–Bendixson criterion, we conclude that there is a periodic orbit in M. △

Example 2.9 The negative-resistance oscillator of Section 1.2.4 is modeled by the second-order differential equation

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0$$

where ε is a positive constant and h satisfies the conditions

$$h(0) = 0, \quad h'(0) < 0, \quad \lim_{v \to \infty} h(v) = \infty, \quad \text{and} \quad \lim_{v \to -\infty} h(v) = -\infty$$

To simplify the analysis, we impose the additional requirements

$$h(v) = -h(-v), \quad h(v) < 0 \text{ for } 0 < v < a, \quad \text{and} \quad h(v) > 0 \text{ for } v > a$$

These additional requirements are satisfied by the typical function of Figure 1.6(b), as well as by the function $h(v) = -v + (1/3)v^3$ of the Van der Pol oscillator. Choose the state variables as

$$x_1 = v \quad \text{and} \quad x_2 = \dot{v} + \varepsilon h(v)$$

to obtain the state model

$$\begin{aligned}\dot{x}_1 &= x_2 - \varepsilon h(x_1) \\ \dot{x}_2 &= -x_1\end{aligned} \tag{2.8}$$

which has a unique equilibrium point at the origin. We start our analysis by showing that every nonequilibrium solution rotates around the equilibrium point in the clockwise direction. To this end, we divide the state plane into four regions, which are determined by the intersection of the two curves

$$x_2 - \varepsilon h(x_1) = 0 \quad \text{and} \quad x_1 = 0$$

as shown in Figure 2.23. The figure also shows the general direction of the vector field $f(x)$ of (2.8) in the four regions as well as on the boundaries between them. It is not difficult to see that a solution starting at point $A = (0, p)$ on the upper half of the x_2-axis describes an orbit with an arc of the general nature shown in Figure 2.24. The point E where the arc intersects the lower half of the x_2-axis depends on the starting point A. Let us denote E by $(0, -\alpha(p))$. We will show that if p is chosen large enough, then $\alpha(p) < p$. Consider the function

$$V(x) = \tfrac{1}{2}(x_1^2 + x_2^2)$$

To show that $\alpha(p) < p$, it is enough to show that $V(E) - V(A) < 0$, since

$$V(E) - V(A) = \tfrac{1}{2}[\alpha^2(p) - p^2] \stackrel{\text{def}}{=} \delta(p)$$

The derivative of $V(x)$ is given by

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2 = -\varepsilon x_1 h(x_1)$$

Thus, $\dot{V}$ is positive for $x_1 < a$ and negative for $x_1 > a$. Now,

$$\delta(p) = V(E) - V(A) = \int_{AE} \dot{V}(x(t))\, dt$$

where the right-hand side integral is taken along the arc from A to E. If p is small, the whole arc will lie inside the strip $0 < x_1 < a$. Then, $\delta(p)$ will be positive. As p increases, a piece of the arc will lie outside the strip, that is, the piece BCD in Figure 2.24. In this case, we evaluate the integral in different ways depending on whether the arc is inside or outside the strip $0 < x_1 < a$. We divide the integral into three parts

$$\delta(p) = \delta_1(p) + \delta_2(p) + \delta_3(p)$$

where

$$\delta_1(p) = \int_{AB} \dot{V}(x(t))\, dt, \quad \delta_2(p) = \int_{BCD} \dot{V}(x(t))\, dt, \quad \delta_3(p) = \int_{DE} \dot{V}(x(t))\, dt$$

Consider first the term

$$\delta_1(p) = -\int_{AB} \varepsilon x_1 h(x_1)\ dt = -\int_{AB} \varepsilon x_1 h(x_1)\ \frac{dt}{dx_1}\ dx_1$$

Substituting for dx_1/dt from (2.8), we obtain

$$\delta_1(p) = -\int_{AB} \varepsilon x_1 h(x_1) \frac{1}{x_2 - \varepsilon h(x_1)}\ dx_1$$

where, along the arc AB, x_2 is a given function of x_1. Clearly, $\delta_1(p)$ is positive. Note that as p increases, $x_2 - \varepsilon h(x_1)$ increases for the arc AB. Hence, $\delta_1(p)$ decreases as $p \longrightarrow \infty$. Similarly, it can be shown that the third term $\delta_3(p)$ is positive and decreases as $p \longrightarrow \infty$. Consider now the second term

$$\delta_2(p) = -\int_{BCD} \varepsilon x_1 h(x_1)\ dt = -\int_{BCD} \varepsilon x_1 h(x_1)\ \frac{dt}{dx_2}\ dx_2$$

Substituting for dx_2/dt from (2.8), we obtain

$$\delta_2(p) = \int_{BCD} \varepsilon h(x_1)\ dx_2$$

where along the arc BCD, x_1 is a given function of x_2. The integral on the right-hand side is negative since $h(x_1) > 0$ and $dx_2 < 0$. As p increases, the arc $ABCDE$ moves to the right and the domain of integration for $\delta_2(p)$ increases. It follows that $\delta_2(p)$ decreases as p increases and evidently $\lim_{p\to\infty} \delta_2(p) = -\infty$. In summary, we have shown that

- $\delta(p) > 0$, if $p < r$, for some $r > 0$.
- $\delta(p)$ decreases monotonically to $-\infty$ as $p \longrightarrow \infty$, $p \geq r$.

A sketch of the function $\delta(p)$ is shown in Figure 2.25. It is now clear that by choosing p large enough, we can ensure that $\delta(p)$ is negative; hence, $\alpha(p) < p$.

Observe that, due to symmetry induced by the fact that $h(\cdot)$ is an odd function, if $(x_1(t), x_2(t))$ is a solution of (2.8), then so is $(-x_1(t), -x_2(t))$. Therefore, if we know that a path $ABCDE$ exists as in Figure 2.24, then the reflection of that path through the origin is another path. Consider $A = (0, p)$ and $E = (0, -\alpha(p))$, where $\alpha(p) < p$. Form a closed curve of the arc $ABCDE$, its reflection through the origin and segments on the x_2-axis connecting these arcs, to form a closed curve. (See Figure 2.26). Let M be the region enclosed by this closed curve. Every trajectory starting in M at $t = 0$ will remain inside for all $t \geq 0$. This is a consequence of the directions of the vector fields on the x_2-axis segments and the fact that trajectories do not intersect each other due to uniqueness of solutions. Now M is closed, bounded, and has a unique equilibrium point at the origin. The Jacobian

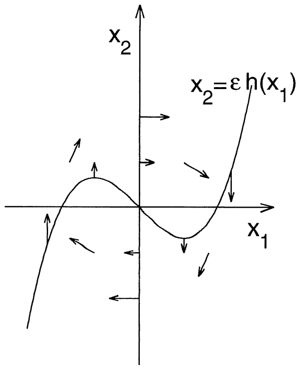

Figure 2.23: Vector field diagram for Example 2.9.

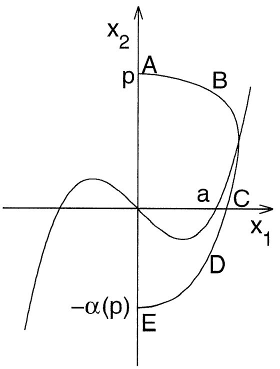

Figure 2.24: The orbit $ABCDE$ of Example 2.9.

matrix at the origin

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

has eigenvalues with positive real parts since $h'(0) < 0$. Thus, by the Poincaré–Bendixson criterion, we conclude that there is a closed orbit in M.

Using the same analysis, we can go beyond the Poincaré–Bendixson criterion and show that this closed orbit is unique. Notice that, due to the symmetry property alluded to earlier, the system can have a closed orbit if and only if $\alpha(p) = p$. From Figure 2.25, it is clear that there is only one value of p for which the condition is satisfied. Hence, there is only one closed orbit. Furthermore, we can show that every nonequilibrium solution spirals toward the unique closed orbit. To argue this point, let $p_0 > 0$ be the unique value of p for which $\alpha(p) = p$. Consider a point $(0, p)$ on the x_2-axis with $p > p_0$. As we argued earlier, the trajectory through $(0, p)$ intersects the lower half of the x_2-axis at a point $(0, -\alpha(p))$, where $\alpha(p) < p$. Due to symmetry, the trajectory through $(0, -\alpha(p))$ will meet the upper half of the x_2-axis at a point $(0, \sigma(p))$, where $p_0 \le \sigma(p) < p$. The upper bound follows from the symmetry property, while the lower bound holds since for $\sigma(p)$ to be less than p_0, the trajectory must intersect the closed orbit. The map $p \to \sigma(p)$ is continuous due to continuous dependence of the solution on the initial states.[16] Starting again at the point $(0, \sigma(p))$, the trajectory comes back to the upper half of the x_2-axis at $(0, \sigma^2(p))$, where $p_0 \le \sigma^2(p) < \sigma(p)$. By induction, we generate a sequence $\sigma^n(p)$, which satisfies

$$p_0 \le \sigma^{n+1}(p) < \sigma^n(p), \quad n = 1, 2, \ldots$$

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[16] This fact is shown in Theorem 3.4.

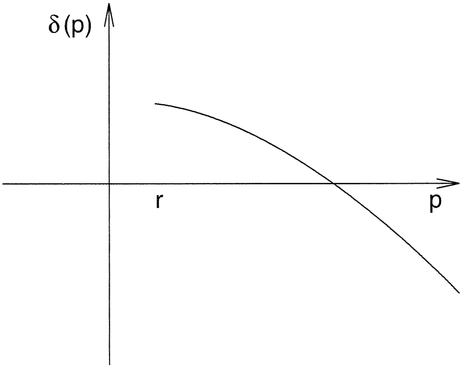


Figure 2.25: A sketch of the function $\delta(p)$ of Example 2.9.

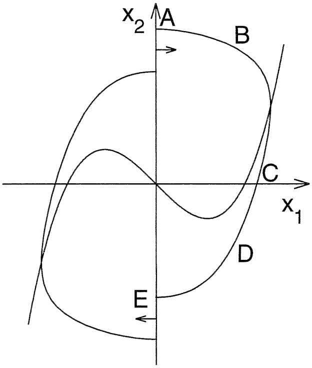


Figure 2.26: The closed curve formed in Example 2.9.

The sequence $\sigma^n(p)$ has a limit $p_1 \geq p_0$. Note that, by continuity of $\sigma(\cdot)$, the limit p_1 satisfies

$$\sigma(p_1) - p_1 = \lim_{n\to\infty} \sigma(\sigma^n(p)) - p_1 = p_1 - p_1 \;=\; 0$$

By uniqueness of the closed orbit, it must be that $p_1 = p_0$. This shows that the trajectory of p spirals toward the unique closed orbit as $t \to \infty$. The same thing is true for $p < p_0$. △

The next result, known as the *Bendixson criterion*, can be used to rule out the existence of periodic orbits in some cases.

Lemma 2.2 (Bendixson Criterion) *If, on a simply connected region*[17] *D of the plane, the expression $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign, then system* (2.7) *has no periodic orbits lying entirely in D.* ◇

Proof: On any orbit of (2.7), we have $dx_2/dx_1 = f_2/f_1$. Therefore, on any closed orbit γ, we have

$$\int_\gamma f_2(x_1, x_2)\, dx_1 - f_1(x_1, x_2)\, dx_2 = 0$$

This implies, by Green's theorem, that

$$\int\!\!\int_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\, dx_1\, dx_2 = 0$$

---

[17] A region D is simply connected if, for every simple closed curve C in D, the inner region of C is a also a subset of D. The interior of any circle is simply connected, but the annular region $0 < c_1 \leq x_1^2 + x_2^2 \leq c_2$ is not simply connected. Intuitively speaking, simple connectedness is equivalent to the absence of "holes."

where S is the interior of γ. If $\partial f_1/\partial x_1 + \partial f_2/\partial x_2 > 0$ (or < 0) on D, then we cannot find a region $S \subset D$ such that the last equality holds. Hence, there can be no closed orbits entirely in D. □

Example 2.10 Consider the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) = ax_1 + bx_2 - x_1^2 x_2 - x_1^3 \end{aligned}$$

and let D be the whole plane. We have

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = b - x_1^2$$

Hence, there can be no periodic orbits if $b < 0$. △

We conclude this section with a useful result that relates the existence of periodic orbits and equilibrium points. The result uses the (Poincaré) index of an equilibrium point. Given the second-order system (2.7), let C be a simple closed curve not passing through any equilibrium point of (2.7). Consider the orientation of the vector field $f(x)$ at a point $p \in C$. Letting p traverse C in the counterclockwise direction, the vector $f(x)$ rotates continuously and, upon returning to the original position, must have rotated an angle $2\pi k$ for some integer k, where the angle is measured counterclockwise. The integer k is called the index of the closed curve C. If C is chosen to encircle a single isolated equilibrium point $\bar{x}$, then k is called the index of $\bar{x}$. It is left to the reader (Exercise 2.25) to verify the next lemma by examining the vector fields.

Lemma 2.3

(a) *The index of a node, a focus, or a center is* $+1$.

(b) *The index of a (hyperbolic) saddle is* -1.

(c) *The index of a closed orbit is* $+1$.

(d) *The index of a closed curve not encircling any equilibrium points is* 0.

(e) *The index of a closed curve is equal to the sum of the indices of the equilibrium points within it.* ◇

As a corollary to this lemma, we have the following:

Corollary 2.1 *Inside any periodic orbit γ, there must be at least one equilibrium point. Suppose the equilibrium points inside γ are hyperbolic, then if N is the number of nodes and foci and S is the number of saddles, it must be that* $N - S = 1$. ◇

Recall that an equilibrium point is hyperbolic if the Jacobian at that point has no eigenvalues on the imaginary axis. If the equilibrium point is not hyperbolic, then its index may differ from ±1. (See Exercise 2.26.)

The index method is usually useful in ruling out the existence of periodic orbits in certain regions of the plane.

Example 2.11 The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2\\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2\end{aligned}$$

has two equilibrium points at $(0,0)$ and $(1,1)$. The Jacobian matrices at these points are

$$\left[\frac{\partial f}{\partial x}\right]_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; \qquad \left[\frac{\partial f}{\partial x}\right]_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Hence, $(0,0)$ is a saddle, while $(1,1)$ is a stable focus. The only combination of equilibrium points that can be encircled by a periodic orbit is a single focus. Other possibilities of periodic orbits, like a periodic orbit encircling both equilibria, are ruled out. △

2.7 Bifurcation

The qualitative behavior of a second-order system is determined by the pattern of its equilibrium points and periodic orbits, as well as by their stability properties. One issue of practical importance is whether the system maintains its qualitative behavior under infinitesimally small perturbations. When it does so, the system is said to be structurally stable. In this section, we are interested in the complement of structural stability. In particular, we are interested in perturbations that will change the equilibrium points or periodic orbits of the system or change their stability properties. Consider, for example, the system

$$\begin{aligned}\dot{x}_1 &= \mu - x_1^2\\ \dot{x}_2 &= -x_2\end{aligned}$$

which depends on a parameter μ. For $\mu > 0$, the system has two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. Linearization at $(\sqrt{\mu}, 0)$ results in the Jacobian matrix

$$\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

which shows that $(\sqrt{\mu}, 0)$ is a stable node, while linearization at $(-\sqrt{\mu}, 0)$ yields the Jacobian matrix

$$\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

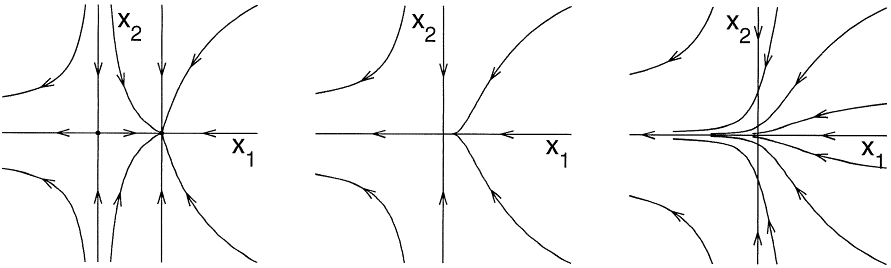
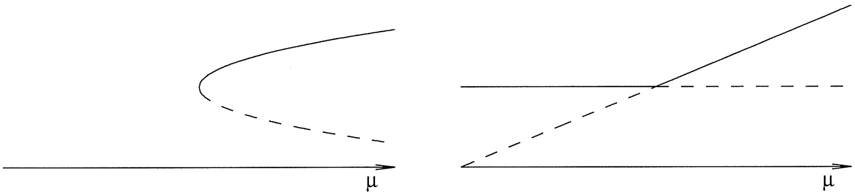


Figure 2.27: Phase portrait of the saddle–node bifurcation example for $\mu > 0$ (left), $\mu = 0$ (center), and $\mu < 0$ (right).

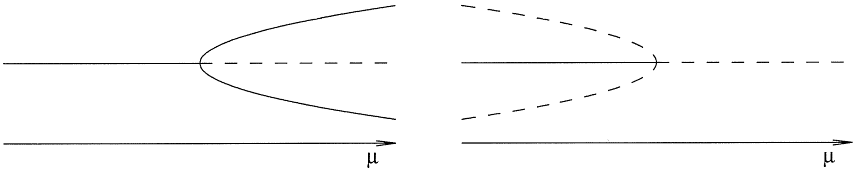
which shows that $(-\sqrt{\mu}, 0)$ is a saddle. As μ decreases, the saddle and node approach each other, collide at $\mu = 0$, and disappear for $\mu < 0$. As μ crosses zero, we witness a dramatic change in the phase portrait of the system. Figure 2.27 shows the phase portrait for positive, zero, and negative values of μ. While for positive μ, no matter how small it is, all trajectories in $\{x_1 > -\sqrt{\mu}\}$ reach steady state at the stable node, for negative μ all trajectories eventually escape to infinity. Such a change in the qualitative behavior of the system is called bifurcation. More generally, *bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied.* The parameter is called *a bifurcation parameter*, and the values at which changes occur are called *bifurcation points.* In the foregoing example, the bifurcation parameter is μ, and the bifurcation point is $\mu = 0$.

The bifurcation we saw in the previous example can be represented by the *bifurcation diagram* shown in Figure 2.28(a). The diagram sketches a measure of the amplitude (or norm) of the equilibrium points versus the bifurcation parameter. The stable node is represented by a solid line, and the saddle by a dashed line. More generally, the ordinate of a bifurcation diagram is a measure of the amplitude of equilibrium points or periodic orbits, with solid lines representing stable nodes, stable foci, and stable limit cycles and dashed lines representing unstable nodes, unstable foci, saddles, and unstable limit cycles. The bifurcation represented in Figure 2.28(a) is called *saddle–node bifurcation* because it results from the collision of a saddle and a node. Note that the Jacobian matrix has a zero eigenvalue at the bifurcation point. This feature is common to the bifurcations shown in Figures 2.28(a) through (d), which are all examples of the so called *zero-eigenvalue bifurcation.* Figure 2.28(b) shows a *transcritical bifurcation*, where equilibrium points persist through the bifurcation, but their stability properties change. For example, the system

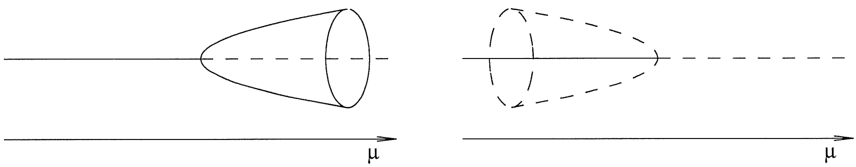
$$\begin{aligned} \dot{x}_1 &= \mu x_1 - x_1^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$



(a) Saddle–node bifurcation (b) Transcritical bifurcation



(c) Supercritical pitchfork bifurcation (d) Subcritical pitchfork bifurcation



(e) Supercritical Hopf bifurcation (f) Subcrtitical Hopf bifurcation

Figure 2.28: Bifurcation diagrams.

has two equilibrium points at $(0,0)$ and $(\mu,0)$. The Jacobian at $(0,0)$ is

$$\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$$

which shows that $(0,0)$ is a stable node for $\mu < 0$ and a saddle for $\mu > 0$. On the other hand, the Jacobian at $(\mu,0)$ is

$$\begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$$

which shows that $(\mu,0)$ is a saddle for $\mu < 0$ and a stable node for $\mu > 0$. So, while the equilibrium points persist through the bifurcation point $\mu = 0$, $(0,0)$ changes from a stable node to a saddle and $(\mu,0)$ changes from a saddle to a stable node.

Before we proceed to describe the other bifurcations in Figure 2.28, let us note an important difference between the foregoing two examples. In the latter example, crossing the bifurcation point causes the equilibrium point at the origin to change from a stable node to a saddle, but at the same time it creates a stable node at $(\mu,0)$, which for small values of μ will be close to the origin. This could mean that the impact of bifurcation on the performance of the system is not dramatic. Suppose, for example, that the nominal system has a negative value of μ so that the origin is a stable node. By sketching the phase portrait, it can be seen that all trajectories in the set $\{x_1 > \mu\}$ approach the origin as time tends to infinity. Suppose the nominal value of μ has a small magnitude so that a small perturbation can cause μ to become positive. Then, the origin will be a saddle and there will be a stable node at $(\mu, 0)$. A sketch of the phase portrait would show that all trajectories in the set $\{x_1 > 0\}$ approach the stable node $(\mu, 0)$ as time tends to infinity. For small values of μ, the steady-state operating point of the system will be close to the origin. So, while the perturbed system does not have the desired steady-state behavior, it comes close to it. The situation is quite different in the saddle–node bifurcation example. Suppose the nominal system has a positive value of μ so that all trajectories in the set $\{x_1 > -\sqrt{\mu}\}$ approach the stable node $(\sqrt{\mu}, 0)$ as time tends to infinity. If the nominal value of μ is small and a small perturbation causes μ to become negative, the stable node disappears all together and trajectories will have to move away from the desired steady-state operating point, even diverge to infinity in this case. Because of the difference in their impact on the steady-state behavior, the bifurcation in the transcritical bifurcation example is said to be *safe* or *soft*, while that in the saddle–node bifurcation example is said to be *dangerous* or *hard*.

The classification of bifurcation into safe versus dangerous cases arises also when we examine the bifurcation diagrams of Figures 2.28(c) and (d), which represent *supercritical pitchfork* and *subcritical pitchfork* bifurcations, respectively. The first of the two cases is exemplified by the system

$$\begin{aligned} \dot{x}_1 &= \mu x_1 - x_1^3 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

For $\mu < 0$, there is a unique equilibrium point at the origin. By calculating the Jacobian, we can see that the origin is a stable node. For $\mu > 0$, there are three equilibrium points at $(0, 0)$, $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$. Jacobian calculations show that $(0, 0)$ is a saddle and the other two equilibria are stable nodes. Thus, as μ crosses the bifurcation point $\mu = 0$, the stable node at the origin bifurcates into a saddle and gives birth to two stable nodes at $(\pm\sqrt{\mu}, 0)$. The amplitude of the newly created stable nodes grows with μ; hence, it is small for small μ. Subcritical pitchfork bifurcation is exemplified by the system

$$\begin{aligned} \dot{x}_1 &= \mu x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

For $\mu < 0$, there are three equilibrium points: a stable node at $(0, 0)$ and two saddles at $(\pm\sqrt{-\mu}, 0)$. For $\mu > 0$, there is a unique equilibrium point at $(0, 0)$, which is a saddle. Thus, as μ crosses the bifurcation point $\mu = 0$, the stable node at the origin collides with the saddles at $(\pm\sqrt{-\mu}, 0)$ and bifurcates into a saddle. Comparing the supercritical and subcritical pitchfork bifurcations, we can easily see that the supercritical bifurcation is safe, while the subcritical one is dangerous. In particular, if the system has a nominal operating point at the stable node $(0, 0)$ for $\mu < 0$, then supercritical pitchfork bifurcation ensures close steady-state operation when μ is perturbed to a small positive value, while in subcritical pitchfork bifurcation trajectories move away from the nominal operating point.

In the simplified examples we used to discuss zero-eigenvalue bifurcations, we noticed that in dangerous cases trajectories diverge to infinity. In more complicated examples, the system may have other equilibrium points or periodic orbits that are not affected by the bifurcation under consideration. Trajectories moving away from the bifurcating equilibrium point could be attracted by another equilibrium point or period orbit rather than diverge to infinity. This situation is illustrated by the next example.

Example 2.12 Consider the tunnel-diode circuit of Section 1.2.2:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}\left[-h(x_1) + x_2\right] \\ \dot{x}_2 &= \frac{1}{L}\left[-x_1 - Rx_2 + \mu\right] \end{aligned}$$

The diode's *v–i* characteristic $h(\cdot)$ is sketched in Figure 1.2 and μ is a constant input. Let us study bifurcation as μ is varied. The equilibrium points of the system are the intersections of the curve $x_2 = h(x_1)$ with the load line $x_2 = (\mu - x_1)/R$. From Figure 2.29(a) and Examples 2.1 and 2.3, we know that for $\mu < A$, there is a stable node on the left branch; for $A < \mu < B$, there are three equilibrium points, a saddle on the middle branch and two stable nodes on the other two branches; and for $\mu > B$, there is a stable node on the right branch. The bifurcation diagram is shown in Figure 2.29(b). There are two saddle–node bifurcations at $\mu = A$ and $\mu = B$. Notice that when a stable node disappears upon collision with a saddle, trajectories that move away are attracted by the other stable node which is not affected by the bifurcation. △

When a stable node loses stability at a bifurcation point, an eigenvalue of the Jacobian passes through zero. What about a stable focus losing stability? In this case, a pair of conjugate complex eigenvalues could pass through the imaginary axis. Figures 2.28(e) and (f) are examples of this situation, where Figures 2.28(e) is called *supercritical Hopf bifurcation* and Figures 2.28(f) is called *subcritical Hopf bifurcation.*[18] The supercritical Hopf bifurcation is exemplified by the system

$$\begin{aligned} \dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1 \end{aligned}$$

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[18] The names Andronov–Hopf bifurcation and Poincaré–Andronov–Hopf bifurcation are also used to acknowledge the earlier contributions of Poincaré and Andronov.

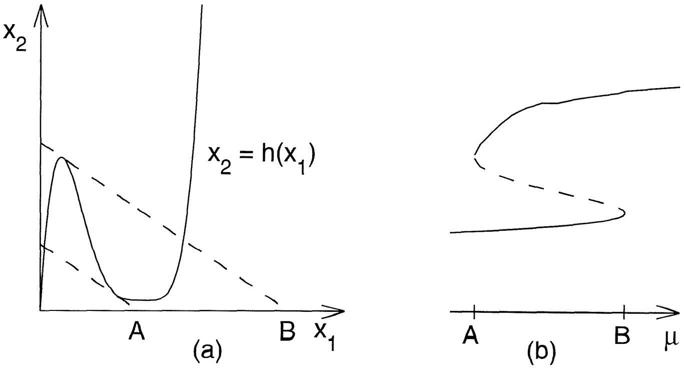

Figure 2.29: Example 2.12: (a) determining equilibrium points; (b) bifurcation diagram.

which is represented in the polar coordinates

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta$$

by

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\theta} = 1$$

The system has a unique equilibrium point at the origin and the phase portraits for opposite signs of μ are shown in Figure 2.30. For $\mu < 0$, the origin is a stable focus and all trajectories are attracted to it, while for $\mu > 0$, the origin is an unstable focus, but there is a stable limit cycle that attracts all trajectories, except the zero solution. The limit cycle is $r = \sqrt{\mu}$, which shows that the amplitude of oscillation grows with μ and is small for small values of μ. This is a safe bifurcation because when the stable focus disappears due to small perturbation, the system will have a steady-state oscillation with a small amplitude. To see how the eigenvalues behave during bifurcation, notice that the Jacobian at the origin

$$\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

has eigenvalues $\mu \pm j$, which cross the imaginary axis from left to right as μ increases from negative to positive values.

The subcritical Hopf bifurcation is exemplified by the system

$$\begin{aligned} \dot{x}_1 &= x_1\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] - x_2 \\ \dot{x}_2 &= x_2\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] + x_1 \end{aligned}$$

which is represented in polar coordinates by

$$\dot{r} = \mu r + r^3 - r^5 \quad \text{and} \quad \dot{\theta} = 1$$

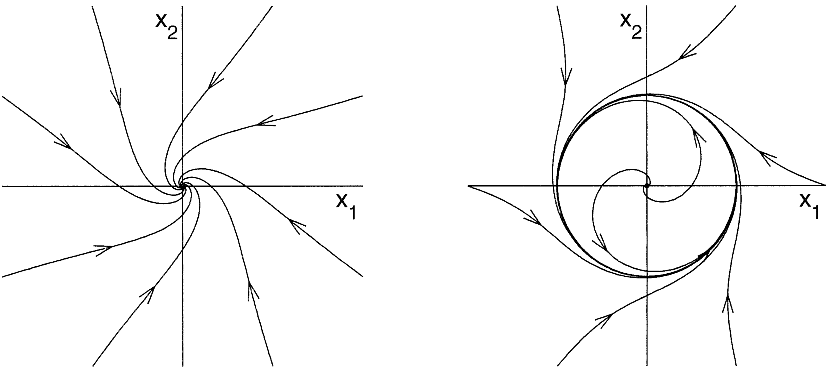

Figure 2.30: Phase portrait of the supercritical Hopf bifurcation example for $\mu < 0$ (left) and $\mu > 0$ (right).

There is a unique equilibrium point at the origin, which is a stable focus for $\mu < 0$ and unstable focus for $\mu > 0$. From the equation

$$0 = \mu + r^2 - r^4$$

we can determine the limit cycles of the system. For $\mu < 0$, there are two limit cycles at $r^2 = (1 \pm \sqrt{1+4\mu})/2$. By sketching $\dot{r} = r(\mu + r^2 - r^4)$ as a function of r (see Figure 2.31), it can be seen that the limit cycle at $r^2 = (1 + \sqrt{1+4\mu})/2$ is stable, while the limit cycle at $r^2 = (1 - \sqrt{1+4\mu})/2$ is unstable. For small $|\mu|$, the unstable limit cycle can be approximated by $r^2 = -\mu$. For $\mu > 0$, there is only one stable limit cycle at $r^2 = (1 + \sqrt{1+4\mu})/2$. Thus, as μ increases from negative to positive values, the stable focus at the origin merges with the unstable limit cycle and bifurcates into an unstable focus, as represented by the bifurcation diagram of Figures 2.28(f). Notice that the stable limit cycle is not shown in the bifurcation diagram because varying μ only varies its amplitude. The subcritical Hopf bifurcation is dangerous because small perturbation of a nominal stable focus at the origin could force trajectories to move away from the origin. Those trajectories are attracted by the stable limit cycle.

All the bifurcations represented in Figure 2.28 occur in the vicinity of an equilibrium point. Therefore, they are called *local bifurcations*. There are also *global bifurcations*, which involve large regions of the state plane and cannot be described in the vicinity of any equilibrium point. We give only one example of global bifurcations.[19] Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2 \end{aligned}$$

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[19] See [187] for other examples.

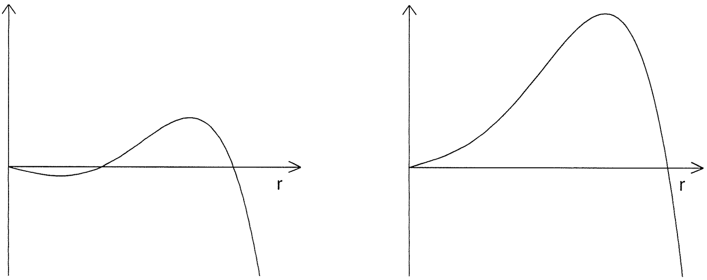

Figure 2.31: A sketch of $\mu r + r^3 - r^5$ for $\mu < 0$ (left) and $\mu > 0$ (right).

There are two equilibrium points at $(0,0)$ and $(1,0)$. By linearization, we can see that $(0,0)$ is always a saddle, while $(1,0)$ is an unstable focus for $-1 < \mu < 1$. Let us limit our analysis to the range $-1 < \mu < 1$. Figure 2.32 shows the phase portrait for four different values of μ. The phase portraits for $\mu = -0.95$ and -0.88 are typical for $\mu < \mu_c \approx -0.8645$, while that for $\mu = -0.8$ is typical for $\mu > \mu_c$. For $\mu < \mu_c$, there is a stable limit cycle that encircles the unstable focus. As μ increases towards μ_c, the limit cycle swells and finally touches the saddle at $\mu = \mu_c$, creating a trajectory that starts and ends at the saddle; such trajectory is called *homoclinic orbit.* For $\mu > \mu_c$, the limit cycle disappears. Note that this bifurcation occurs without any changes to the equilibrium points at $(0,0)$ and $(1,0)$. This type of global bifurcation is called *saddle–connection* or *homoclinic bifurcation.*

2.8 Exercises

2.1 For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium:

(1) $\dot{x}_1 = -x_1 + 2x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2$

(2) $\dot{x}_1 = x_1 + x_1 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$

(3) $\dot{x}_1 = [1 - x_1 - 2h(x)]\, x_1, \qquad \dot{x}_2 = [2 - h(x)]\, x_2$

(4) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 + 0.1x_1^4)$

(5) $\dot{x}_1 = (x_1 - x_2)(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$

(6) $\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1 - x_2^3$

In the third system, $h(x) = x_2/(1 + x_1)$.

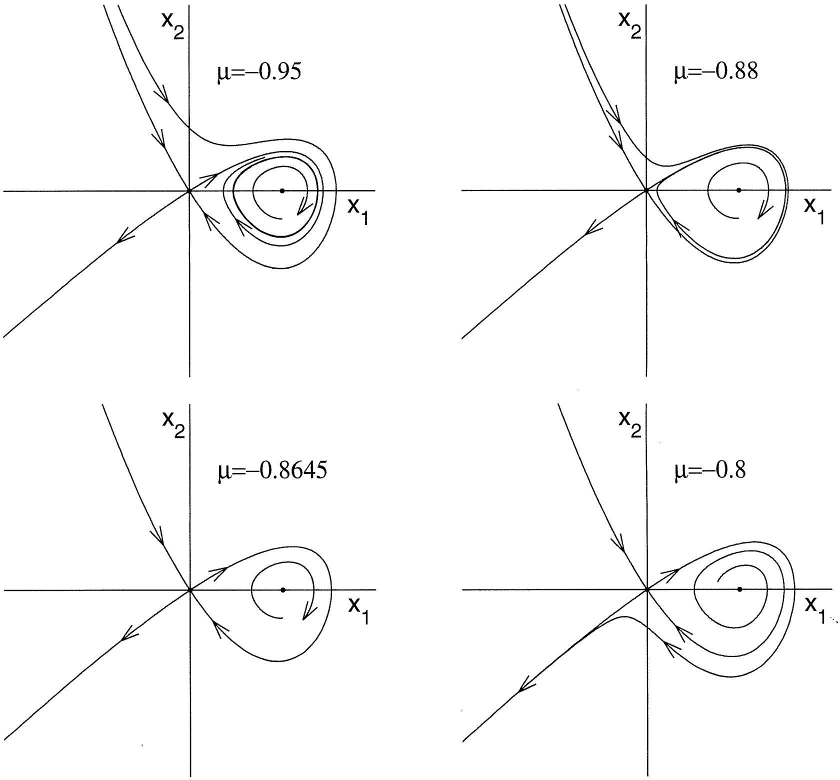

Figure 2.32: Saddle–connection bifurcation.

2.2 For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium:

$$\begin{array}{llll} \textbf{(1)} & \dot{x}_1 = x_2, & & \dot{x}_2 = -x_1 + \frac{1}{16}x_1^5 - x_2 \\ \textbf{(2)} & \dot{x}_1 = 2x_1 - x_1x_2, & & \dot{x}_2 = 2x_1^2 - x_2 \\ \textbf{(3)} & \dot{x}_1 = x_2, & & \dot{x}_2 = -x_2 - \psi(x_1 - x_2) \end{array}$$

In the third system, $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5\ \mathrm{sgn}(y)$ if $|y| > 1$.

2.3 For each of the systems in Exercise 2.1, construct the phase portrait and discuss the qualitative behavior of the system.

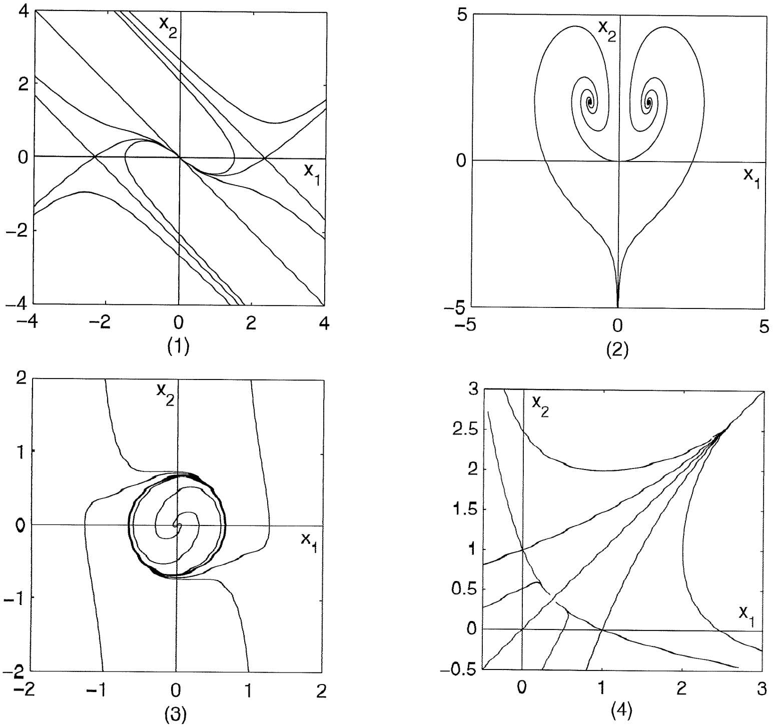

Figure 2.33: Phase portraits of Exercise 2.4.

2.4 The phase portraits of the following four systems are shown in Figure 2.33.

$$\textbf{(1)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - 2\tan^{-1}(x_1 + x_2)$$

$$\textbf{(2)} \qquad \dot{x}_1 = 2x_1 - x_1x_2, \qquad \dot{x}_2 = 2x_1^2 - x_2$$

$$\textbf{(3)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

$$\textbf{(4)} \qquad \dot{x}_1 = -(x_1 - x_1^2) + h(x), \qquad \dot{x}_2 = -(x_2 - x_2^2) + h(x)$$

In the fourth system, $h(x) = 1 - x_1 - x_2$. Mark the arrowheads and discuss the qualitative behavior of each system.

2.5 The system

$$\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}}, \qquad \dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}}$$

has an equilibrium point at the origin.

(a) Linearize the system about the origin and show that the origin is a stable node of the linear system.

(b) Find the phase portrait of the nonlinear system near the origin and show that the portrait resembles a stable focus.
Hint: Transform the equations into polar coordinates.

(c) Explain the discrepancy between the results of parts (a) and (b).

2.6 Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + ax_2 - bx_1x_2 + x_2^2\\ \dot{x}_2 &= -(a+b)x_1 + bx_1^2 - x_1x_2\end{aligned}$$

where $a > 0$ and $b \neq 0$.

(a) Find all equilibrium points of the system.

(b) Determine the type of each isolated equilibrium point, for all values of $a > 0$ and $b \neq 0$.

(c) For each of the following cases, construct the phase portrait and discuss the qualitative behavior of the system:

i. $a = b = 1$.
ii. $a = 1,\ b = -\frac{1}{2}$.
iii. $a = 1,\ b = -2$.

2.7 Consider the negative resistance oscillator of Section 1.2.4 with

$$h(v) = -v + v^3 - \tfrac{1}{5}v^5 + \tfrac{1}{105}v^7$$

and $\varepsilon = 1$. Construct the phase portrait in the x-coordinates and discuss the qualitative behavior of system.

2.8 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{16}x_1^5 - x_2$$

(a) Find the equilibrium points and determine the type of each isolated one.

(b) Without using a computer program, sketch the phase portrait of the system.

2.9 For cruise control, the longitudinal motion of a vehicle on a flat road can be modeled, with the use of Newton's second law, by the first-order differential equation

$$m\dot{v} = u - K_c\,\mathrm{sgn}(v) - K_f v - K_a v^2$$

where m is the vehicle's mass, v is its speed, u is the tractive force generated by the engine, $K_c \operatorname{sgn}(v)$ is the coulomb friction force, $K_f v$ is the viscous friction force, and $K_a v^2$ is the aerodynamic drag. The coefficients K_c, K_f and K_a are nonnegative. When a PI controller is used, $u = K_I \sigma + K_P(v_d - v)$, where v_d is the desired speed, σ is the state of the integrator $\dot{\sigma} = v_d - v$, and K_I and K_P are positive constants. We are only interested in the region $v \geq 0$.

(a) Using $x_1 = \sigma$ and $x_2 = v$ as the state variables, find the state model of the system.

(b) Let v_d be a positive constant. Find all equilibrium points and determine the type of each point.

(c) Construct the phase portrait and discuss the qualitative behavior of the system, for the following numerical data: $m = 1500$ kg, $K_c = 110$ N, $K_f = 2.5$ N/m/sec, $K_a = 1$ N/m^2/sec^2, $K_I = 15$, $K_P = 500$, and $v_d = 30$ m/sec.

(d) Repeat part (c) when K_I is increased to 150. Compare with the behavior in part (c).

(e) Repeat part (c) when saturation is used to limit u to the range $0 \leq u \leq 1800$ N. Compare with the behavior in part (c).

2.10 Consider the tunnel-diode circuit of Section 1.2.2 with the numerical data used in Example 2.1, except for R and E, which are taken as $E = 0.2V$ and $R = 0.2k\Omega$.

(a) Find all equilibrium points and determine the type of each point.

(b) Construct the phase portrait and discuss the qualitative behavior of the circuit.

2.11 Repeat the previous problem with $E = 0.4V$ and $R = 0.2k\Omega$.

2.12 Consider the Hopfield neural network model of Section 1.2.4 with $n = 2$, $V_M = 1$, and $T_{21} = T_{12} = 1$. For $i = 1, 2$, take $I_i = 0$, $C_i = 1$, $\rho_i = 1$, $T_{ii} = 0$, and $g_i(u) = (2/\pi)\tan^{-1}(\lambda\pi u/2)$.

(a) Find all equilibrium points and determine the type of each point.

(b) For $\lambda = 5$, construct the phase portrait and discuss the qualitative behavior of the system.

2.13 An equivalent circuit of the Wien–Bridge oscillator is shown in Figure 2.34 [40], where $g(v_2)$ is a nonlinear voltage-controlled voltage source.

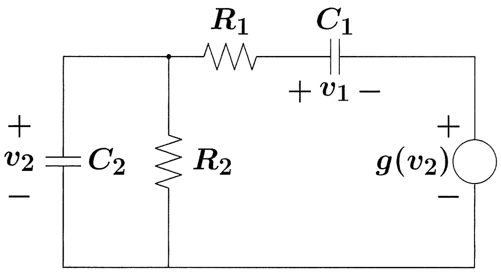

Figure 2.34: Exercise 2.13.

(a) With $x_1 = v_1$ and $x_2 = v_2$ as state variables, show that the state model is given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C_1 R_1}[-x_1 + x_2 - g(x_2)] \\ \dot{x}_2 &= -\frac{1}{C_2 R_1}[-x_1 + x_2 - g(x_2)] - \frac{1}{C_2 R_2} x_2 \end{aligned}$$

(b) Let $C_1 = C_2 = R_1 = R_2 = 1$ and $g(v) = 3.234v - 2.195v^3 + 0.666v^5$. Construct the phase portrait and discuss the qualitative behavior of the system.

2.14 Consider the mass-spring system with Coulomb friction

$$\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = 0$$

where η is defined in Section 1.2.3. Use piecewise linear analysis to sketch the phase portrait (without numerical data) and discuss the qualitative behavior of the system.

2.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u$$

where the control input u can take the values ±1.

(a) Sketch the phase portrait when $u = 1$.

(b) Sketch the phase portrait when $u = -1$.

(c) By superimposing the two phase portraits, develop a strategy for switching the control between ±1 so that any point in the state plane can be moved to the origin in finite time.

2.16 A prey–predatory system may be modeled by [202]

$$\dot{x}_1 = x_1(1 - x_1 - ax_2), \qquad \dot{x}_2 = bx_2(x_1 - x_2)$$

where x_1 and x_2 are dimensionless variables proportional to the prey and predator populations, respectively, and a and b are positive constants.

(a) Find all equilibrium points and determine the type of each point.

(b) Construct the phase portrait in the first quadrant ($x_1 \geq 0$, $x_2 \geq 0$) when $a = 1$, $b = 0.5$ and discuss the qualitative behavior of the system.

2.17 For each of the following systems, use the Poincaré–Bendixson's criterion to show that the system has a periodic orbit:

$$
\begin{array}{llllll}
(1) & \ddot{y} + y & = & \varepsilon\dot{y}(1 - y^2 - \dot{y}^2) & & \\
(2) & \dot{x}_1 & = & x_2, & \dot{x}_2 & = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2) \\
(3) & \dot{x}_1 & = & x_2, & \dot{x}_2 & = -x_1 + x_2 - 2(x_1 + 2x_2)x_2^2 \\
(4) & \dot{x}_1 & = & x_1 + x_2 - x_1h(x), & \dot{x}_2 & = -2x_1 + x_2 - x_2h(x)
\end{array}
$$

In the fourth system, $h(x) = \max\{|x_1|, |x_2|\}$.

2.18 (Conservative Systems) Consider the second-order system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)$$

where g is continuously differentiable and $zg(z) > 0$ for $z \neq 0$ Consider the energy function

$$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} g(z)\ dz$$

(a) Show that $V(x)$ remains constant during the motion of the system.

(b) Show that, for sufficiently small $\|x(0)\|$, every solution is periodic.

(c) Suppose that $zg(z) > 0$ for $z \in (-\infty, \infty)$ and

$$\int_0^y g(z)\ dz \rightarrow \infty \text{ as } |y| \rightarrow \infty$$

Show that every solution is a periodic solution.

(d) Suppose that $g(z) = -g(-z)$ and let $G(y) = \int_0^y g(z)\ dz$. Show that the trajectory through $(A, 0)$ is given by

$$x_2 = \pm\sqrt{2[G(A) - G(x_1)]}$$

(e) Using part (d), show that the period of oscillation of a closed trajectory through $(A, 0)$ is

$$T(A) = 2\sqrt{2} \int_0^A \frac{dy}{[G(A) - G(y)]^{1/2}}$$

(f) Discuss how the trajectory equation in part (d) can be used to construct the phase portrait of the system.

2.19 Use the previous exercise to construct the phase portrait and study periodic solutions for each of the following systems:

$$\textbf{(1)}\;\; g(x_1) = \sin x_1, \qquad \textbf{(2)}\;\; g(x_1) = x_1 + x_1^3, \qquad \textbf{(3)}\;\; g(x_1) = x_1^3$$

In each case, give the period of oscillation of the periodic orbit through the point $(1, 0)$.

2.20 For each of the following systems, show that the system has no limit cycles:

$$\begin{array}{lrcllrcl}
\textbf{(1)} & \dot{x}_1 & = & -x_1 + x_2, & & \dot{x}_2 & = & g(x_1) + a x_2, \;\; a \neq 1 \\
\textbf{(2)} & \dot{x}_1 & = & -x_1 + x_1^3 + x_1 x_2^2, & & \dot{x}_2 & = & -x_2 + x_2^3 + x_1^2 x_2 \\
\textbf{(3)} & \dot{x}_1 & = & 1 - x_1 x_2^2, & & \dot{x}_2 & = & x_1 \\
\textbf{(4)} & \dot{x}_1 & = & x_1 x_2, & & \dot{x}_2 & = & x_2 \\
\textbf{(5)} & \dot{x}_1 & = & x_2 \cos(x_1), & & \dot{x}_2 & = & \sin x_1
\end{array}$$

2.21 Consider the system

$$\dot{x}_1 = -x_1 + x_2(x_1 + a) - b, \qquad \dot{x}_2 = -c x_1 (x_1 + a)$$

where a, b, and c are positive constants with $b > a$. Let

$$D = \left\{ x \in R^2 \mid x_1 < -a \text{ and } x_2 < \frac{x_1 + b}{x_1 + a} \right\}$$

(a) Show that every trajectory starting in D stays in D for all future time.

(b) Show that there can be no periodic orbits through any point $x \in D$.

2.22 Consider the system

$$\dot{x}_1 = a x_1 - x_1 x_2, \qquad \dot{x}_2 = b x_1^2 - c x_2$$

where a, b, and c are positive constants with $c > a$. Let $D = \{x \in R^2 \mid x_2 \geq 0\}$.

(a) Show that every trajectory starting in D stays in D for all future time.

(b) Show that there can be no periodic orbits through any point $x \in D$.

2.23 **([85])** Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -[2b - g(x_1)]ax_2 - a^2 x_1$$

where a and b are positive constants, and

$$g(x_1) = \begin{cases} 0, & |x_1| > 1 \\ k, & |x_1| \le 1 \end{cases}$$

(a) Show, using Bendixson's criterion, that there are no periodic orbits if $k < 2b$.

(b) Show, using the Poincaré–Bendixson criterion, that there is a periodic orbit if $k > 2b$.

2.24 Consider a second-order system and suppose that the set $M = \{x_1^2 + x_2^2 \le a^2\}$ has the property that every trajectory starting in M stays in M for all future time. Show that M contains an equilibrium point.

2.25 Verify Lemma 2.3 by examining the vector fields.

2.26 **([70])** For each of the following systems, show that the origin is not hyperbolic, find the index of the origin, and verify that it is different from ± 1:

$$\text{(1)} \qquad \dot{x}_1 = x_1^2, \qquad \dot{x}_2 = -x_2$$

$$\text{(2)} \qquad \dot{x}_1 = x_1^2 - x_2^2, \qquad \dot{x}_2 = 2x_1x_2$$

2.27 For each of the following systems, find and classify bifurcations that occur as μ varies:

$$\text{(1)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^3 - 3x_1^2 x_2$$

$$\text{(2)} \qquad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -(1+\mu^2)x_1 + 2\mu x_2 - \mu x_1^3 + 2(x_2 - \mu x_1)^3$$

$$\text{(3)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu - x_2 - x_1^2 - 2x_1x_2$$

$$\text{(4)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = -(1+\mu^2)x_1 + 2\mu x_2 + \mu x_1^3 - x_1^2 x_2$$

$$\text{(5)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^3 + 3x_1^2 x_2$$

$$\text{(6)} \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^2 - 2x_1x_2$$

2.28 The model that follows is used to analyze the interaction between inhibitory and excitatory neurons in a biological system [195]. In its simplest form, the model describes the interaction of two neurons with x_1 as the output of the excitatory neuron and x_2 as the output of the inhibitory neuron. The evolution of x_1 and x_2 is described by

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)\\ \dot{x}_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)\end{aligned}$$

where $\tau > 0$ is a characteristic time constant and $\lambda > 0$ is an amplification gain.

(a) Using the Poincaré–Bendixson criterion, show that the system has a periodic orbit when $\lambda\tau > 1$.

(b) Construct the phase portrait for $\tau = 1$ and $\lambda = 2$ and discuss the qualitative behavior of the system.

(c) Repeat part (b) for $\tau = 1$ and $\lambda = 1/2$.

(d) Find and classify bifurcations that occur as $\mu = \lambda\tau$ varies.

2.29 A model that is used to analyze a class of experimental systems known as chemical oscillators [187] is given by

$$\dot{x}_1 = a - x_1 - \frac{4x_1x_2}{1+x_1^2}, \qquad \dot{x}_2 = bx_1\left(1 - \frac{x_2}{1+x_1^2}\right)$$

where x_1 and x_2 are dimensionaless concentrations of certain chemicals and a, b are positive constants.

(a) Using the Poincaré–Bendixson criterion, show that the system has a periodic orbit when $b < 3a/5 - 25/a$.

(b) Construct the phase portrait in the first quadrant for $a = 10$, $b = 2$ and discuss the qualitative behavior of the system.

(c) Repeat part (b) for $a = 10$, $b = 4$.

(d) Find and classify bifurcations that occur as b varies, while a is fixed.

2.30 A biochemical reactor can be represented by the model

$$\dot{x}_1 = \left(\frac{\mu_m x_2}{k_m + x_2} - d\right)x_1, \qquad \dot{x}_2 = d(x_{2f} - x_2) - \frac{\mu_m x_1 x_2}{Y(k_m + x_2)}$$

where the state variables and the nonnegative constants d, μ_m, k_m, Y, and x_{2f} are defined in Exercise 1.22. Let $\mu_m = 0.5$, $k_m = 0.1$, $Y = 0.4$, and $x_{2f} = 4$.

(a) Find all equilibrium points for $d > 0$ and determine the type of each point.

(b) Study bifurcation as d varies.

(c) Construct the phase portrait and discuss the qualitative behavior of the system when $d = 0.4$.

2.31 A biochemical reactor can be represented by the model

$$\dot{x}_1 = \left(\frac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2} - d \right) x_1, \qquad \dot{x}_2 = d(x_{2f} - x_2) - \frac{\mu_m x_1 x_2}{Y(k_m + x_2 + k_1 x_2^2)}$$

where the state variables and the nonnegative constants d, μ_m, k_m, k_1, Y, and x_{2f} are defined in Exercise 1.22. Let $\mu_m = 0.5$, $k_m = 0.1$, $k_1 = 0.5$, $Y = 0.4$, and $x_{2f} = 4$.

(a) Find all equilibrium points for $d > 0$ and determine the type of each point.

(b) Study bifurcation as d varies.

(c) Construct the phase portrait and discuss the qualitative behavior of the system when $d = 0.1$.

(d) Repeat part (c) when $d = 0.25$.

(e) Repeat part (c) when $d = 0.5$

Chapter 3

Fundamental Properties

This chapter states some fundamental properties of the solutions of ordinary differential equations, like existence, uniqueness, continuous dependence on initial conditions, and continuous dependence on parameters. These properties are essential for the state equation $\dot{x} = f(t, x)$ to be a useful mathematical model of a physical system. In experimenting with a physical system such as the pendulum, we expect that starting the experiment from a given initial state at time t_0, the system will move and its state will be defined in the (at least immediate) future time $t > t_0$. Moreover, with a deterministic system, we expect that if we could repeat the experiment exactly, we would get exactly the same motion and the same state at $t > t_0$. For the mathematical model to predict the future state of the system from its current state at t_0, the initial-value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \tag{3.1}$$

must have a unique solution. This is the question of existence and uniqueness that is addressed in Section 3.1. It is shown that existence and uniqueness can be ensured by imposing some constraints on the right-hand side function $f(t, x)$. The key constraint used in Section 3.1 is the Lipschitz condition, whereby $f(t, x)$ satisfies the inequality[1]

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\| \tag{3.2}$$

for all (t, x) and (t, y) in some neighborhood of (t_0, x_0).

An essential factor in the validity of any mathematical model is the continuous dependence of its solutions on the data of the problem. The least we should expect from a mathematical model is that arbitrarily small errors in the data will not result in large errors in the solutions obtained by the model. The data of the initial-value problem (3.1) are the initial state x_0, the initial time t_0, and the right-hand side function $f(t,x)$. Continuous dependence on the initial conditions (t_0, x_0) and on the parameters of f are studied in Section 3.2. If f is differentiable with respect to its parameters, then the solution will be differentiable with respect to these parameters. This is shown in Section 3.3 and is used to derive sensitivity equations that describe the effect of small parameter variations on the performance of the system. The continuity and differentiability results of Sections 3.2 and 3.3 are valid only on finite time intervals. Continuity results on the infinite time interval will be given later, after stability concepts have been introduced.[2]

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[1] $\|\cdot\|$ denotes any p-norm, as defined in Appendix A.

The chapter ends with a brief statement of a comparison principle that bounds the solution of a scalar differential inequality $\dot{v} \leq f(t,v)$ by the solution of the differential equation $\dot{u} = f(t,u)$.

3.1 Existence and Uniqueness

In this section, we derive sufficient conditions for the existence and uniqueness of the solution of the initial-value problem (3.1). By a solution of (3.1) over an interval $[t_0, t_1]$, we mean a continuous function $x : [t_0, t_1] \to R^n$ such that $\dot{x}(t)$ is defined and $\dot{x}(t) = f(t, x(t))$ for all $t \in [t_0, t_1]$. If $f(t,x)$ is continuous in t and x, then the solution $x(t)$ will be continuously differentiable. We will assume that $f(t,x)$ is continuous in x, but only piecewise continuous in t, in which case, a solution $x(t)$ could only be piecewise continuously differentiable. The assumption that $f(t,x)$ be piecewise continuous in t allows us to include the case when $f(t,x)$ depends on a time-varying input that may experience step changes with time.

A differential equation with a given initial condition might have several solutions. For example, the scalar equation

$$\dot{x} = x^{1/3}, \quad \text{with} \quad x(0) = 0 \tag{3.3}$$

has a solution $x(t) = (2t/3)^{3/2}$. This solution is not unique, since $x(t) \equiv 0$ is another solution. In noting that the right-hand side of (3.3) is continuous in x, it is clear that continuity of $f(t,x)$ in its arguments is not sufficient to ensure uniqueness of the solution. Extra conditions must be imposed on the function f. The question of existence of a solution is less stringent. In fact, continuity of $f(t,x)$ in its arguments ensures that there is at least one solution. We will not prove this fact here.[3] Instead, we prove an easier theorem that employs the Lipschitz condition to show existence and uniqueness.

Theorem 3.1 (Local Existence and Uniqueness) *Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

---

[2] See, in particular, Section 9.4.

[3] See [135, Theorem 2.3] for a proof.

$\forall\, x, y \in B = \{x \in R^n \mid \|x - x_0\| \leq r\}$, $\forall\, t \in [t_0, t_1]$. *Then, there exists some* $\delta > 0$ *such that the state equation* $\dot{x} = f(t, x)$ *with* $x(t_0) = x_0$ *has a unique solution over* $[t_0, t_0 + \delta]$. ◇

Proof: See Appendix C.1.

The key assumption in Theorem 3.1 is the Lipschitz condition (3.2). A function satisfying (3.2) is said to be *Lipschitz* in x, and the positive constant L is called a *Lipschitz constant.* We also use the words *locally Lipschitz* and *globally Lipschitz* to indicate the domain over which the Lipschitz condition holds. Let us introduce the terminology first for the case when f depends only on x. A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset R^n$ if each point of D has a neighborhood D_0 such that f satisfies the Lipschitz condition (3.2) for all points in D_0 with some Lipschitz constant L_0. We say that f is Lipschitz on a set W if it satisfies (3.2) for all points in W, with the same Lipschitz constant L. A locally Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D (Exercise 3.19). A function $f(x)$ is said to be *globally Lipschitz* if it is Lipschitz on R^n. The same terminology is extended to a function $f(t, x)$, provided the Lipschitz condition holds uniformly in t for all t in a given interval of time. For example, $f(t, x)$ is locally Lipschitz in x on $[a, b] \times D \subset R \times R^n$ if each point $x \in D$ has a neighborhood D_0 such that f satisfies (3.2) on $[a, b] \times D_0$ with some Lipschitz constant L_0. We say that $f(t, x)$ is locally Lipschitz in x on $[t_0, \infty) \times D$ if it is locally Lipschitz in x on $[a, b] \times D$ for every compact interval $[a, b] \subset [t_0, \infty)$. A function $f(t, x)$ is Lipschitz in x on $[a, b] \times W$ if it satisfies (3.2) for all $t \in [a, b]$ and all points in W, with the same Lipschitz constant L.

When $f : R \to R$, the Lipschitz condition can be written as

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

which implies that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L. Therefore, any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point. For example, any discontinuous function is not locally Lipschitz at the point of discontinuity. As another example, the function $f(x) = x^{1/3}$, which was used in (3.3), is not locally Lipschitz at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \to \infty$ as $x \to 0$. On the other hand, if $|f'(x)|$ is bounded by a constant k over the interval of interest, then $f(x)$ is Lipschitz on the same interval with Lipschitz constant $L = k$. This observation extends to vector-valued functions, as demonstrated by Lemma 3.1.

Lemma 3.1 *Let* $f : [a, b] \times D \to R^m$ *be continuous for some domain* $D \subset R^n$. *Suppose that* $[\partial f / \partial x]$ *exists and is continuous on* $[a, b] \times D$. *If, for a convex subset*

$W \subset D$, there is a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L$$

on $[a,b] \times W$, then

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

for all $t \in [a,b]$, $x \in W$, and $y \in W$. ◇

Proof: Let $\|\cdot\|_p$ be the underlying norm for any $p \in [1,\infty]$, and determine $q \in [1,\infty]$ from the relationship $1/p + 1/q = 1$. Fix $t \in [a,b]$, $x \in W$, and $y \in W$. Define $\gamma(s) = (1-s)x + sy$ for all $s \in R$ such that $\gamma(s) \in D$. Since $W \subset D$ is convex, $\gamma(s) \in W$ for $0 \leq s \leq 1$. Take $z \in R^m$ such that[4]

$$\|z\|_q = 1 \quad \text{and} \quad z^T[f(t,y) - f(t,x)] = \|f(t,y) - f(t,x)\|_p$$

Set $g(s) = z^T f(t,\gamma(s))$. Since $g(s)$ is a real-valued function, which is continuously differentiable in an open interval that includes $[0,1]$, we conclude by the mean value theorem that there is $s_1 \in (0,1)$ such that

$$g(1) - g(0) = g'(s_1)$$

Evaluating g at $s=0$, $s=1$, and calculating $g'(s)$ by using the chain rule, we obtain

$$\begin{aligned} z^T[f(t,y) - f(t,x)] &= z^T \frac{\partial f}{\partial x}(t,\gamma(s_1))(y-x) \\ \|f(t,y) - f(t,x)\|_p &\leq \|z\|_q \left\| \frac{\partial f}{\partial x}(t,\gamma(s_1)) \right\|_p \|y-x\|_p \leq L\|y-x\|_p \end{aligned}$$

where we used the Hölder inequality $|z^T w| \leq \|z\|_q \|w\|_p$. □

The lemma shows how a Lipschitz constant can be calculated using knowledge of $[\partial f/\partial x]$.

The Lipschitz property of a function is stronger than continuity. It can be easily seen that if $f(x)$ is Lipschitz on W, then it is uniformly continuous on W (Exercise 3.20). The converse is not true, as seen from the function $f(x) = x^{1/3}$, which is continuous, but not locally Lipschitz at $x=0$. The Lipschitz property is weaker than continuous differentiability, as stated in the next lemma.

Lemma 3.2 *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times D$, for some domain $D \subset R^n$, then f is locally Lipschitz in x on $[a,b] \times D$.* ◇

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[4] Such z always exists. (See Exercise 3.21.)

Proof: For $x_0 \in D$, let r be so small that the ball $D_0 = \{x \in R^n \mid \|x - x_0\| \le r\}$ is contained in D. The set D_0 is convex and compact. By continuity, $[\partial f/\partial x]$ is bounded on $[a,b] \times D_0$. Let L_0 be a bound for $\|\partial f/\partial x\|$ on $[a,b] \times D_0$. By Lemma 3.1, f is Lipschitz on $[a,b] \times D_0$ with Lipschitz constant L_0. □

It is left to the reader (Exercise 3.22) to extend the proof of Lemma 3.1 to prove the next lemma.

Lemma 3.3 *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lipschitz in x on $[a,b] \times R^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b] \times R^n$.* ◇

Example 3.1 The function

$$f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix}$$

is continuously differentiable on R^2. Hence, it is locally Lipschitz on R^2. It is not globally Lipschitz since $[\partial f/\partial x]$ is not uniformly bounded on R^2. On any compact subset of R^2, f is Lipschitz. Suppose that we are interested in calculating a Lipschitz constant over the convex set $W = \{x \in R^2 \mid |x_1| \le a_1,\ |x_2| \le a_2\}$. The Jacobian matrix is given by

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$$

Using $\|.\|_\infty$ for vectors in R^2 and the induced matrix norm for matrices, we have

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\{|-1 + x_2| + |x_1|,\ |x_2| + |1 - x_1|\}$$

All points in W satisfy

$$|-1 + x_2| + |x_1| \le 1 + a_2 + a_1 \quad \text{and} \quad |x_2| + |1 - x_1| \le a_2 + 1 + a_1$$

Hence,

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty \le 1 + a_1 + a_2$$

and a Lipschitz constant can be taken as $L = 1 + a_1 + a_2$. △

Example 3.2 The function

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

is not continuously differentiable on R^2. Let us check its Lipschitz property by examining $f(x) - f(y)$. Using $\|.\|_2$ for vectors in R^2 and the fact that the saturation function $\text{sat}(\cdot)$ satisfies

$$|\text{sat}(\eta) - \text{sat}(\xi)| \le |\eta - \xi|$$

we obtain

$$\begin{aligned} \|f(x) - f(y)\|_2^2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

Using the inequality

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right\} \times \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2^2$$

we conclude that

$$\|f(x) - f(y)\|_2 \leq \sqrt{2.618}\ \|x - y\|_2, \quad \forall\ x, y \in R^2$$

Here we have used a property of positive semidefinite symmetric matrices; that is, $x^T P x \leq \lambda_{\max}(P)\ x^T x$, for all $x \in R^n$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the matrix. A more conservative (larger) Lipschitz constant will be obtained if we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

resulting in a Lipschitz constant $L = \sqrt{3}$. △

In these two examples, we have used $\|\cdot\|_\infty$ in one case and $\|\cdot\|_2$ in the other. Due to equivalence of norms, the choice of a norm on R^n does not affect the Lipschitz property of a function. It only affects the value of the Lipschitz constant (Exercise 3.5). Example 3.2 illustrates the fact that the Lipschitz condition (3.2) does not uniquely define the Lipschitz constant L. If (3.2) is satisfied with some positive constant L, it is satisfied with any constant larger than L. This nonuniqueness can be removed by defining L to be the smallest constant for which (3.2) is satisfied, but we seldom need to do that.

Theorem 3.1 is a local theorem since it guarantees existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where δ may be very small. In other words, we have no control on δ; hence, we cannot ensure existence and uniqueness over a given time interval $[t_0, t_1]$. However, one may try to extend the interval of existence by repeated applications of the local theorem. Starting at a time t_0, with an initial state $x(t_0) = x_0$, Theorem 3.1 shows that there is a positive constant δ (dependent on x_0) such that the state equation (3.1) has a unique solution over the time interval $[t_0, t_0 + \delta]$. Now, taking $t_0 + \delta$ as a new initial time and $x(t_0 + \delta)$ as a new initial state, one may try to apply Theorem 3.1 to establish existence of the solution beyond $t_0 + \delta$. If the conditions of the theorem are satisfied at $(t_0 + \delta, x(t_0 + \delta))$, then there exists $\delta_2 > 0$ such that the equation has a unique solution over $[t_0 + \delta, t_0 + \delta + \delta_2]$ that passes through the point $(t_0 + \delta, x(t_0 + \delta))$. We piece together the solutions over $[t_0, t_0 + \delta]$ and $[t_0 + \delta, t_0 + \delta + \delta_2]$ to establish the existence of a unique solution over $[t_0, t_0 + \delta + \delta_2]$. This idea can be repeated to keep extending the solution. However, in general, the interval of existence of the solution cannot be extended indefinitely because the conditions of Theorem 3.1 may cease to hold. There is a maximum interval $[t_0, T)$ where the unique solution starting at (t_0, x_0) exists.[5] In general, T may be less than t_1, in which case as $t \rightarrow T$, the solution leaves any compact set over which f is locally Lipschitz in x (Exercise 3.26).

Example 3.3 Consider the scalar system

$$\dot{x} = -x^2, \quad \text{with } \; x(0) = -1$$

The function $f(x) = -x^2$ is locally Lipschitz for all $x \in R$. Hence, it is Lipschitz on any compact subset of R. The unique solution

$$x(t) = \frac{1}{t-1}$$

exists over $[0, 1)$. As $t \rightarrow 1$, $x(t)$ leaves any compact set. △

The phrase "finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time. In Example 3.3, we say that the trajectory has a finite escape time at $t = 1$.

In view of the discussion preceding Example 3.3, one may pose the following question: When is it guaranteed that the solution can be extended indefinitely? One way to answer the question is to require additional conditions which ensure that the solution $x(t)$ will always be in a set where $f(t, x)$ is uniformly Lipschitz in x. This is done in the next theorem by requiring f to satisfy a global Lipschitz condition. The theorem establishes the existence of a unique solution over $[t_0, t_1]$, where t_1 may be arbitrarily large.

Theorem 3.2 (Global Existence and Uniqueness) *Suppose that $f(t, x)$ is piecewise continuous in t and satisfies*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

$\forall\, x, y \in R^n$, $\forall\, t \in [t_0, t_1]$. *Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.* ◇

Proof: See Appendix C.1.

Example 3.4 Consider the linear system

$$\dot{x} = A(t)x + g(t) = f(t, x)$$

where $A(t)$ and $g(t)$ are piecewise continuous functions of t. Over any finite interval of time $[t_0, t_1]$, the elements of $A(t)$ are bounded. Hence, $\|A(t)\| \leq a$, where $\|A\|$ is any induced matrix norm. The conditions of Theorem 3.2 are satisfied since

$$\|f(t, x) - f(t, y)\| = \|A(t)(x - y)\| \leq \|A(t)\|\; \|x - y\| \leq a\|x - y\|$$

---

[5] For a proof of this statement, see [81, Section 8.5] or [135, Section 2.3].

for all $x, y \in R^n$ and $t \in [t_0, t_1]$. Therefore, Theorem 3.2 shows that the linear system has a unique solution over $[t_0, t_1]$. Since t_1 can be arbitrarily large, we can also conclude that if $A(t)$ and $g(t)$ are piecewise continuous $\forall\ t \geq t_0$, then the system has a unique solution $\forall\ t \geq t_0$. Hence, the system cannot have a finite escape time. △

For the linear system of Example 3.4, the global Lipschitz condition of Theorem 3.2 is a reasonable requirement. This may not be the case for nonlinear systems, in general. We should distinguish between the local Lipschitz requirement of Theorem 3.1 and the global Lipschitz requirement of Theorem 3.2. Local Lipschitz property of a function is basically a smoothness requirement. It is implied by continuous differentiability. Except for discontinuous nonlinearities, which are idealizations of physical phenomena, it is reasonable to expect models of physical systems to have locally Lipschitz right-hand side functions. Examples of continuous functions that are not locally Lipschitz are quite exceptional and rarely arise in practice. The global Lipschitz property, on the other hand, is restrictive. Models of many physical systems fail to satisfy it. One can easily construct smooth meaningful examples that do not have the global Lipschitz property, but do have unique global solutions, which is an indication of the conservative nature of Theorem 3.2.

Example 3.5 Consider the scalar system

$$\dot{x} = -x^3 = f(x)$$

The function $f(x)$ does not satisfy a global Lipschitz condition since the Jacobian $\partial f/\partial x = -3x^2$ is not globally bounded. Nevertheless, for any initial state $x(t_0) = x_0$, the equation has the unique solution

$$x(t) = \text{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

which is well defined for all $t \geq t_0$. △

In view of the conservative nature of the global Lipschitz condition, it would be useful to have a global existence and uniqueness theorem that requires the function f to be only locally Lipschitz. The next theorem achieves that at the expense of having to know more about the solution of the system.

Theorem 3.3 *Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$. ◇

Proof: Recall the discussion on extending solutions, preceding Example 3.3. By Theorem 3.1, there is a unique local solution over $[t_0, t_0 + \delta]$. Let $[t_0, T)$ be its maximal interval of existence. We want to show that $T = \infty$. Recall (Exercise 3.26) the fact that if T is finite, then the solution must leave any compact subset of D. Since the solution never leaves the compact set W, we conclude that $T = \infty$. □

The trick in applying Theorem 3.3 is in checking the assumption that every solution lies in a compact set without actually solving the differential equation. We will see in Chapter 4 that Lyapunov's method for studying stability is very valuable in that regard. For now, let us illustrate the application of the theorem by a simple example.

Example 3.6 Consider again the system

$$\dot{x} = -x^3 = f(x)$$

of Example 3.5. The function $f(x)$ is locally Lipschitz on R. If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$. Thus, without calculating the solution, we conclude by Theorem 3.3 that the equation has a unique solution for all $t \geq 0$. △

3.2 Continuous Dependence on Initial Conditions and Parameters

For the solution of the state equation (3.1) to be of any interest, it must depend continuously on the initial state x_0, the initial time t_0, and the right-hand side function $f(t, x)$. Continuous dependence on the initial time t_0 is obvious from the integral expression

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Therefore, we leave it as an exercise (Exercise 3.28) and concentrate our attention on continuous dependence on the initial state x_0 and the function f. Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$. The solution depends continuously on y_0 if solutions starting at nearby points are defined on the same time interval and remain close to each other in that interval. This statement can be made precise with the ε–δ argument: Given $\varepsilon > 0$, there is $\delta > 0$ such that for all z_0 in the ball $\{x \in R^n \mid \|x - y_0\| < \delta\}$, the equation $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $\|z(t) - y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$. Continuous dependence on the right-hand side function f is defined similarly, but to state the definition precisely, we need a mathematical representation of the perturbation of f. One possible representation is to replace f by a sequence of functions f_m, which converge uniformly to f as $m \to \infty$. For each function f_m, the solution of $\dot{x} = f_m(t, x)$ with $x(t_0) = x_0$ is denoted by $x_m(t)$. The solution is said to depend continuously on the right-hand side function if $x_m(t) \to x(t)$ as $m \to \infty$. This approach is a little bit involved, and will not be pursued here.[6] A more restrictive, but simpler, mathematical representation is to assume that f depends continuously on a set of constant parameters; that is, $f = f(t, x, \lambda)$, where $\lambda \in R^p$. The constant parameters could represent physical parameters of the system, and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging. Let $x(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda_0) = x_0$. The solution is said to depend continuously on λ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all λ in the ball $\{\lambda \in R^p \mid \|\lambda - \lambda_0\| < \delta\}$, the equation $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda) = x_0$, and satisfies $\|x(t, \lambda) - x(t, \lambda_0)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

Continuous dependence on initial states and continuous dependence on parameters can be studied simultaneously. We start with a simple result that bypasses the issue of existence and uniqueness and concentrates on the closeness of solutions.

Theorem 3.4 *Let $f(t, x)$ be piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with a Lipschitz constant L, where $W \subset R^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of*

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

and

$$\dot{z} = f(t, z) + g(t, z), \quad z(t_0) = z_0$$

such that $y(t),\ z(t) \in W$ for all $t \in [t_0, t_1]$. Suppose that

$$\|g(t, x)\| \le \mu, \quad \forall\ (t, x) \in [t_0, t_1] \times W$$

for some $\mu > 0$. Then,

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L}\{\exp[L(t - t_0)] - 1\}$$

$\forall\ t \in [t_0, t_1]$. ◇

Proof: The solutions $y(t)$ and $z(t)$ are given by

$$\begin{aligned} y(t) &= y_0 + \int_{t_0}^{t} f(s, y(s))\, ds \\ z(t) &= z_0 + \int_{t_0}^{t} [f(s, z(s)) + g(s, z(s))]\, ds \end{aligned}$$

---

[6] See [43, Section 1.3], [75, Section 1.3], or [135, Section 2.5] for results on continuous dependence on parameters using this approach.

Subtracting the two equations and taking norms yield

$$\begin{aligned}\|y(t)-z(t)\| &\leq \|y_0-z_0\|+\int_{t_0}^{t}\|f(s,y(s))-f(s,z(s))\|\,ds\\ &\quad+\int_{t_0}^{t}\|g(s,z(s))\|\,ds\\ &\leq \gamma+\mu(t-t_0)+\int_{t_0}^{t}L\|y(s)-z(s)\|\,ds\end{aligned}$$

where $\gamma=\|y_0-z_0\|$. Application of the Gronwall–Bellman inequality (Lemma A.1) to the function $\|y(t)-z(t)\|$ results in

$$\|y(t)-z(t)\|\leq\gamma+\mu(t-t_0)+\int_{t_0}^{t}L[\gamma+\mu(s-t_0)]\exp[L(t-s)]\,ds$$

Integrating the right-hand side by parts, we obtain

$$\begin{aligned}\|y(t)-z(t)\| &\leq \gamma+\mu(t-t_0)-\gamma-\mu(t-t_0)+\gamma\exp[L(t-t_0)]\\ &\quad+\int_{t_0}^{t}\mu\exp[L(t-s)]\,ds\\ &= \gamma\exp[L(t-t_0)]+\frac{\mu}{L}\{\exp[L(t-t_0)]-1\}\end{aligned}$$

which completes the proof of the theorem. □

With Theorem 3.4 in hand, we can prove the next theorem on the continuity of solutions in terms of initial states and parameters.

Theorem 3.5 *Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0,t_1]\times D\times\{\|\lambda-\lambda_0\|\leq c\}$, where $D\subset R^n$ is an open connected set. Let $y(t,\lambda_0)$ be a solution of $\dot{x}=f(t,x,\lambda_0)$ with $y(t_0,\lambda_0)=y_0\in D$. Suppose $y(t,\lambda_0)$ is defined and belongs to D for all $t\in[t_0,t_1]$. Then, given $\varepsilon>0$, there is $\delta>0$ such that if*

$$\|z_0-y_0\|<\delta \quad\text{and}\quad \|\lambda-\lambda_0\|<\delta$$

then there is a unique solution $z(t,\lambda)$ of $\dot{x}=f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $z(t_0,\lambda)=z_0$, and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda)-y(t,\lambda_0)\|<\varepsilon,\quad \forall\ t\in[t_0,t_1]$$

◇

Proof: By continuity of $y(t,\lambda_0)$ in t and the compactness of $[t_0,t_1]$, we know that $y(t,\lambda_0)$ is bounded on $[t_0,t_1]$. Define a "tube" U around the solution $y(t,\lambda_0)$ (see Figure 3.1) by

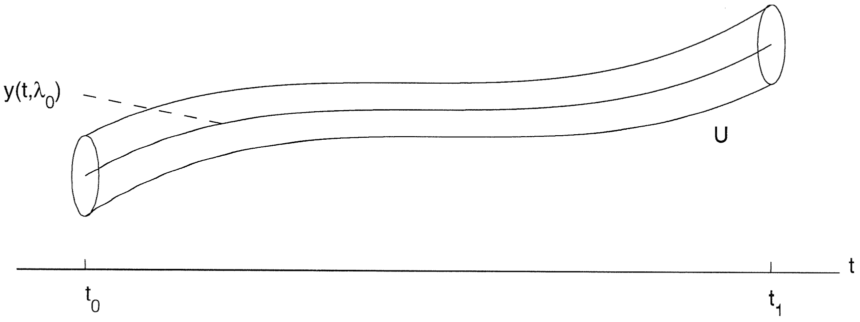

Figure 3.1: A tube constructed around the solution $y(t, \lambda_0)$.

$$U = \{(t, x) \in [t_0, t_1] \times R^n \mid \|x - y(t, \lambda_0)\| \leq \varepsilon\}$$

Suppose that $U \subset [t_0, t_1] \times D$; if not, replace ε by $\varepsilon_1 < \varepsilon$ that is small enough to ensure that $U \subset [t_0, t_1] \times D$ and continue the proof with ε_1. The set U is compact; hence, $f(t, x, \lambda)$ is Lipschitz in x on U with a Lipschitz constant, say, L. By continuity of f in λ, for any $\alpha > 0$, there is $\beta > 0$ (with $\beta < c$) such that

$$\|f(t, x, \lambda) - f(t, x, \lambda_0)\| < \alpha, \quad \forall\ (t, x) \in U, \ \forall\ \|\lambda - \lambda_0\| < \beta$$

Take $\alpha < \varepsilon$ and $\|z_0 - y_0\| < \alpha$. By the local existence and uniqueness theorem, there is a unique solution $z(t, \lambda)$ on some time interval $[t_0, t_0 + \Delta]$. The solution starts inside the tube U, and as long as it remains in the tube, it can be extended. We will show that, by choosing a small enough α, the solution remains in U for all $t \in [t_0, t_1]$. In particular, we let τ be the first time the solution leaves the tube and show that we can make $\tau > t_1$. On the time interval $[t_0, \tau]$, the conditions of Theorem 3.4 are satisfied with $\mu = \alpha$. Hence,

$$\begin{aligned} \|z(t, \lambda) - y(t, \lambda_0)\| &< \alpha \exp[L(t - t_0)] + \frac{\alpha}{L}\{\exp[L(t - t_0)] - 1\} \\ &< \alpha\left(1 + \frac{1}{L}\right) \exp[L(t - t_0)] \end{aligned}$$

Choosing $\alpha \leq \varepsilon L \exp[-L(t_1 - t_0)]/(1 + L)$ ensures that the solution $z(t, \lambda)$ cannot leave the tube during the interval $[t_0, t_1]$. Therefore, $z(t, \lambda)$ is defined on $[t_0, t_1]$ and satisfies $\|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon$. Taking $\delta = \min\{\alpha, \beta\}$ completes the proof of the theorem. □

3.3 Differentiability of Solutions and Sensitivity Equations

Suppose that $f(t, x, \lambda)$ is continuous in (t, x, λ) and has continuous first partial derivatives with respect to x and λ for all $(t, x, \lambda) \in [t_0, t_1] \times R^n \times R^p$. Let λ_0 be a nominal value of λ, and suppose that the nominal state equation

$$\dot{x} = f(t, x, \lambda_0), \quad \text{with} \;\; x(t_0) = x_0$$

has a unique solution $x(t, \lambda_0)$ over $[t_0, t_1]$. From Theorem 3.5, we know that for all λ sufficiently close to λ_0, that is, $\|\lambda - \lambda_0\|$ sufficiently small, the state equation

$$\dot{x} = f(t, x, \lambda), \quad \text{with} \;\; x(t_0) = x_0$$

has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$ that is close to the nominal solution $x(t, \lambda_0)$. The continuous differentiability of f with respect to x and λ implies the additional property that the solution $x(t, \lambda)$ is differentiable with respect to λ near λ_0. To see this, write

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) \; ds$$

Taking partial derivatives with respect to λ yields

$$x_\lambda(t, \lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) \; x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] \; ds$$

where $x_\lambda(t, \lambda) = [\partial x(t, \lambda)/\partial \lambda]$ and $[\partial x_0/\partial \lambda] = 0$, since x_0 is independent of λ. Differentiating with respect to t, it can be seen that $x_\lambda(t, \lambda)$ satisfies the differential equation

$$\frac{\partial}{\partial t} x_\lambda(t, \lambda) = A(t, \lambda) x_\lambda(t, \lambda) + B(t, \lambda), \quad x_\lambda(t_0, \lambda) = 0 \tag{3.4}$$

where

$$A(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right|_{x = x(t, \lambda)}, \quad B(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right|_{x = x(t, \lambda)}$$

For λ sufficiently close to λ_0, the matrices $A(t, \lambda)$ and $B(t, \lambda)$ are defined on $[t_0, t_1]$. Hence, $x_\lambda(t, \lambda)$ is defined on the same interval. At $\lambda = \lambda_0$, the right-hand side of (3.4) depends only on the nominal solution $x(t, \lambda_0)$. Let $S(t) = x_\lambda(t, \lambda_0)$; then $S(t)$ is the unique solution of the equation

$$\dot{S}(t) = A(t, \lambda_0) S(t) + B(t, \lambda_0), \quad S(t_0) = 0 \tag{3.5}$$

The function $S(t)$ is called the *sensitivity function*, and (3.5) is called the *sensitivity equation*. Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions. They can also be used to approximate the solution when λ is sufficiently close to its nominal value λ_0. For small $\|\lambda - \lambda_0\|$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$ to obtain

$$x(t, \lambda) = x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + \text{ higher-order terms}$$

Neglecting the higher-order terms, the solution $x(t, \lambda)$ can be approximated by

$$x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0) \tag{3.6}$$

We will not justify this approximation here. It will be justified in Chapter 10 when we study the perturbation theory. The significance of (3.6) is in the fact that knowledge of the nominal solution and the sensitivity function suffices to approximate the solution for all values of λ in a (small) ball centered at λ_0.

The procedure for calculating the sensitivity function $S(t)$ is summarized by the following steps:

- Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$.
- Evaluate the Jacobian matrices

$$A(t, \lambda_0) = \left.\frac{\partial f(t, x, \lambda)}{\partial x}\right|_{x = x(t,\lambda_0), \lambda = \lambda_0}, \quad B(t, \lambda_0) = \left.\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right|_{x = x(t,\lambda_0), \lambda = \lambda_0}$$

- Solve the sensitivity equation (3.5) for $S(t)$.

In this procedure, we need to solve the nonlinear nominal state equation and the linear time-varying sensitivity equation. Except for some trivial cases, we will be forced to solve these equations numerically. An alternative approach for calculating $S(t)$ is to solve for the nominal solution and the sensitivity function simultaneously. This can be done by appending the variational equation (3.4) with the original state equation, then setting $\lambda = \lambda_0$ to obtain the $(n + np)$ augmented equation

$$\begin{array}{rcl@{\qquad}rcl} \dot{x} &=& f(t, x, \lambda_0), & x(t_0) &=& x_0 \\ \dot{S} &=& \left[\frac{\partial f(t,x,\lambda)}{\partial x}\right]_{\lambda=\lambda_0} S + \left[\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_0}, & S(t_0) &=& 0 \end{array} \tag{3.7}$$

which is solved numerically. Notice that if the original state equation is autonomous, that is, $f(t, x, \lambda) = f(x, \lambda)$, then the augmented equation (3.7) will be autonomous as well. We illustrate the latter procedure by the next example.

Example 3.7 Consider the phase-locked-loop model

$$\begin{array}{rclcl} \dot{x}_1 &=& x_2 &=& f_1(x_1, x_2) \\ \dot{x}_2 &=& -c \sin x_1 - (a + b \cos x_1)x_2 &=& f_2(x_1, x_2) \end{array}$$

and suppose the parameters a, b, and c have the nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$. The nominal system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \end{aligned}$$

The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \lambda]$ are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c\cos x_1 + bx_2 \sin x_1 & -(a + b\cos x_1) \end{bmatrix}$$

$$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} & \frac{\partial f}{\partial c} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$$

Evaluate these Jacobian matrices at the nominal parameters $a = 1$, $b = 0$, and $c = 1$ to obtain

$$\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$$

Let

$$S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix}\right|_{\text{nominal}}$$

Then (3.7) is given by

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(0) &= x_{10} \\
\dot{x}_2 &= -\sin x_1 - x_2, & x_2(0) &= x_{20} \\
\dot{x}_3 &= x_4, & x_3(0) &= 0 \\
\dot{x}_4 &= -x_3 \cos x_1 - x_4 - x_2, & x_4(0) &= 0 \\
\dot{x}_5 &= x_6, & x_5(0) &= 0 \\
\dot{x}_6 &= -x_5 \cos x_1 - x_6 - x_2 \cos x_1, & x_6(0) &= 0 \\
\dot{x}_7 &= x_8, & x_7(0) &= 0 \\
\dot{x}_8 &= -x_7 \cos x_1 - x_8 - \sin x_1, & x_8(0) &= 0
\end{aligned}$$

The solution of this equation was computed for the initial state $x_{10} = x_{20} = 1$. Figure 3.2(a) shows x_3, x_5, and x_7, which are the sensitivities of x_1 with respect to a, b, and c, respectively. Figure 3.2(b) shows the corresponding quantities for x_2. Inspection of these figures shows that the solution is more sensitive to variations in the parameter c than to variations in the parameters a and b. This pattern is consistent when we solve for other initial states. △

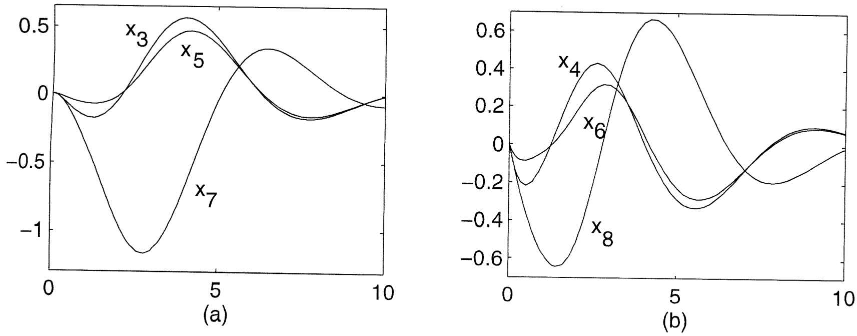

Figure 3.2: Sensitivity function for Example 3.7.

3.4 Comparison Principle

Quite often when we study the state equation $\dot{x} = f(t,x)$ we need to compute bounds on the solution $x(t)$ without computing the solution itself. The Gronwall–Bellman inequality (Lemma A.1) is one tool that can be used toward that goal. Another tool is the comparison lemma. It applies to a situation where the derivative of a scalar differentiable function $v(t)$ satisfies inequality of the form $\dot{v}(t) \leq f(t, v(t))$ for all t in a certain time interval. Such inequality is called a *differential inequality* and a function $v(t)$ satisfying the inequality is called a solution of the differential inequality. The comparison lemma compares the solution of the differential inequality $\dot{v}(t) \leq f(t, v(t))$ with the solution of the differential equation $\dot{u} = f(t,u)$. The lemma applies even when $v(t)$ is not differentiable, but has an upper right-hand derivative $D^+v(t)$, which satisfies a differential inequality. The upper right-hand derivative $D^+v(t)$ is defined in Appendix C.2. For our purposes, it is enough to know two facts:

- If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$.
- If

  $$\frac{1}{h}\,|v(t+h) - v(t)| \leq g(t,h), \quad \forall\, h \in (0, b]$$

  and

  $$\lim_{h \to 0^+} g(t,h) = g_0(t)$$

  then $D^+v(t) \leq g_0(t)$.

The limit $h \to 0^+$ means that h approaches zero from above.

Lemma 3.4 (Comparison Lemma) *Consider the scalar differential equation*

$$\dot{u} = f(t,u), \quad u(t_0) = u_0$$

where $f(t, u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in J \subset R$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$. ◇

Proof: See Appendix C.2.

Example 3.8 The scalar differential equation

$$\dot{x} = f(x) = -(1 + x^2)x, \quad x(0) = a$$

has a unique solution on $[0, t_1)$, for some $t_1 > 0$, because $f(x)$ is locally Lipschitz. Let $v(t) = x^2(t)$. The function $v(t)$ is differentiable and its derivative is given by

$$\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t)$$

Hence, $v(t)$ satisfies the differential inequality

$$\dot{v}(t) \leq -2v(t), \quad v(0) = a^2$$

Let $u(t)$ be the solution of the differential equation

$$\dot{u} = -2u, \quad u(0) = a^2 \quad \Rightarrow \quad u(t) = a^2e^{-2t}$$

Then, by the comparison lemma, the solution $x(t)$ is defined for all $t \geq 0$ and satisfies

$$|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \quad \forall\, t \geq 0$$

△

Example 3.9 The scalar differential equation

$$\dot{x} = f(t, x) = -(1 + x^2)x + e^t, \quad x(0) = a$$

has a unique solution on $[0, t_1)$ for some $t_1 > 0$, because $f(t, x)$ is locally Lipschitz in x. We want to find an upper bound on $|x(t)|$ similar to the one we obtained in the previous example. Let us start with $v(t) = x^2(t)$ as in Example 3.8. The derivative of v is given by

$$\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) + 2x(t)e^t \leq -2v(t) + 2\sqrt{v(t)}e^t$$

We can apply the comparison lemma to this differential inequality, but the resulting differential equation will not be easy to solve. Instead, we consider a different choice of $v(t)$. Let $v(t) = |x(t)|$. For $x(t) \neq 0$, the function $v(t)$ is differentiable and its derivative is given by

$$\dot{v}(t) = \frac{d}{dt}\sqrt{x^2(t)} = \frac{x(t)\dot{x}(t)}{|x(t)|} = -|x(t)|[1 + x^2(t)] + \frac{x(t)}{|x(t)|}e^t$$

Since $1 + x^2(t) \geq 1$, we have $-|x(t)|[1 + x^2(t)] \leq -|x(t)|$ and $\dot{v}(t) \leq -v(t) + e^t$. On the other hand, when $x(t) = 0$, we have

$$\begin{aligned} \frac{|v(t+h) - v(t)|}{h} &= \frac{|x(t+h)|}{h} = \frac{1}{h}\left|\int_t^{t+h} f(\tau, x(\tau))\, d\tau\right| \\ &= \left|f(t,0) + \frac{1}{h}\int_t^{t+h} [f(\tau, x(\tau)) - f(t, x(t))]\, d\tau\right| \\ &\leq |f(t,0)| + \frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\, d\tau \end{aligned}$$

Since $f(t, x(t))$ is a continuous function of t, given any $\varepsilon > 0$ there is $\delta > 0$ such that for all $|\tau - t| < \delta$, $|f(\tau, x(\tau)) - f(t, x(t))| < \varepsilon$. Hence, for all $h < \delta$,

$$\frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\, d\tau < \varepsilon$$

which shows that

$$\lim_{h \to 0^+} \frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\, d\tau = 0$$

Thus, $D^+v(t) \leq |f(t,0)| = e^t$ whenever $x(t) = 0$. Hence, for all $t \in [0, t_1)$, we have

$$D^+v(t) \leq -v(t) + e^t, \quad v(0) = |a|$$

Letting $u(t)$ be the solution of the linear differential equation

$$\dot{u} = -u + e^t, \quad u(0) = |a|$$

we conclude by the comparison lemma that

$$v(t) \leq u(t) = e^{-t}|a| + \tfrac{1}{2}\left[e^t - e^{-t}\right], \quad \forall\, t \in [0, t_1)$$

The upper bound on $v(t)$ is finite for every finite t_1 and approaches infinity only as $t_1 \to \infty$. Therefore, the solution $x(t)$ is defined for all $t \geq 0$ and satisfies

$$|x(t)| \leq e^{-t}|a| + \tfrac{1}{2}\left[e^t - e^{-t}\right], \quad \forall\, t \geq 0$$

$\triangle$

3.5 Exercises

3.1 For each of the functions $f(x)$ given next, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

(1) $f(x) = x^2 + |x|$.
(2) $f(x) = x + \text{sgn}(x)$.
(3) $f(x) = \sin(x)\ \text{sgn}(x)$.
(4) $f(x) = -x + a\sin(x)$.
(5) $f(x) = -x + 2|x|$.
(6) $f(x) = \tan(x)$.

(7) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$.

(8) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$.

3.2 Let $D_r = \{x \in R^n \mid \|x\| < r\}$. For each of the following systems, represented as $\dot{x} = f(t, x)$, find whether (a) f is locally Lipschitz in x on D_r, for sufficiently small r; (b) f is locally Lipschitz in x on D_r, for any finite $r > 0$; (c) f is globally Lipschitz in x:

(1) The pendulum equation with friction and constant input torque (Section 1.2.1).

(2) The tunnel-diode circuit (Example 2.1).

(3) The mass–spring equation with linear spring, linear viscous damping, Coulomb friction, and zero external force (Section 1.2.3).

(4) The Van der Pol oscillator (Example 2.6).

(5) The closed-loop equation of a third-order adaptive control system (Section 1.2.5).

(6) The system $\dot{x} = Ax - B\psi(Cx)$, where A, B, and C are $n \times n$, $n \times 1$, and $1 \times n$ matrices, respectively, and $\psi(\cdot)$ is the dead-zone nonlinearity of Figure 1.10(c).

3.3 Show that if $f_1 : R \to R$ and $f_2 : R \to R$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$ and $f_2 \circ f_1$ are locally Lipschitz.

3.4 Let $f : R^n \to R^n$ be defined by

$$f(x) = \begin{cases} \frac{1}{\|Kx\|} Kx, & \text{if } g(x)\|Kx\| \geq \mu > 0 \\ \\ \frac{g(x)}{\mu} Kx, & \text{if } g(x)\|Kx\| < \mu \end{cases}$$

where $g : R^n \to R$ is locally Lipschitz and nonnegative, and K is a constant matrix. Show that $f(x)$ is Lipschitz on any compact subset of R^n.

3.5 Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different p-norms on R^n. Show that $f : R^n \to R^m$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

3.6 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \le k_1 + k_2\|x\|, \quad \forall\ (t,x) \in [t_0,\infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}$$

for all $t \ge t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

3.7 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)} g(x)$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \ge 0$.

3.8 Show that the state equation

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b \end{aligned}$$

has a unique solution defined for all $t \ge 0$.

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

3.10 Derive the sensitivity equations for the tunnel-diode circuit of Example 2.1 as L and C vary from their nominal values.

3.11 Derive the sensitivity equations for the Van der Pol oscillator of Example 2.6 as ε varies from its nominal value. Use the state equation in the x-coordinates.

3.12 Repeat the previous exercise by using the state equation in the z-coordinates.

3.13 Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

3.14 Consider the system

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)\\ \dot{x}_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)\end{aligned}$$

where λ and τ are positive constants.

(a) Derive the sensitivity equations as λ and τ vary from their nominal values λ_0 and τ_0.

(b) Show that $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \le -\frac{1}{\tau}r + 2\sqrt{2}$$

(c) Using the comparison lemma, show that the solution of the state equation satisfies the inequality

$$\|x(t)\|_2 \le e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau})$$

3.15 Using the comparison lemma, show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_2^2}, \qquad \dot{x}_2 = -x_2 + \frac{2x_1}{1 + x_1^2}$$

satisfies the inequality

$$\|x(t)\|_2 \le e^{-t}\|x(0)\|_2 + \sqrt{2}\left(1 - e^{-t}\right)$$

3.16 Using the comparison lemma, find an upper bound on the solution of the scalar equation

$$\dot{x} = -x + \frac{\sin t}{1 + x^2}, \quad x(0) = 2$$

3.17 Consider the initial-value problem (3.1) and let $D \subset R^n$ be a domain that contains $x = 0$. Suppose $x(t)$, the solution of (3.1), belongs to D for all $t \ge t_0$ and $\|f(t,x)\|_2 \le L\|x\|_2$ on $[t_0, \infty) \times D$. Show that

(a)

$$\left|\frac{d}{dt}\left[x^T(t)x(t)\right]\right| \le 2L\|x(t)\|_2^2$$

(b)

$$\|x_0\|_2 \exp[-L(t - t_0)] \le \|x(t)\|_2 \le \|x_0\|_2 \exp[L(t - t_0)]$$

3.18 Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3] \, d\tau$$

where k_1, k_2, and k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall–Bellman inequality, show that

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

3.19 Let $f : R^n \to R^n$ be locally Lipschitz in a domain $D \subset R^n$. Let $S \subset D$ be a compact set. Show that there is a positive constant L such that for all $x, y \in S$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Hint: The set S can be covered by a finite number of neighborhoods; that is,

$$S \subset N(a_1, r_1) \cup N(a_2, r_2) \cup \cdots \cup N(a_k, r_k)$$

Consider the following two cases separately:

- $x, y \in S \cap N(a_i, r_i)$ for some i.
- $x, y \not\in S \cap N(a_i, r_i)$ for any i; in this case, $\|x - y\| \geq \min_i r_i$.

In the second case, use the fact that $f(x)$ is uniformly bounded on S.

3.20 Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

3.21 For any $x \in R^n - \{0\}$ and any $p \in [1, \infty)$, define $y \in R^n$ by

$$y_i = \frac{x_i^{p-1}}{\|x\|_p^{p-1}} \operatorname{sign}(x_i^p)$$

Show that $y^T x = \|x\|_p$ and $\|y\|_q = 1$, where $q \in (1, \infty]$ is determined from $1/p + 1/q = 1$. For $p = \infty$, find a vector y such that $y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$.

3.22 Prove Lemma 3.3.

3.23 Let $f(x)$ be a continuously differentiable function that maps a convex domain $D \subset R^n$ into R^n. Suppose D contains the origin $x = 0$ and $f(0) = 0$. Show that

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) \, d\sigma \; x, \quad \forall \; x \in D$$

Hint: Set $g(\sigma) = f(\sigma x)$ for $0 \leq \sigma \leq 1$ and use the fact that $g(1) - g(0) = \int_0^1 g'(\sigma) \, d\sigma$.

3.24 Let $V : R \times R^n \to R$ be continuously differentiable. Suppose that $V(t, 0) = 0$ for all $t \geq 0$ and

$$V(t,x) \geq c_1\|x\|^2; \quad \left\| \frac{\partial V}{\partial x}(t,x) \right\| \leq c_4\|x\|, \quad \forall\ (t,x) \in [0,\infty) \times D$$

where c_1 and c_4 are positive constants and $D \subset R^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \leq \frac{1}{2}c_4\|x\|^2$ for all $x \in D$.
Hint: Use the representation $V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x)\ d\sigma\ x$.

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \leq c_4$.

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$|W(t,x_2) - W(t,x_1)| \leq \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\|, \quad \forall\ t \geq 0,\ \forall\ x_1, x_2 \in D$$

3.25 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0, t_1] \times D$, for some domain $D \subset R^n$. Let W be a compact subset of D. Let $x(t)$ be the solution of $\dot{x} = f(t,x)$ starting at $x(t_0) = x_0 \in W$. Suppose that $x(t)$ is defined and $x(t) \in W$ for all $t \in [t_0, T)$, $T < t_1$.

(a) Show that $x(t)$ is uniformly continuous on $[t_0, T)$.

(b) Show that $x(T)$ is defined and belongs to W and $x(t)$ is a solution on $[t_0, T]$.

(c) Show that there is $\delta > 0$ such that the solution can be extended to $[t_0, T + \delta]$.

3.26 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0, t_1] \times D$, for some domain $D \subset R^n$. Let $y(t)$ be a solution of (3.1) on a maximal open interval $[t_0, T) \subset [t_0, t_1]$ with $T < \infty$. Let W be any compact subset of D. Show that there is some $t \in [t_0, T)$ with $y(t) \notin W$.
Hint: Use the previous exercise.

3.27 (**[43]**) Let $x_1 : R \to R^n$ and $x_2 : R \to R^n$ be differentiable functions such that

$$\|x_1(a) - x_2(a)\| \leq \gamma, \quad \|\dot{x}_i(t) - f((t, x_i(t))\| \leq \mu_i, \ \text{for } i = 1, 2$$

for $a \leq t \leq b$. If f satisfies the Lipschitz condition (3.2), show that

$$\|x_1(t) - x_2(t)\| \leq \gamma e^{L(t-a)} + (\mu_1 + \mu_2)\left[\frac{e^{L(t-a)} - 1}{L}\right], \quad \text{for } a \leq t \leq b$$

3.28 Show, under the assumptions of Theorem 3.5, that the solution of (3.1) depends continuously on the initial time t_0.

3.29 Let $f(t,x)$ and its partial derivatives with respect to x be continuous in (t,x) for all $(t,x) \in [t_0,t_1] \times R^n$. Let $x(t,\eta)$ be the solution of (3.1) that starts at $x(t_0) = \eta$ and suppose $x(t,\eta)$ is defined on $[t_0,t_1]$. Show that $x(t,\eta)$ is continuously differentiable with respect to η and find the variational equation satisfied by $[\partial x/\partial\eta]$. Hint: Put $y = x - \eta$ to transform (3.1) into

$$\dot{y} = f(t, y + \eta), \quad y(t_0) = 0$$

with η as a parameter.

3.30 Let $f(t,x)$ and its partial derivative with respect to x be continuous in (t,x) for all $(t,x) \in R \times R^n$. Let $x(t,a,\eta)$ be the solution of (3.1) that starts at $x(a) = \eta$ and suppose that $x(t,a,\eta)$ is defined on $[a,t_1]$. Show that $x(t,a,\eta)$ is continuously differentiable with respect to a and η and let $x_a(t)$ and $x_\eta(t)$ denote $[\partial x/\partial a]$ and $[\partial x/\partial\eta]$, respectively. Show that $x_a(t)$ and $x_\eta(t)$ satisfy the identity

$$x_a(t) + x_\eta(t) f(a,\eta) \equiv 0, \quad \forall\ t \in [a,t_1]$$

3.31 ([43]) Let $f : R \times R \to R$ be a continuous function. Suppose that $f(t,x)$ is locally Lipschitz and nondecreasing in x for each fixed value of t. Let $x(t)$ be a solution of $\dot{x} = f(t,x)$ on an interval $[a,b]$. If a continuous function $y(t)$ satisfies the integral inequality

$$y(t) \le x(a) + \int_a^t f(s, y(s))\ ds$$

for $a \le t \le b$, show that $y(t) \le x(t)$ throughout this interval.

Chapter 4

Lyapunov Stability

Stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This chapter is concerned mainly with stability of equilibrium points. In later chapters, we shall see other kinds of stability, such as input–output stability and stability of periodic orbits. Stability of equilibrium points is usually characterized in the sense of Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory, which now carries his name. An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. These notions are made precise in Section 4.1, where the basic theorems of Lyapunov's method for autonomous systems are given. An extension of the basic theory, due to LaSalle, is given in Section 4.2. For a linear time-invariant system $\dot{x}(t) = Ax(t)$, the stability of the equilibrium point $x = 0$ can be completely characterized by the location of the eigenvalues of A. This is discussed in Section 4.3. In the same section, it is shown when and how the stability of an equilibrium point can be determined by linearization about that point. In Section 4.4, we introduce class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which are used extensively in the rest of the chapter, and indeed the rest of the book. In Sections 4.5 and 4.6, we extend Lyapunov's method to nonautonomous systems. In Section 4.5, we define the concepts of uniform stability, uniform asymptotic stability, and exponential stability for nonautonomous systems, and give Lyapunov's method for testing them. In Section 4.6, we study linear time-varying systems and linearization.

Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and so on. They do not say whether the given conditions are also necessary. There are theorems which establish, at least conceptually, that for many of Lyapunov stability theorems, the given conditions are indeed necessary. Such theorems are usually called converse theorems. We present three converse theorems in Section 4.7. Moreover, we use the converse theorem for exponential stability to show that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin.

Lyapunov stability analysis can be used to show boundedness of the solution, even when the system has no equilibrium points. This is shown in Section 4.8 where the notions of uniform boundedness and ultimate boundedness are introduced. Finally, in Section 4.9, we introduce the notion of input-to-state stability, which provides a natural extension of Lyapunov stability to systems with inputs.

4.1 Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{4.1}$$

where $f : D \rightarrow R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} \in D$ is an equilibrium point of (4.1); that is, $f(\bar{x}) = 0$. Our goal is to characterize and study the stability of $\bar{x}$. For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and consider the change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

In the new variable y, the system has equilibrium at the origin. Therefore, without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and study the stability of the origin $x = 0$.

Definition 4.1 *The equilibrium point $x = 0$ of (4.1) is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq 0$$

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0$$

The ε–δ requirement for stability takes a challenge-answer form. To demonstrate that the origin is stable, then, for any value of ε that a challenger may care to designate, we must produce a value of δ, possibly dependent on ε, such that a trajectory starting in a δ neighborhood of the origin will never leave the ε neighborhood. The three types of stability properties can be illustrated by the pendulum example of Section 1.2.1. The pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Neglecting friction, by setting $b = 0$, we have seen in Chapter 2 (Figure 2.2) that trajectories in the neighborhood of the first equilibrium are closed orbits. Therefore, by starting sufficiently close to the equilibrium point, trajectories can be guaranteed to stay within any specified ball centered at the equilibrium point. Hence, the ε–δ requirement for stability is satisfied. The equilibrium point, however, is not asymptotically stable since trajectories starting off the equilibrium point do not tend to it eventually. Instead, they remain in their closed orbits. When friction is taken into consideration $(b > 0)$, the equilibrium point at the origin becomes a stable focus. Inspection of the phase portrait of a stable focus shows that the ε–δ requirement for stability is satisfied. In addition, trajectories starting close to the equilibrium point tend to it as t tends to ∞. The second equilibrium point at $x_1 = \pi$ is a saddle point. Clearly the ε–δ requirement cannot be satisfied since, for any $\varepsilon > 0$, there is always a trajectory that will leave the ball $\{x \in R^n \mid \|x - \bar{x}\| \le \varepsilon\}$ even when $x(0)$ is arbitrarily close to the equilibrium point $\bar{x}$.

Implicit in Definition 4.1 is a requirement that solutions of (4.1) be defined for all $t \ge 0$.[1] Such global existence of the solution is not guaranteed by the local Lipschitz property of f. It will be shown, however, that the additional conditions needed in Lyapunov's theorem will ensure global existence of the solution. This will come as an application of Theorem 3.3.

Having defined stability and asymptotic stability of equilibrium points, our task now is to find ways to determine stability. The approach we used in the pendulum example relied on our knowledge of the phase portrait of the pendulum equation. Trying to generalize that approach amounts to actually finding all solutions of (4.1), which may be difficult or even impossible. However, the conclusions we reached about the stable equilibrium point of the pendulum can also be reached by using energy concepts. Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

$$E(x) = \int_0^{x_1} a\sin y\ dy + \tfrac{1}{2}x_2^2 = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

When friction is neglected $(b = 0)$, the system is conservative; that is, there is no dissipation of energy. Hence, $E =$ constant during the motion of the system or, in other words, $dE/dt = 0$ along the trajectories of the system. Since $E(x) = c$ forms a closed contour around $x = 0$ for small c, we can again arrive at the conclusion that $x = 0$ is a stable equilibrium point. When friction is accounted for ($b > 0$), energy will dissipate during the motion of the system, that is, $dE/dt \leq 0$ along the trajectories of the system. Due to friction, E cannot remain constant indefinitely while the system is in motion. Hence, it keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞. Thus, by examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point. In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point. Let $V : D \to R$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of (4.1), denoted by $\dot{V}(x)$, is given by

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[1] It is possible to change the definition to alleviate the implication of global existence of the solution. In [154], stability is defined on the maximal interval of existence $[0, t_1)$, without assuming that $t_1 = \infty$.

$$\begin{aligned}\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\ \frac{\partial V}{\partial x_1},\ \frac{\partial V}{\partial x_2},\ \ldots,\ \frac{\partial V}{\partial x_n}\ \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x)\end{aligned}$$

The derivative of V along the trajectories of a system is dependent on the system's equation. Hence, $\dot{V}(x)$ will be different for different systems. If $\phi(t; x)$ is the solution of (4.1) that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \left.\frac{d}{dt} V(\phi(t; x))\right|_{t=0}$$

Therefore, if $\dot{V}(x)$ is negative, V will decrease along the solution of (4.1). We are now ready to state Lyapunov's stability theorem.

Theorem 4.1 *Let* $x = 0$ *be an equilibrium point for* (4.1) *and* $D \subset R^n$ *be a domain containing* $x = 0$. *Let* $V : D \to R$ *be a continuously differentiable function such that*

$$V(0) = 0 \ \text{ and } \ V(x) > 0 \text{ in } D - \{0\} \tag{4.2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{4.3}$$

Then, $x = 0$ *is stable. Moreover, if*

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{4.4}$$

then $x = 0$ *is asymptotically stable.* ◇

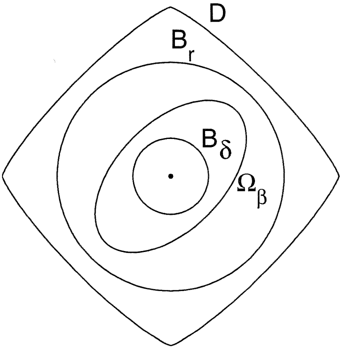

Figure 4.1: Geometric representation of sets in the proof of Theorem 4.1.

Proof: Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset D$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by (4.2). Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r.[2] (See Figure 4.1.) The set Ω_β has the property that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t \geq 0$. This follows from (4.3) since

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \ \forall\, t \geq 0$$

Because Ω_β is a compact set,[3] we conclude from Theorem 3.3 that (4.1) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \Rightarrow V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \quad \forall\, t \geq 0$$

---

[2] This fact can be shown by contradiction. Suppose Ω_β is not in the interior of B_r, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. At this point, $V(p) \geq \alpha > \beta$, but for all $x \in \Omega_\beta$, $V(x) \leq \beta$, which is a contradiction.

[3] Ω_β is closed by definition and bounded, since it is contained in B_r.

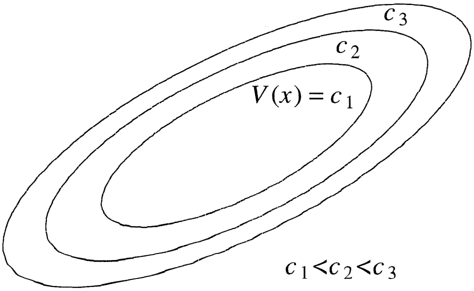

Figure 4.2: Level surfaces of a Lyapunov function.

which shows that the equilibrium point $x = 0$ is stable. Now, assume that (4.4) holds as well. To show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to \infty$; that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$. By repetition of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geq 0 \ \text{ as } \ t \to \infty$$

To show that $c = 0$, we use a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leq \|x\| \leq r\}$.[4] By (4.4), $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) \, d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. □

A continuously differentiable function $V(x)$ satisfying (4.2) and (4.3) is called a *Lyapunov function.* The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface.* Using Lyapunov surfaces, we notice that Figure 4.2 makes the theorem intuitively clear. It shows Lyapunov surfaces for increasing values of c. The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses. If we only know that $\dot{V} \leq 0$, we cannot be sure that the trajectory will approach the origin,[5] but we can conclude that the origin is stable since the trajectory can be contained inside any ball B_ε by requiring the initial state $x(0)$ to lie inside a Lyapunov surface contained in that ball.

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[4] See [10, Theorem 4-20].

A function $V(x)$ satisfying condition (4.2)—that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$—is said to be *positive definite.* If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is said to be *positive semidefinite.* A function $V(x)$ is said to be *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively. If $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. With this terminology, we can rephrase Lyapunov's theorem to say that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.*

A class of scalar functions for which sign definiteness can be easily checked is the class of functions of the quadratic form

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix. In this case, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principal minors of P are positive (all principal minors of P are nonnegative).[6] If $V(x) = x^T P x$ is positive definite (positive semidefinite), we say that the matrix P is positive definite (positive semidefinite) and write $P > 0$ $(P \geq 0)$.

Example 4.1 Consider

$$\begin{aligned} V(x) &= a x_1^2 + 2 x_1 x_3 + a x_2^2 + 4 x_2 x_3 + a x_3^2 \\ &= [x_1 \ x_2 \ x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x \end{aligned}$$

The leading principal minors of P are a, a^2, and $a(a^2-5)$. Therefore, $V(x)$ is positive definite if $a > \sqrt{5}$. For negative definiteness, the leading principal minors of $-P$ should be positive; that is, the leading principal minors of P should have alternating signs, with the odd-numbered minors being negative and the even-numbered minors being positive. Consequently, $V(x)$ is negative definite if $a < -\sqrt{5}$. By calculating all principal minors, it can be seen that $V(x)$ is positive semidefinite if $a \geq \sqrt{5}$ and negative semidefinite if $a \leq -\sqrt{5}$. For $a \in (-\sqrt{5}, \sqrt{5})$, $V(x)$ is indefinite. △

Lyapunov's theorem can be applied without solving the differential equation (4.1). On the other hand, there is no systematic method for finding Lyapunov

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[5] See, however, LaSalle's theorem in Section 4.2

[6] This is a well-known fact in matrix theory. Its proof can be found in [21] or [63].

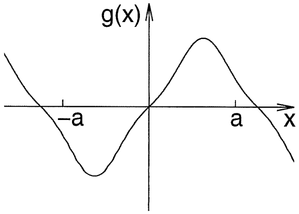

Figure 4.3: A possible nonlinearity in Example 4.2.

functions. In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error. The situation, however, is not as bad as it might seem. As we go over various examples and applications throughout the book, some ideas and approaches for searching for Lyapunov functions will be delineated.

Example 4.2 Consider the first-order differential equation

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0 \;\text{ and }\; x \in (-a, a)$$

A sketch of a possible $g(x)$ is shown in Figure 4.3. The system has an isolated equilibrium point at the origin. It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$. To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)\, dy$$

Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\, x \in D - \{0\}$$

Hence, by Theorem 4.1 we conclude that the origin is asymptotically stable. △

Example 4.3 Consider the pendulum equation without friction, namely,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\, a \sin x_1 \end{aligned}$$

and let us study the stability of the equilibrium point at the origin. A natural Lyapunov function candidate is the energy function

$$V(x) = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

Thus, conditions (4.2) and (4.3) of Theorem 4.1 are satisfied, and we conclude that the origin is stable. Since $\dot{V}(x) \equiv 0$, we can also conclude that the origin is not asymptotically stable; for trajectories starting on a Lyapunov surface $V(x) = c$ remain on the same surface for all future time. △

Example 4.4 Consider again the pendulum equation, but this time with friction, namely,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2 \end{aligned}$$

Again, let us try $V(x) = a(1 - \cos x_1) + (1/2)x_2^2$ as a Lyapunov function candidate.

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The derivative $\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $b > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. We will see later in Section 4.2 that LaSalle's theorem will enable us to arrive at a different conclusion. For now, let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy Lyapunov function, let us replace the term $(1/2)x_2^2$ by the more general quadratic form $(1/2)x^TPx$ for some 2×2 positive definite matrix P:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^TPx + a(1 - \cos x_1) \\ &= \tfrac{1}{2}[x_1 \; x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

For the quadratic form $(1/2)x^TPx$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a \sin x_1)\, x_2 + (p_{12}x_1 + p_{22}x_2)\,(-a \sin x_1 - bx_2) \\ &= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)\, x_1x_2 + (p_{12} - p_{22}b)\, x_2^2 \end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = bp_{12}$. With these choices, p_{12} must satisfy $0 < p_{12} < b$ for $V(x)$ to be positive definite. Let us take $p_{12} = b/2$. Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\ \tfrac{1}{2}abx_1 \sin x_1 - \tfrac{1}{2}bx_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, by Theorem 4.1, we conclude that the origin is asymptotically stable. △

This example emphasizes an important feature of Lyapunov's stability theorem; namely, *the theorem's conditions are only sufficient.* Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

In searching for a Lyapunov function in Example 4.4, we approached the problem in a backward manner. We investigated an expression for the derivative $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method.* To describe the procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories of (4.1) is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite. It is not difficult (Exercise 4.5) to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric; that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x) f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\ dy = \int_0^x \sum_{i=1}^n g_i(y)\ dy_i$$

The integration is taken over any path joining the origin to x.[7] Usually, this is done along the axes; that is,

$$V(x) \ = \ \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\ dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\ dy_2$$

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[7] The line integral of a gradient vector is independent of the path. (See [10, Theorem 10-37].)

$$+ \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\; dy_n$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 4.4. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 4.5 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 \end{aligned}$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$, $y \in (-b, c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$$

and

$$V(x) = \int_0^x g^T(y)\; dy > 0, \quad \text{for } x \neq 0$$

Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

where the scalar functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\delta(\cdot)$ are to be determined. To satisfy the symmetry requirement, we must have

$$\beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$$

The derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 - a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)$$

To cancel the cross-product terms, we choose

$$\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$$

so that

$$\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

To simplify our choices, let us take $\delta(x) = \delta = \text{constant}$, $\gamma(x) = \gamma = \text{constant}$, and $\beta(x) = \beta = \text{constant}$. Then, $\alpha(x)$ depends only on x_1, and the symmetry requirement is satisfied by choosing $\beta = \gamma$. The expression for $g(x)$ reduces to

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\ dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\ dy_2 \\ &= \tfrac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\ dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 \ = \ \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\ dy \end{aligned}$$

where

$$P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. For example, taking $\gamma = ak\delta$ for $0 < k < 1$ yields the Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\ dy$$

which satisfies conditions (4.2) and (4.4) of Theorem 4.1 over the domain $D = \{x \in R^2 \mid -b < x_1 < c\}$. Therefore, the origin is asymptotically stable. △

When the origin $x = 0$ is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (also called *region of asymptotic stability*, *domain of attraction*, or *basin*). Let $\phi(t; x)$ be the solution of (4.1) that starts at initial state x at time $t = 0$. Then, the region of attraction is defined as the set of all points x such that $\phi(t; x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \phi(t; x) = 0$. Finding the exact region of attraction analytically might be difficult or even impossible. However, Lyapunov functions can be used to estimate the region of attraction, that is, to find sets contained in the region of attraction. From the proof of Theorem 4.1, we see that if there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D and, if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$. Thus, Ω_c is an estimate of the region of attraction. The estimate, however, may be conservative; that is, it may be much smaller than the actual region of attraction. In Section 8.2, we will solve examples on estimating the region of attraction and see some ideas to enlarge the estimates. Here, we want to pursue another question: Under what conditions will the region of attraction be the whole space R^n? It will be the case if we can show that for any initial state x, the trajectory $\phi(t; x)$ approaches the origin

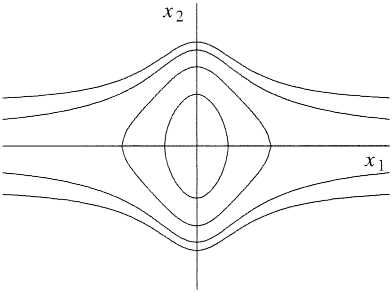

Figure 4.4: Lyapunov surfaces for $V(x) = x_1^2/(1+x_1^2) + x_2^2$.

as $t \to \infty$, no matter how large $\|x\|$ is. If an asymptotically stable equilibrium point at the origin has this property, it is said to be *globally asymptotically stable.* Recalling again the proof of Theorem 4.1, we can see that global asymptotic stability can be established if any point $x \in R^n$ can be included in the interior of a bounded set Ω_c. It is obvious that for this condition to hold, the conditions of the theorem must hold globally, that is, $D = R^n$; but, is that enough? It turns out that we need more conditions to ensure that any point in R^n can be included in a bounded set Ω_c. The problem is that for large c, the set Ω_c need not be bounded. Consider, for example, the function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

Figure 4.4 shows the surfaces $V(x) = c$ for various positive values of c. For small c, the surface $V(x) = c$ is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$. This is a consequence of the continuity and positive definiteness of $V(x)$. As c increases, a value is reached after which the surface $V(x) = c$ is open and Ω_c is unbounded. For Ω_c to be in the interior of a ball B_r, c must satisfy $c < \inf_{\|x\| \geq r} V(x)$. If

$$l = \lim_{r \to \infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then Ω_c will be bounded if $c < l$. In the preceding example,

$$l = \lim_{r \to \infty} \min_{\|x\| = r} \left[\frac{x_1^2}{1+x_1^2} + x_2^2 \right] = \lim_{|x_1| \to \infty} \frac{x_1^2}{1+x_1^2} = 1$$

Thus, Ω_c is bounded only for $c < 1$. An extra condition that ensures that Ω_c is bounded for all values of $c > 0$ is

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

A function satisfying this condition is said to be *radially unbounded.*

Theorem 4.2 *Let* $x = 0$ *be an equilibrium point for* (4.1). *Let* $V : R^n \to R$ *be a continuously differentiable function such that*

$$V(0) = 0 \ \text{ and } \ V(x) > 0, \ \ \forall \ x \neq 0 \tag{4.5}$$

$$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty \tag{4.6}$$

$$\dot{V}(x) < 0, \ \ \forall \ x \neq 0 \tag{4.7}$$

then $x = 0$ *is globally asymptotically stable.* ◇

Proof: Given any point $p \in R^n$, let $c = V(p)$. Condition (4.6) implies that for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$, which implies that Ω_c is bounded. The rest of the proof is similar to that of Theorem 4.1. □

Theorem 4.2 is known as Barbashin–Krasovskii theorem. Exercise 4.8 gives a counterexample to show that the radial unboundedness condition of the theorem is indeed needed.

Example 4.6 Consider again the system of Example 4.5, but this time, assume that the condition $yh(y) > 0$ holds for all $y \neq 0$. The Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y) \, dy$$

is positive definite for all $x \in R^2$ and radially unbounded. The derivative

$$\dot{V}(x) = -a\delta(1-k)x_2^2 - a\delta k x_1 h(x_1)$$

is negative definite for all $x \in R^2$ since $0 < k < 1$. Therefore, the origin is globally asymptotically stable. △

If the origin $x = 0$ is a globally asymptotically stable equilibrium point of a system, then it must be the unique equilibrium point of the system. For if there were another equilibrium point $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$ for all $t \geq 0$; hence, it would not approach the origin, which contradicts the claim that the origin is globally asymptotically stable. Therefore, global asymptotic stability is not studied for multiple equilibria systems like the pendulum equation.

Theorems 4.1 and 4.2 are concerned with establishing the stability or asymptotic stability of an equilibrium point. There are also instability theorems for establishing that an equilibrium point is unstable. The most powerful of these theorems is Chetaev's theorem, which will be stated as Theorem 4.3. Before we state the theorem, let us introduce some terminology that will be used in the theorem's statement. Let $V : D \to R$ be a continuously differentiable function on a domain $D \subset R^n$ that contains the origin $x = 0$. Suppose $V(0) = 0$ and there is a point x_0

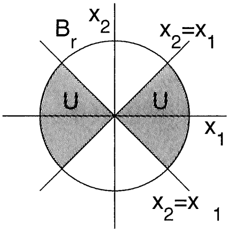

Figure 4.5: The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

arbitrarily close to the origin such that $V(x_0) > 0$. Choose $r > 0$ such that the ball $B_r = \{x \in R^n \mid \|x\| \le r\}$ is contained in D, and let

$$U = \{x \in B_r \mid V(x) > 0\} \tag{4.8}$$

The set U is a nonempty set contained in B_r. Its boundary is the surface $V(x) = 0$ and the sphere $\|x\| = r$. Since $V(0) = 0$, the origin lies on the boundary of U inside B_r. Notice that U may contain more than one component. For example, Figure 4.5 shows the set U for $V(x) = (x_1^2 - x_2^2)/2$. The set U can be always constructed provided that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to the origin.

Theorem 4.3 *Let $x = 0$ be an equilibrium point for (4.1). Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set U as in (4.8) and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.* ◇

Proof: The point x_0 is in the interior of U and $V(x_0) = a > 0$. The trajectory $x(t)$ starting at $x(0) = x_0$ must leave the set U. To see this point, notice that as long as $x(t)$ is inside U, $V(x(t)) \ge a$, since $\dot{V}(x) > 0$ in U. Let

$$\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \ge a\}$$

which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \ge a\} = \{x \in B_r \text{ and } V(x) \ge a\}$.[8] Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\, ds \ge a + \int_0^t \gamma\, ds = a + \gamma t$$

This inequality shows that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \ge a$. Hence, it must leave U through the sphere $\|x\| = r$. Because this can happen for an arbitrarily small $\|x_0\|$, the origin is unstable. □

There are other instability theorems that were proved before Chetaev's theorem, but they are corollaries of the theorem. (See Exercises 4.11 and 4.12.)

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[8] See [10, Theorem 4-20].

Example 4.7 Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_1 + g_1(x)\\ \dot{x}_2 &= -x_2 + g_2(x)\end{aligned}$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are locally Lipschitz functions that satisfy the inequalities

$$|g_1(x)| \le k\|x\|_2^2, \qquad |g_2(x)| \le k\|x\|_2^2$$

in a neighborhood D of the origin. These inequalities imply that $g_1(0) = g_2(0) = 0$. Hence, the origin is an equilibrium point. Consider the function

$$V(x) = \tfrac{1}{2}(x_1^2 - x_2^2)$$

On the line $x_2 = 0$, $V(x) > 0$ at points arbitrarily close to the origin. The set U is shown in Figure 4.5. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

The magnitude of the term $x_1 g_1(x) - x_2 g_2(x)$ satisfies the inequality

$$|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \le 2k\|x\|_2^3$$

Hence,

$$\dot{V}(x) \ge \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$$

Choosing r such that $B_r \subset D$ and $r < 1/(2k)$, it is seen that all the conditions of Theorem 4.3 are satisfied. Therefore, the origin is unstable. △

4.2 The Invariance Principle

In our study of the pendulum equation with friction (Example 4.4), we saw that the energy Lyapunov function fails to satisfy the asymptotic stability condition of Theorem 4.1 because $\dot{V}(x) = -bx_2^2$ is only negative semidefinite. Notice, however, that $\dot{V}(x)$ is negative everywhere, except on the line $x_2 = 0$, where $\dot{V}(x) = 0$. For the system to maintain the $\dot{V}(x) = 0$ condition, the trajectory of the system must be confined to the line $x_2 = 0$. Unless $x_1 = 0$, this is impossible because from the pendulum equation

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; \sin x_1(t) \equiv 0$$

Hence, on the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The foregoing argument shows, formally, that if in a domain about the origin we can find a Lyapunov function whose derivative along the trajectories of the system is negative semidefinite, and if we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$, except at the origin, then the origin is asymptotically stable. This idea follows from LaSalle's *invariance principle*, which is the subject of this section. To state and prove LaSalle's invariance theorem, we need to introduce a few definitions. Let $x(t)$ be a solution of (4.1). A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $x(t_n) \rightarrow p$ as $n \rightarrow \infty$. The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$. A set M is said to be an *invariant set* with respect to (4.1) if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \in R$$

That is, if a solution belongs to M at some time instant, then it belongs to M for all future and past time. A set M is said to be a *positively invariant set* if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \geq 0$$

We also say that $x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \quad \forall\, t > T$$

where $\text{dist}(p, M)$ denotes the distance from a point p to a set M, that is, the smallest distance from p to any point in M. More precisely,

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

These few concepts can be illustrated by examining an asymptotically stable equilibrium point and a stable limit cycle in the plane. The asymptotically stable equilibrium is the positive limit set of every solution starting sufficiently near the equilibrium point. The stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle. The solution approaches the limit cycle as $t \rightarrow \infty$. Notice, however, that the solution does not approach any specific point on the limit cycle. In other words, the statement $x(t)$ approaches M as $t \rightarrow \infty$ does not imply that $\lim_{t\rightarrow\infty} x(t)$ exists. The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for all $t \in R$. The set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ for all $x \in \Omega_c$ is a positively invariant set since, as we saw in the proof of Theorem 4.1, a solution starting in Ω_c remains in Ω_c for all $t \geq 0$.

A fundamental property of limit sets is stated in the next lemma, whose proof is given in Appendix C.3.

Lemma 4.1 *If a solution $x(t)$ of* (4.1) *is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \rightarrow \infty$.* ◇

We are now ready to state LaSalle's theorem.

Theorem 4.4 *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to* (4.1). *Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.* ◇

Proof: Let $x(t)$ be a solution of (4.1) starting in Ω. Since $\dot{V}(x) \le 0$ in Ω, $V(x(t))$ is a decreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$. Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on L^+. Since (by Lemma 4.1) L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Thus,

$$L^+ \subset M \subset E \subset \Omega$$

Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 4.1). Hence, $x(t)$ approaches M as $t \to \infty$. □

Unlike Lyapunov's theorem, Theorem 4.4 does not require the function $V(x)$ to be positive definite. Note also that the construction of the set Ω does not have to be tied in with the construction of the function $V(x)$. However, in many applications the construction of $V(x)$ will itself guarantee the existence of a set Ω. In particular, if $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is bounded and $\dot{V}(x) \le 0$ in Ω_c, then we can take $\Omega = \Omega_c$. When $V(x)$ is positive definite, Ω_c is bounded for sufficiently small $c > 0$. This is not necessarily true when $V(x)$ is not positive definite. For example, if $V(x) = (x_1 - x_2)^2$, the set Ω_c is not bounded no matter how small c is. If $V(x)$ is radially unbounded—that is, $V(x) \to \infty$ as $\|x\| \to \infty$—the set Ω_c is bounded for all values of c. This is true whether or not $V(x)$ is positive definite.

When our interest is in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin. This is done by showing that no solution can stay identically in E, other than the trivial solution $x(t) \equiv 0$. Specializing Theorem 4.4 to this case and taking $V(x)$ to be positive definite, we obtain the following two corollaries that extend Theorems 4.1 and 4.2.[9]

Corollary 4.1 *Let $x = 0$ be an equilibrium point for* (4.1). *Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \le 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.* ◇

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[9] Corollaries 4.1 and 4.2 are known as the theorems of Barbashin and Krasovskii, who proved them before the introduction of LaSalle's invariance principle.

Corollary 4.2 *Let $x = 0$ be an equilibrium point for (4.1). Let $V : R^n \to R$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \le 0$ for all $x \in R^n$. Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.* ◇

When $\dot{V}(x)$ is negative definite, $S = \{0\}$. Then, Corollaries 4.1 and 4.2 coincide with Theorems 4.1 and 4.2, respectively.

Example 4.8 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2) \end{aligned}$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are locally Lipschitz and satisfy

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \forall\ y \ne 0 \text{ and } y \in (-a, a)$$

The system has an isolated equilibrium point at the origin. Depending upon the functions $h_1(\cdot)$ and $h_2(\cdot)$, it might have other equilibrium points. The system can be viewed as a generalized pendulum with $h_2(x_2)$ as the friction term. Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} h_1(y)\ dy\ +\ \tfrac{1}{2}x_2^2$$

Let $D = \{x \in R^2 \mid -a < x_i < a\}$; $V(x)$ is positive definite in D and

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

is negative semidefinite. To find $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \ \Rightarrow\ x_2 h_2(x_2) = 0 \ \Rightarrow\ x_2 = 0, \quad \text{since } -a < x_2 < a$$

Hence, $S = \{x \in D \mid x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) \equiv 0$. Thus, the origin is asymptotically stable. △

Example 4.9 Consider again the system of Example 4.8, but this time let $a = \infty$ and assume that $h_1(\cdot)$ satisfies the additional condition:

$$\int_0^y h_1(z)\ dz\ \to\ \infty \ \text{ as } \ |y| \to \infty$$

The Lyapunov function $V(x) = \int_0^{x_1} h_1(y)\, dy + (1/2)x_2^2$ is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x) \leq 0$ in R^2, and the set

$$S = \{x \in R^2 \mid \dot{V}(x) = 0\} = \{x \in R^2 \mid x_2 = 0\}$$

contains no solutions other than the trivial solution. Hence, the origin is globally asymptotically stable. △

Not only does LaSalle's theorem relax the negative definiteness requirement of Lyapunov's theorem, but it also extends Lyapunov's theorem in three different directions. First, it gives an estimate of the region of attraction, which is not necessarily of the form $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$. The set Ω of Theorem 4.4 can be any compact positively invariant set. We will use this feature in Section 8.2 to obtain less conservative estimates of the region of attraction. Second, LaSalle's theorem can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point. This will be illustrated by an application to a simple adaptive control example from Section 1.2.6. Third, the function $V(x)$ does not have to be positive definite. The utility of this feature will be illustrated by an application to the neural network example of Section 1.2.5.

Example 4.10 Consider the first-order system

$$\dot{y} = ay + u$$

together with the adaptive control law

$$u = -ky, \qquad \dot{k} = \gamma y^2, \quad \gamma > 0$$

Taking $x_1 = y$ and $x_2 = k$, the closed-loop system is represented by

$$\begin{aligned} \dot{x}_1 &= -(x_2 - a)x_1 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}$$

The line $x_1 = 0$ is an equilibrium set. We want to show that the trajectories approach this equilibrium set as $t \to \infty$, which means that the adaptive controller regulates y to zero. Consider the Lyapunov function candidate

$$V(x) = \tfrac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

where $b > a$. The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 = -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \leq 0$$

Hence, $\dot{V}(x) \leq 0$. Since $V(x)$ is radially unbounded, the set $\Omega_c = \{x \in R^2 \mid V(x) \leq c\}$ is a compact, positively invariant set. Thus, taking $\Omega = \Omega_c$, all the conditions of Theorem 4.4 are satisfied. The set E is given by $E = \{x \in \Omega_c \mid x_1 = 0\}$. Because any point on the line $x_1 = 0$ is an equilibrium point, E is an invariant set. Therefore, in this example, $M = E$. From Theorem 4.4, we conclude that every trajectory starting in Ω_c approaches E as $t \to \infty$; that is, $x_1(t) \to 0$ as $t \to \infty$. Moreover, since $V(x)$ is radially unbounded, the conclusion is global; that is, it holds for all initial conditions $x(0)$ because for any $x(0)$, the constant c can be chosen large enough that $x(0) \in \Omega_c$. △

Note that the Lyapunov function in Example 4.10 is dependent on a constant b, which is required to satisfy $b > a$. Since in the adaptive control problem the constant a is not known, we may not know the constant b explicitly, but we know that it always exists. This highlights another feature of Lyapunov's method, which we have not seen before; namely, in some situations, we may be able to assert the existence of a Lyapunov function that satisfies the conditions of a certain theorem even though we may not explicitly know that function. In Example 4.10, we can determine the Lyapunov function explicitly if we know some bound on a. For example, if we know that $|a| \leq \alpha$, where the bound α is known, we can choose $b > \alpha$.

Example 4.11 The neural network of Section 1.2.5 is represented by

$$\dot{x}_i = \frac{1}{C_i} h_i(x_i) \left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right]$$

for $i = 1, 2, \ldots, n$, where the state variables x_i are the voltages at the amplifier outputs. They can only take values in the set

$$H = \{x \in R^n \mid -V_M < x_i < V_M\}$$

The functions $g_i : R \to (-V_M, V_M)$ are sigmoid functions,

$$h_i(x_i) = \left. \frac{dg_i}{du_i} \right|_{u_i = g_i^{-1}(x_i)} > 0, \quad \forall\ x_i \in (-V_M, V_M)$$

I_i are constant current inputs, $R_i > 0$, and $C_i > 0$. Assume that the symmetry condition $T_{ij} = T_{ji}$ is satisfied. The system may have several equilibrium points in H. We assume that all equilibrium points in H are isolated. Due to the symmetry property $T_{ij} = T_{ji}$, the vector whose ith component is

$$-\left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right]$$

is a gradient vector of a scalar function. By integration, similar to what we have done in the variable gradient method, it can be shown that this scalar function is given by

$$V(x) = -\tfrac{1}{2}\sum_i \sum_j T_{ij} x_i x_j + \sum_i \frac{1}{R_i} \int_0^{x_i} g_i^{-1}(y)\,dy - \sum_i I_i x_i$$

This function is continuously differentiable, but (typically) not positive definite. We rewrite the state equations as

$$\dot{x}_i = -\frac{1}{C_i} h_i(x_i) \frac{\partial V}{\partial x_i}$$

Let us now apply Theorem 4.4 with $V(x)$ as a candidate function. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = -\sum_{i=1}^{n} \frac{1}{C_i} h_i(x_i) \left(\frac{\partial V}{\partial x_i}\right)^2 \leq 0$$

Moreover,

$$\dot{V}(x) = 0 \Rightarrow \frac{\partial V}{\partial x_i} = 0 \Rightarrow \dot{x}_i = 0, \quad \forall\, i$$

Hence, $\dot{V}(x) = 0$ only at equilibrium points. To apply Theorem 4.4, we need to construct a set Ω. Let

$$\Omega(\varepsilon) = \{x \in R^n \mid -(V_M - \varepsilon) \leq x_i \leq (V_M - \varepsilon)\}$$

where $\varepsilon > 0$ is arbitrarily small. The set $\Omega(\varepsilon)$ is closed and bounded, and $\dot{V}(x) \leq 0$ in $\Omega(\varepsilon)$. It remains to show that $\Omega(\varepsilon)$ is a positively invariant set; that is, every trajectory starting in $\Omega(\varepsilon)$ stays for all future time in $\Omega(\varepsilon)$. To simplify the task, we assume a specific form for the sigmoid function $g_i(\cdot)$. Let

$$g_i(u_i) = \frac{2V_M}{\pi} \tan^{-1}\left(\frac{\lambda \pi u_i}{2V_M}\right), \quad \lambda > 0$$

Then,

$$\dot{x}_i = \frac{1}{C_i} h_i(x_i) \left[\sum_j T_{ij} x_j - \frac{2V_M}{\lambda \pi R_i} \tan\left(\frac{\pi x_i}{2V_M}\right) + I_i\right]$$

For $|x_i| \geq V_M - \varepsilon$,

$$\left|\tan\left(\frac{\pi x_i}{2V_M}\right)\right| \geq \tan\left(\frac{\pi(V_M - \varepsilon)}{2V_M}\right) \to \infty \text{ as } \varepsilon \to 0$$

Since x_i and I_i are bounded, ε can be chosen small enough to ensure that

$$x_i \sum_j T_{ij} x_j - \frac{2V_M x_i}{\lambda \pi R_i} \tan\left(\frac{\pi x_i}{2V_M}\right) + x_i I_i < 0, \quad \text{for } V_M - \varepsilon \leq |x_i| < V_M$$

Hence,

$$\frac{d}{dt}\left(x_i^2\right) = 2x_i\dot{x}_i < 0, \quad \text{for } V_M - \varepsilon \le |x_i| < V_M,\ \forall\ i$$

Consequently, trajectories starting in $\Omega(\varepsilon)$ will stay in $\Omega(\varepsilon)$ for all future time. In fact, trajectories starting in $H - \Omega(\varepsilon)$ will converge to $\Omega(\varepsilon)$, implying that all equilibrium points lie in the compact set $\Omega(\varepsilon)$. Hence, there can be only a finite number of isolated equilibrium points. In $\Omega(\varepsilon)$, $E = M =$ the set of equilibrium points inside $\Omega(\varepsilon)$. By Theorem 4.4, we know that every trajectory in $\Omega(\varepsilon)$ approaches M as $t \to \infty$. Since M consists of isolated equilibrium points, it can be shown (Exercise 4.20) that a trajectory approaching M must approach one of these equilibria. Hence, the system will not oscillate. △

4.3 Linear Systems and Linearization

The linear time-invariant system

$$\dot{x} = Ax \tag{4.9}$$

has an equilibrium point at the origin. The equilibrium point is isolated if and only if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix A has a nontrivial null space. Every point in the null space of A is an equilibrium point for the system (4.9). In other words, if $\det(A) = 0$, the system has an equilibrium subspace. Notice that a linear system cannot have multiple isolated equilibrium points. For, if $\bar{x}_1$ and $\bar{x}_2$ are two equilibrium points for (4.9), then by linearity, every point on the line connecting $\bar{x}_1$ and $\bar{x}_2$ is an equilibrium point for the system. Stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory[10] that the solution of (4.9) for a given initial state $x(0)$ is given by

$$x(t) = \exp(At)x(0) \tag{4.10}$$

and that for any matrix A there is a nonsingular matrix P (possibly complex) that transforms A into its Jordan form; that is,

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is a Jordan block associated with the eigenvalue λ_i of A. A Jordan block of order one takes the form $J_i = \lambda_i$, while a Jordan block of order $m > 1$ takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m\times m}$$

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[10] See, for example, [9], [35], [81], [94], or [158].

Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik} \tag{4.11}$$

where m_i is the order of the Jordan block J_i. If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i,[11] then the Jordan blocks associated with λ_i have order one if and only if $\text{rank}(A-\lambda_i I) = n-q_i$. The next theorem characterizes the stability properties of the origin.

Theorem 4.5 *The equilibrium point* $x = 0$ *of* $\dot{x} = Ax$ *is stable if and only if all eigenvalues of* A *satisfy* $\text{Re}\lambda_i \leq 0$ *and for every eigenvalue with* $\text{Re}\lambda_i = 0$ *and algebraic multiplicity* $q_i \geq 2$, $\text{rank}(A - \lambda_i I) = n - q_i$, *where* n *is the dimension of* x. *The equilibrium point* $x = 0$ *is (globally) asymptotically stable if and only if all eigenvalues of* A *satisfy* $\text{Re}\lambda_i < 0$. ◇

Proof: From (4.10), we can see that the origin is stable if and only if $\exp(At)$ is a bounded function of t for all $t \geq 0$. If one of the eigenvalues of A is in the open right-half complex plane, the corresponding exponential term $\exp(\lambda_i t)$ in (4.11) will grow unbounded as $t \to \infty$. Therefore, we must restrict the eigenvalues to be in the closed left-half complex plane. However, those eigenvalues on the imaginary axis (if any) could give rise to unbounded terms if the order of an associated Jordan block is higher than one, due to the term t^{k-1} in (4.11). Therefore, we must restrict eigenvalues on the imaginary axis to have Jordan blocks of order one, which is equivalent to the rank condition $\text{rank}(A - \lambda_i I) = n - q_i$. Thus, we conclude that the condition for stability is a necessary one. It is clear that the condition is also sufficient to ensure that $\exp(At)$ is bounded. For asymptotic stability of the origin, $\exp(At)$ must approach 0 as $t \to \infty$. From (4.11), this is the case if and only if $\text{Re}\lambda_i < 0$, $\forall\ i$. Since $x(t)$ depends linearly on the initial state $x(0)$, asymptotic stability of the origin is global. □

The proof shows, mathematically, why repeated eigenvalues on the imaginary axis must satisfy the rank condition $\text{rank}(A-\lambda_i I) = n-q_i$. The next example may shed some light on the physical meaning of this requirement.

Example 4.12 Figure 4.6 shows a series connection and a parallel connection of two identical systems. Each system is represented by the state model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

where u and y are the input and output, respectively. Let A_s and A_p be the matrices of the series and parallel connections, when modeled in the form (4.9) (no driving

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[11] Equivalently, q_i is the multiplicity of λ_i as a zero of $\det(\lambda I - A)$.

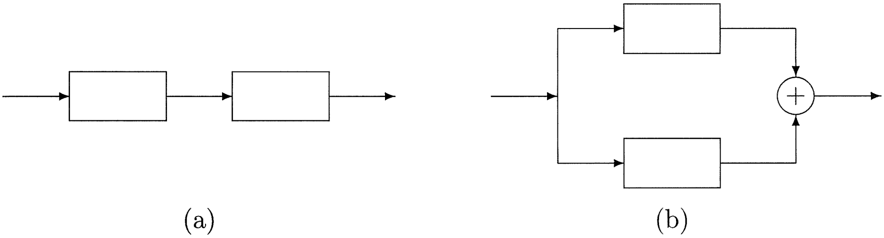

Figure 4.6: (a) Series connection; (b) parallel connection.

inputs). Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The matrices A_p and A_s have the same eigenvalues on the imaginary axis, $\pm j$ with algebraic multiplicity $q_i = 2$, where $j = \sqrt{-1}$. It can be easily checked that $\text{rank}(A_p - jI) = 2 = n - q_i$, while $\text{rank}(A_s - jI) = 3 \neq n - q_i$. Thus, by Theorem 4.5, the origin of the parallel connection is stable, while the origin of the series connection is unstable. To physically see the difference between the two cases, notice that in the parallel connection, nonzero initial conditions produce sinusoidal oscillations of frequency 1 rad/sec, which are bounded functions of time. The sum of these sinusoidal signals remains bounded. On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of frequency 1 rad/sec, which acts as a driving input for the second component. Since the second component has an undamped natural frequency of 1 rad/sec, the driving input causes "resonance" and the response grows unbounded. △

When all eigenvalues of A satisfy $\text{Re}\lambda_i < 0$, A is called a *Hurwitz matrix* or a *stability matrix.* The origin of (4.9) is asymptotically stable if and only if A is Hurwitz. Asymptotic stability of the origin can be also investigated by using Lyapunov's method. Consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x$$

where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of the linear system (4.9) is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{4.12}$$

If Q is positive definite, we can conclude by Theorem 4.1 that the origin is asymptotically stable; that is, $\text{Re}\lambda_i < 0$ for all eigenvalues of A. Here we follow the usual procedure of Lyapunov's method, where we choose $V(x)$ to be positive definite and then check the negative definiteness of $\dot{V}(x)$. In the case of linear systems, we can reverse the order of these two steps. Suppose we start by choosing Q as a real symmetric positive definite matrix, and then solve (4.12) for P. If (4.12) has a positive definite solution, then we can again conclude that the origin is asymptotically stable. Equation (4.12) is called the *Lyapunov equation.* The next theorem characterizes asymptotic stability of the origin in terms of the solution of the Lyapunov equation.

Theorem 4.6 *A matrix A is Hurwitz; that is,* $\text{Re}\lambda_i < 0$ *for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation* (4.12). *Moreover, if A is Hurwitz, then P is the unique solution of* (4.12). ◇

Proof: Sufficiency follows from Theorem 4.1 with the Lyapunov function $V(x) = x^T Px$, as we have already shown. To prove necessity, assume that all eigenvalues of A satisfy $\text{Re}\lambda_i < 0$ and consider the matrix P, defined by

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) \; dt \tag{4.13}$$

The integrand is a sum of terms of the form $t^{k-1}\exp(\lambda_i t)$, where $\text{Re}\lambda_i < 0$. Therefore, the integral exists. The matrix P is symmetric and positive definite. The fact that it is positive definite can be shown as follows: Supposing it is not so, there is a vector $x \neq 0$ such that $x^T Px = 0$. However,

$$\begin{aligned} x^T Px = 0 \quad &\Rightarrow \quad \int_0^\infty x^T \exp(A^T t) Q \exp(At) x \; dt = 0 \\ &\Rightarrow \quad \exp(At)x \equiv 0, \; \forall \, t \geq 0 \;\Rightarrow\; x = 0 \end{aligned}$$

since $\exp(At)$ is nonsingular for all t. This contradiction shows that P is positive definite. Now, substituting (4.13) in the left-hand side of (4.12) yields

$$\begin{aligned} PA + A^T P \;&=\; \int_0^\infty \exp(A^T t) Q \exp(At) A \; dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) \; dt \\ &=\; \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) \; dt \;=\; \exp(A^T t) Q \exp(At) \Big|_0^\infty \;=\; -Q \end{aligned}$$

which shows that P is indeed a solution of (4.12). To show that it is the unique solution, suppose there is another solution $\tilde{P} \neq P$. Then,

$$(P - \tilde{P})A + A^T(P - \tilde{P}) = 0$$

Premultiplying by $\exp(A^T t)$ and postmultiplying by $\exp(At)$, we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})]\exp(At) = \frac{d}{dt}\left\{\exp(A^T t)(P - \tilde{P})\exp(At)\right\}$$

Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{a constant } \forall\ t$$

In particular, since $\exp(A0) = I$, we have

$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P})\exp(At) \to 0 \text{ as } t \to \infty$$

Therefore, $\tilde{P} = P$. □

The positive definiteness requirement on Q can be relaxed. It is left to the reader (Exercise 4.22) to verify that Q can be taken as a positive semidefinite matrix of the form $Q = C^T C$, where the pair (A, C) is observable.

Equation (4.12) is a linear algebraic equation that can be solved by rearranging it in the form $Mx = y$, where x and y are defined by stacking the elements of P and Q in vectors, as will be illustrated in the next example. There are numerically efficient methods for solving such equations.[12]

Example 4.13 Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{21}$. The Lyapunov equation (4.12) can be rewritten as

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

The unique solution of this equation is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

The matrix P is positive definite since its leading principal minors (1.5 and 1.25) are positive. Hence, all eigenvalues of A are in the open left-half complex plane. △

The Lyapunov equation can be used to test whether or not a matrix A is Hurwitz, as an alternative to calculating the eigenvalues of A. One starts by choosing a positive definite matrix Q (for example, $Q = I$) and solves the Lyapunov equation (4.12) for P. If the equation has a positive definite solution, we conclude that A is Hurwitz; otherwise, it is not so. However, there is no computational advantage in solving the Lyapunov equation over calculating the eigenvalues of A.[13] Besides, the eigenvalues provide more direct information about the response of the linear system. The interest in the Lyapunov equation is not in its use as a stability test for linear systems;[14] rather, it is in the fact that it provides a procedure for finding a Lyapunov function for any linear system $\dot{x} = Ax$ when A is Hurwitz. The mere existence of a Lyapunov function will allow us to draw conclusions about the system when the right-hand side Ax is perturbed, whether such perturbation is a linear perturbation in the coefficients of A or a nonlinear perturbation. This advantage will unfold as we continue our study of Lyapunov's method.

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[12] Consult [67] on numerical methods for solving linear algebraic equations. The Lyapunov equation can also be solved by viewing it as a special case of the Sylvester equation $PA + BP + C = 0$, which is treated in [67]. Almost all commercial software programs for control systems include commands for solving the Lyapunov equation.

Let us go back to the nonlinear system

$$\dot{x} = f(x) \tag{4.14}$$

where $f : D \to R^n$ is a continuously differentiable map from a domain $D \subset R^n$ into R^n. Suppose the origin $x = 0$ is in D and is an equilibrium point for the system; that is, $f(0) = 0$. By the mean value theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)\, x$$

where z_i is a point on the line segment connecting x to the origin. The foregoing equality is valid for any point $x \in D$ such that the line segment connecting x to the origin lies entirely in D. Since $f(0) = 0$, we can write

$$f_i(x) = \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

Hence,

$$f(x) = Ax + g(x)$$

where

$$A = \frac{\partial f}{\partial x}(0) \quad \text{and} \quad g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

The function $g_i(x)$ satisfies

$$|g_i(x)| \le \left\| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \right\| \, \|x\|$$

By continuity of $[\partial f/\partial x]$, we see that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \ \text{ as } \ \|x\| \to 0$$

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[13] A typical procedure for solving the Lyapunov equation, the Bartels–Stewart algorithm [67], starts by transforming A into its real Schur form, which gives the eigenvalues of A. Hence, the computational effort for solving the Lyapunov equation is more than calculating the eigenvalues of A. Other algorithms for solving the Lyapunov equation take an amount of computations comparable to the Bartels–Stewart algorithm.

[14] It might be of interest, however, to know that one can use the Lyapunov equation to derive the classical Routh–Hurwitz criterion. (See [35, pp. 417–419].)

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system (4.14) by its linearization about the origin

$$\dot{x} = Ax, \quad \text{where} \quad A = \frac{\partial f}{\partial x}(0)$$

The next theorem spells out conditions under which we can draw conclusions about the stability of the origin as an equilibrium point for the nonlinear system by investigating its stability as an equilibrium point for the linear system. The theorem is known as *Lyapunov's indirect method.*

Theorem 4.7 *Let* $x = 0$ *be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ *is continuously differentiable and* D *is a neighborhood of the origin. Let*

$$A = \left.\frac{\partial f}{\partial x}(x)\right|_{x=0}$$

Then,

1. *The origin is asymptotically stable if* $\text{Re}\lambda_i < 0$ *for all eigenvalues of* A.
2. *The origin is unstable if* $\text{Re}\lambda_i > 0$ *for one or more of the eigenvalues of* A.

◇

Proof: To prove the first part, let A be a Hurwitz matrix. Then, by Theorem 4.6, we know that for any positive definite symmetric matrix Q, the solution P of the Lyapunov equation (4.12) is positive definite. We use $V(x) = x^T Px$ as a Lyapunov function candidate for the nonlinear system. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= x^T Pf(x) + f^T(x)Px \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T Pg(x) \\ &= -x^T Qx + 2x^T Pg(x) \end{aligned}$$

The first term on the right-hand side is negative definite, while the second term is (in general) indefinite. The function $g(x)$ satisfies

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g(x)\|_2 < \gamma\|x\|_2, \quad \forall\ \|x\|_2 < r$$

Hence,

$$\dot{V}(x) < -x^T Qx + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall\ \|x\|_2 < r$$

But

$$x^T Qx \geq \lambda_{\min}(Q) \|x\|_2^2$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Note that $\lambda_{\min}(Q)$ is real and positive since Q is symmetric and positive definite. Thus,

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma \|P\|_2] \|x\|_2^2, \quad \forall\ \|x\|_2 < r$$

Choosing $\gamma < (1/2)\lambda_{\min}(Q)/\|P\|_2$ ensures that $\dot{V}(x)$ is negative definite. By Theorem 4.1, we conclude that the origin is asymptotically stable. To prove the second part of the theorem, let us consider first the special case when A has no eigenvalues on the imaginary axis. If the eigenvalues of A cluster into a group of eigenvalues in the open right-half plane and a group of eigenvalues in the open left-half plane, then there is a nonsingular matrix T such that[15]

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 and A_2 are Hurwitz matrices. Let

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where the partition of z is compatible with the dimensions of A_1 and A_2. The change of variables $z = Tx$ transforms the system

$$\dot{x} = Ax + g(x)$$

into the form

$$\begin{aligned} \dot{z}_1 &= -A_1 z_1 + g_1(z) \\ \dot{z}_2 &= A_2 z_2 + g_2(z) \end{aligned}$$

where the functions $g_i(z)$ have the property that for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g_i(z)\|_2 < \gamma \|z\|_2, \quad \forall\ \|z\|_2 \leq r, \ i = 1, 2$$

The origin $z = 0$ is an equilibrium point for the system in the z-coordinates. Clearly, any conclusion we arrive at concerning the stability properties of $z = 0$ carries over to the equilibrium point $x = 0$ in the x-coordinates, since T is nonsingular.[16] To show that the origin is unstable, we apply Theorem 4.3. The construction of a function $V(z)$ will be done basically as in Example 4.7, except for working with vectors, instead of scalars. Let Q_1 and Q_2 be positive definite symmetric matrices of the dimensions of A_1 and A_2, respectively. Since A_1 and A_2 are Hurwitz, we know from Theorem 4.6 that the Lyapunov equations

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[15] There are several methods for finding the matrix T, one of which is to transform the matrix A into its real Jordan form [67].

[16] See Exercise 4.26 for a general discussion of stability preserving maps.

$$P_iA_i + A_i^TP_i = -Q_i, \quad i = 1, 2$$

have unique positive definite solutions P_1 and P_2. Let

$$V(z) = z_1^TP_1z_1 - z_2^TP_2z_2 = z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z$$

In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin. Let

$$U = \{z \in R^n \mid \|z\|_2 \leq r \text{ and } V(z) > 0\}$$

In U,

$$\begin{aligned} \dot{V}(z) &= -z_1^T(P_1A_1 + A_1^TP_1)z_1 + 2z_1^TP_1g_1(z) \\ &\quad - z_2^T(P_2A_2 + A_2^TP_2)z_2 - 2z_2^TP_2g_2(z) \\ &= z_1^TQ_1z_1 + z_2^TQ_2z_2 + 2z^T \begin{bmatrix} P_1g_1(z) \\ -P_2g_2(z) \end{bmatrix} \\ &\geq \lambda_{\min}(Q_1)\|z_1\|_2^2 + \lambda_{\min}(Q_2)\|z_2\|_2^2 \\ &\quad - 2\|z\|_2\sqrt{\|P_1\|_2^2\|g_1(z)\|_2^2 + \|P_2\|_2^2\|g_2(z)\|_2^2} \\ &> (\alpha - 2\sqrt{2}\beta\gamma)\|z\|_2^2 \end{aligned}$$

where

$$\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\} \quad \text{and} \quad \beta = \max\{\|P_1\|_2, \|P_2\|_2\}$$

Thus, choosing $\gamma < \alpha/(2\sqrt{2}\beta)$ ensures that $\dot{V}(z) > 0$ in U. Therefore, by Theorem 4.3, the origin is unstable. Notice that we could have applied Theorem 4.3 in the original coordinates by defining the matrices

$$P = T^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} T; \quad Q = T^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} T$$

which satisfy the equation

$$PA + A^TP = Q$$

The matrix Q is positive definite, and $V(x) = x^TPx$ is positive for points arbitrarily close to the origin $x = 0$. Let us consider now the general case when A may have eigenvalues on the imaginary axis, in addition to eigenvalues in the open right-half complex plane. We can reduce this case to the special case we have just studied by a simple trick of shifting the imaginary axis. Suppose A has m eigenvalues with

$\mathrm{Re}\lambda_i > \delta > 0$. Then, the matrix $[A - (\delta/2)I]$ has m eigenvalues in the open right-half plane, but no eigenvalues on the imaginary axis. By previous arguments, there exist matrices $P = P^T$ and $Q = Q^T > 0$ such that

$$P\left[A - \frac{\delta}{2}I\right] + \left[A - \frac{\delta}{2}I\right]^T P = Q$$

where $V(x) = x^T Px$ is positive for points arbitrarily close to the origin. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= x^T(PA + A^TP)x + 2x^TPg(x) \\ &= x^T\left[P\left(A - \frac{\delta}{2}I\right) + \left(A - \frac{\delta}{2}I\right)^T P\right]x + \delta x^TPx + 2x^TPg(x) \\ &= x^TQx + \delta V(x) + 2x^TPg(x)\end{aligned}$$

In the set

$$\{x \in R^n \mid \|x\|_2 \le r \text{ and } V(x) > 0\}$$

where r is chosen such that $\|g(x)\|_2 \le \gamma\|x\|_2$ for $\|x\|_2 < r$, $\dot{V}(x)$ satisfies

$$\dot{V}(x) \ge \lambda_{\min}(Q)\|x\|_2^2 - 2\|P\|_2\|x\|_2\|g(x)\|_2 \ge (\lambda_{\min}(Q) - 2\gamma\|P\|_2)\,\|x\|_2^2$$

which is positive for $\gamma < (1/2)\lambda_{\min}(Q)/\|P\|_2$. Applying Theorem 4.3 concludes the proof. □

Theorem 4.7 provides us with a simple procedure for determining the stability of an equilibrium point at the origin. We calculate the *Jacobian matrix*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

and test its eigenvalues. If $\mathrm{Re}\lambda_i < 0$ for all i or $\mathrm{Re}\lambda_i > 0$ for some i, we conclude that the origin is asymptotically stable or unstable, respectively. Moreover, the proof of the theorem shows that when $\mathrm{Re}\lambda_i < 0$ for all i, we can also find a Lyapunov function for the system that will work locally in some neighborhood of the origin. The Lyapunov function is the quadratic form $V(x) = x^TPx$, where P is the solution of the Lyapunov equation (4.12) for any positive definite symmetric matrix Q. Note that Theorem 4.7 does not say anything about the case when $\mathrm{Re}\lambda_i \le 0$ for all i, with $\mathrm{Re}\lambda_i = 0$ for some i. In this case, linearization fails to determine the stability of the equilibrium point.[17]

Example 4.14 Consider the scalar system

$$\dot{x} = ax^3$$

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[17] See Section 8.1 for further investigation of the critical case when linearization fails.

Linearizing the system about the origin $x = 0$ yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

There is one eigenvalue that lies on the imaginary axis. Hence, linearization fails to determine the stability of the origin. This failure is genuine in the sense that the origin could be asymptotically stable, stable, or unstable, depending on the value of the parameter a. If $a < 0$, the origin is asymptotically stable as can be seen from the Lyapunov function $V(x) = x^4$, whose derivative $\dot{V}(x) = 4ax^6 < 0$ for $x \neq 0$. If $a = 0$, the system is linear and the origin is stable according to Theorem 4.5. If $a > 0$, the origin is unstable as can be seen from Theorem 4.3 and the function $V(x) = x^4$, whose derivative $\dot{V}(x) = 4ax^6 > 0$ for $x \neq 0$. △

Example 4.15 The pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\,a\sin x_1 - bx_2\end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Let us investigate stability of both points by using linearization. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{bmatrix}$$

To determine the stability of the origin, we evaluate the Jacobian at $x = 0$:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The eigenvalues of A are

$$\lambda_{1,2} = -\,\tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{b^2 - 4a}$$

For all $a,\ b > 0$, the eigenvalues satisfy $\mathrm{Re}\lambda_i < 0$. Consequently, the equilibrium point at the origin is asymptotically stable. In the absence of friction $(b = 0)$, both eigenvalues are on the imaginary axis. Thus, we cannot determine the stability of the origin through linearization. We have seen in Example 4.3 that, in this case, the origin is a stable equilibrium point as determined by an energy Lyapunov function. To determine the stability of the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$, we evaluate the Jacobian at that point. This is equivalent to performing a change of variables $z_1 = x_1 - \pi,\ z_2 = x_2$ to shift the equilibrium point to the origin, and evaluating the Jacobian $[\partial f/\partial z]$ at $z = 0$:

$$\tilde{A} = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi,x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_{1,2} = -\tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{b^2 + 4a}$$

For all $a > 0$ and $b \geq 0$, there is one eigenvalue in the open right-half plane. Hence, the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$ is unstable. △

4.4 Comparison Functions

As we move from autonomous to nonautonomous systems, one degree of difficulty will arise from the fact that the solution of the nonautonomous system $\dot{x} = f(t, x)$, starting at $x(t_0) = x_0$, depends on both t and t_0. To cope with this new situation, we will refine the definitions of stability and asymptotic stability so that they hold uniformly in the initial time t_0. While we can refine Definition 4.1 to achieve the required uniformity, it turns out that there are more transparent definitions which use special comparison functions, known as class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Definition 4.2 *A continuous function* $\alpha : [0, a) \to [0, \infty)$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$. *It is said to belong to class* $\mathcal{K}_\infty$ *if* $a = \infty$ *and* $\alpha(r) \to \infty$ *as* $r \to \infty$.

Definition 4.3 *A continuous function* $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is said to belong to class* $\mathcal{KL}$ *if, for each fixed* s, *the mapping* $\beta(r, s)$ *belongs to class* $\mathcal{K}$ *with respect to* r *and, for each fixed* r, *the mapping* $\beta(r, s)$ *is decreasing with respect to* s *and* $\beta(r, s) \to 0$ *as* $s \to \infty$.

Example 4.16

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.

- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$.

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. Notice that $\alpha(r)$ is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function.

- $\beta(r, s) = r/(ksr + 1)$, for any positive real number k, is strictly increasing in r since

  $$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0$$

  and strictly decreasing in s since

  $$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr + 1)^2} < 0$$

Moreover, $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. Therefore, it belongs to class $\mathcal{KL}$.

- $\beta(r, s) = r^c e^{-s}$, for any positive real number c, belongs to class $\mathcal{KL}$. △

The next lemma states some useful properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which will be needed later on. The proof of the lemma is left as an exercise for the reader (Exercise 4.34).

Lemma 4.2 *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.*
- *$\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.*
- *$\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$.* ◇

Class $\mathcal{K}$ and class $\mathcal{KL}$ functions enter into Lyapunov analysis through the next two lemmas.

Lemma 4.3 *Let $V : D \rightarrow R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in R^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$. ◇

Proof: See Appendix C.4.

For a quadratic positive definite function $V(x) = x^T P x$, Lemma 4.3 follows from the inequalities

$$\lambda_{\min}(P)\|x\|_2^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|_2^2$$

Lemma 4.4 *Consider the scalar autonomous differential equation*

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where α is a locally Lipschitz class $\mathcal{K}$ function defined on $[0, a)$. For all $0 \leq y_0 < a$, this equation has a unique solution $y(t)$ defined for all $t \geq t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0)$$

where σ is a class $\mathcal{KL}$ function defined on $[0, a) \times [0, \infty)$. ◇

Proof: See Appendix C.5.

We can see that the claim of this lemma is true by examining specific examples, where a closed-form solution of the scalar equation can be found. For example, if $\dot{y} = -ky$, $k > 0$, then the solution is

$$y(t) = y_0 \exp[-k(t - t_0)] \;\Rightarrow\; \sigma(r, s) = r \exp(-ks)$$

As another example, if $\dot{y} = -ky^2$, $k > 0$, then the solution is

$$y(t) = \frac{y_0}{ky_0(t - t_0) + 1} \;\Rightarrow\; \sigma(r, s) = \frac{r}{krs + 1}$$

To see how class $\mathcal{K}$ and class $\mathcal{KL}$ functions enter into Lyapunov analysis, let us see how they could have been used in the proof of Theorem 4.1. In the proof, we wanted to choose β and δ such that $B_\delta \subset \Omega_\beta \subset B_r$. Using the fact that a positive definite function $V(x)$ satisfies

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

we can choose $\beta \le \alpha_1(r)$ and $\delta \le \alpha_2^{-1}(\beta)$. This is so because

$$V(x) \le \beta \;\Rightarrow\; \alpha_1(\|x\|) \le \alpha_1(r) \;\Leftrightarrow\; \|x\| \le r$$

and

$$\|x\| \le \delta \;\Rightarrow\; V(x) \le \alpha_2(\delta) \le \beta$$

In the same proof, we wanted to show that when $\dot{V}(x)$ is negative definite, the solution $x(t)$ tends to zero as t tends to infinity. Using Lemma 4.3 we see that there is a class $\mathcal{K}$ function α_3 such that $\dot{V}(x) \le -\alpha_3(\|x\|)$. Hence, V satisfies the differential inequality

$$\dot{V} \le -\alpha_3(\alpha_2^{-1}(V))$$

The comparison lemma (Lemma 3.4) shows that $V(x(t))$ is bounded by the solution of the scalar differential equation

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))$$

Lemma 4.2 shows that $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function and Lemma 4.4 shows that the solution of the scalar equation is $y(t) = \beta(y(0), t)$, where β is a class $\mathcal{KL}$ function. Consequently, $V(x(t))$ satisfies the inequality $V(x(t)) \le \beta(V(x(0)), t)$, which shows that $V(x(t))$ tends to zero as t tends to infinity. In fact, we can go beyond the proof of Theorem 4.1 to provide estimates of $\|x(t)\|$ that are not provided in that proof. The inequality $V(x(t)) \le V(x(0))$ implies that

$$\alpha_1(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \alpha_2(\|x(0)\|)$$

Hence, $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|))$, where $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function. Similarly, the inequality $V(x(t)) \leq \beta(V(x(0)), t)$ implies that

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(\|x(0)\|), t)$$

Therefore, $\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t))$, where ${\alpha_1}^{-1}(\beta(\alpha_2(r), t))$ is a class $\mathcal{KL}$ function.

4.5 Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x) \tag{4.15}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (4.15) at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\ t \geq 0$$

An equilibrium point at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of a nonzero solution of the system. To see the latter point, suppose $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$. The change of variables

$$x = y - \bar{y}(\tau); \quad t = \tau - a$$

transforms the system into the form

$$\dot{x} = g(\tau, y) - \dot{\bar{y}}(\tau) = g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \overset{\text{def}}{=} f(t, x)$$

Since

$$\dot{\bar{y}}(t + a) = g(t + a, \bar{y}(t + a)), \quad \forall\ t \geq 0$$

the origin $x = 0$ is an equilibrium point of the transformed system at $t = 0$. Therefore, by examining the stability behavior of the origin as an equilibrium point for the transformed system, we determine the stability behavior of the solution $\bar{y}(\tau)$ of the original system. Notice that if $\bar{y}(\tau)$ is not constant, the transformed system will be nonautonomous even when the original system is autonomous, that is, even when $g(\tau, y) = g(y)$. This is why studying the stability behavior of solutions in the sense of Lyapunov can be done only in the context of studying the stability behavior of the equilibria of nonautonomous systems.

The notions of stability and asymptotic stability of equilibrium points of nonautonomous systems are basically the same as those introduced in Definition 4.1 for autonomous systems. The new element here is that, while the solution of an autonomous system depends only on $(t - t_0)$, the solution of a nonautonomous system may depend on both t and t_0. Therefore, the stability behavior of the equilibrium point will, in general, be dependent on t_0. The origin $x = 0$ is a stable equilibrium point for (4.15) if, for each $\varepsilon > 0$, and any $t_0 \geq 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\ t \geq t_0$$

The constant δ is, in general, dependent on the initial time t_0. The existence of δ for every t_0 does not necessarily guarantee that there is one constant δ, dependent only on ε, that would work for all t_0, as illustrated by the next example.

Example 4.17 The linear first-order system

$$\dot{x} = (6t \sin t - 2t)x$$

has the solution

$$\begin{aligned} x(t) &= x(t_0) \exp\left[\int_{t_0}^{t} (6\tau \sin\tau - 2\tau)\ d\tau\right] \\ &= x(t_0) \exp\left[6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2\right] \end{aligned}$$

For any t_0, the term $-t^2$ will eventually dominate, which shows that the exponential term is bounded for all $t \geq t_0$ by a constant $c(t_0)$ dependent on t_0. Hence,

$$|x(t)| < |x(t_0)| c(t_0), \quad \forall\ t \geq t_0$$

For any $\varepsilon > 0$, the choice $\delta = \varepsilon / c(t_0)$ shows that the origin is stable. Now, suppose t_0 takes on the successive values $t_0 = 2n\pi$, for $n = 0, 1, 2, \ldots$, and $x(t)$ is evaluated π seconds later in each case. Then,

$$x(t_0 + \pi) = x(t_0) \exp\left[(4n + 1)(6 - \pi)\pi\right]$$

which implies that, for $x(t_0) \neq 0$,

$$\frac{x(t_0 + \pi)}{x(t_0)} \to \infty \quad \text{as} \quad n \to \infty$$

Thus, given $\varepsilon > 0$, there is no δ independent of t_0 that would satisfy the stability requirement uniformly in t_0. △

Nonuniformity with respect to t_0 could also appear in studying asymptotic stability of the origin, as the next example shows.

Example 4.18 The linear first-order system

$$\dot{x} = -\frac{x}{1+t}$$

has the solution

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau}\, d\tau\right) = x(t_0)\frac{1+t_0}{1+t}$$

Since $|x(t)| \leq |x(t_0)|$, $\forall\ t \geq t_0$, the origin is clearly stable. Actually, given any $\varepsilon > 0$, we can choose δ independent of t_0. It is also clear that

$$x(t) \to 0 \ \text{ as } \ t \to \infty$$

Consequently, according to Definition 4.1, the origin is asymptotically stable. Notice, however, that the convergence of $x(t)$ to the origin is not uniform with respect to the initial time t_0. Recall that convergence of $x(t)$ to the origin is equivalent to saying that, given any $\varepsilon > 0$, there is $T = T(\varepsilon, t_0) > 0$ such that $|x(t)| < \varepsilon$ for all $t \geq t_0 + T$. Although this is true for every t_0, the constant T cannot be chosen independent of t_0. △

As a consequence, we need to refine Definition 4.1 to emphasize the dependence of the stability behavior of the origin on the initial time t_0. We are interested in a refinement that defines stability and asymptotic stability of the origin as uniform properties with respect to the initial time.[18]

Definition 4.4 *The equilibrium point $x = 0$ of* (4.15) *is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\ t \geq t_0 \geq 0 \tag{4.16}$$

- *uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that* (4.16) *is satisfied.*

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.*

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[18] See [72] or [95] for other refinements of Definition 4.1. It is worthwhile to note that, for autonomous systems, the definition of global uniform asymptotic stability given here is equivalent to global asymptotic stability as defined in Section 4.1. In particular, $\delta(\varepsilon)$ can be always chosen such that $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$. This is shown in the proof of Theorem 4.17. Lemma C.2 shows that, when the origin of an autonomous system is globally asymptotically stable, its solution $x(t)$ satisfies $\|x(t)\| \leq \beta(\|x(t_0)\|, 0)$ for all $x(t_0)$, where $\beta(r, 0)$ is a class $\mathcal{K}_\infty$ function. The function $\delta(\varepsilon)$ can be taken as $\delta(\varepsilon) = \beta^{-1}(\varepsilon, 0)$.

- *uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall\ t \geq t_0 + T(\eta),\ \forall\ \|x(t_0)\| < c \tag{4.17}$$

- *globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall\ t \geq t_0 + T(\eta, c),\ \forall\ \|x(t_0)\| < c \tag{4.18}$$

The next lemma gives equivalent, more transparent, definitions of uniform stability and uniform asymptotic stability by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Lemma 4.5 *The equilibrium point $x = 0$ of* (4.15) *is*

- *uniformly stable if and only if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall\ t \geq t_0 \geq 0,\ \forall\ \|x(t_0)\| < c \tag{4.19}$$

- *uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \geq t_0 \geq 0,\ \forall\ \|x(t_0)\| < c \tag{4.20}$$

- *globally uniformly asymptotically stable if and only if inequality* (4.20) *is satisfied for any initial state $x(t_0)$.* ◇

Proof: See Appendix C.6.

As a consequence of Lemma 4.5, we see that in the case of autonomous systems stability and asymptotic stability per Definition 4.1 imply the existence of class $\mathcal{K}$ and class $\mathcal{KL}$ functions that satisfy inequalities (4.19) and (4.20). This is the case because, for autonomous systems, stability and asymptotic stability of the origin are uniform with respect to the initial time t_0.

A special case of uniform asymptotic stability arises when the class $\mathcal{KL}$ function β in (4.20) takes the form $\beta(r, s) = kre^{-\lambda s}$. This case is very important and will be designated as a distinct stability property of equilibrium points.

Definition 4.5 *The equilibrium point $x = 0$ of* (4.15) *is exponentially stable if there exist positive constants c, k, and λ such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall\ \|x(t_0)\| < c \tag{4.21}$$

and globally exponentially stable if (4.21) is satisfied for any initial state $x(t_0)$.

Lyapunov theory for autonomous systems can be extended to nonautonomous systems. For each of Theorems 4.1 through 4.4, one can state various extensions to nonautonomous systems. We will not document all these extensions here.[19] Instead, we concentrate on uniform stability and uniform asymptotic stability. These are the cases we encounter in most nonautonomous applications of Lyapunov's method.

Theorem 4.8 *Let* $x = 0$ *be an equilibrium point for* (4.15) *and* $D \subset R^n$ *be a domain containing* $x = 0$. *Let* $V : [0, \infty) \times D \to R$ *be a continuously differentiable function such that*

$$W_1(x) \le V(t,x) \le W_2(x) \tag{4.22}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0 \tag{4.23}$$

$\forall\ t \ge 0$ *and* $\forall\ x \in D$, *where* $W_1(x)$ *and* $W_2(x)$ *are continuous positive definite functions on* D. *Then,* $x = 0$ *is uniformly stable.* ◇

Proof: The derivative of V along the trajectories of (4.15) is given by

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \ \le\ 0$$

Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then, $\{x \in B_r \mid W_1(x) \le c\}$ is in the interior of B_r. Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r \mid V(t,x) \le c\}$$

The set $\Omega_{t,c}$ contains $\{x \in B_r \mid W_2(x) \le c\}$ since

$$W_2(x) \le c \Rightarrow V(t,x) \le c$$

On the other hand, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \le c\}$ since

$$V(t,x) \le c \Rightarrow W_1(x) \le c$$

Thus,

$$\{x \in B_r \mid W_2(x) \le c\} \subset \Omega_{t,c} \subset \{x \in B_r \mid W_1(x) \le c\} \subset B_r \subset D$$

for all $t \ge 0$. These five nested sets are sketched in Figure 4.7. The setup of Figure 4.7 is similar to that of Figure 4.1, except that the surface $V(t,x) = c$ is now dependent on t, and that is why it is surrounded by the time-independent surfaces $W_1(x) = c$ and $W_2(x) = c$.

Since $\dot{V}(t,x) \le 0$ on D, for any $t_0 \ge 0$ and any $x_0 \in \Omega_{t_0,c}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,c}$ for all $t \ge t_0$. Therefore, any solution starting in $\{x \in$

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[19] Lyapunov theory for nonautonomous systems is well documented in the literature. Good references on the subject include [72] and [154], while good introductions can be found in [201] and [135].

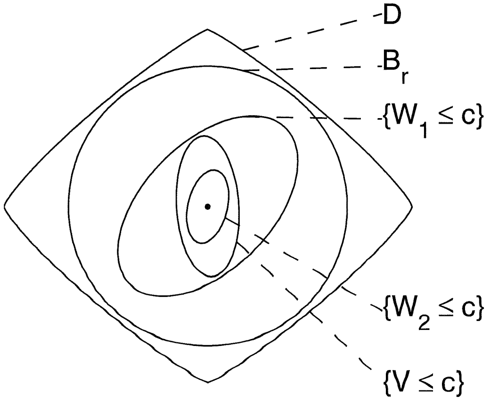

Figure 4.7: Geometric representation of sets in the proof of Theorem 4.8.

$B_r \mid W_2(x) \le c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \le c\}$, for all future time. Hence, the solution is bounded and defined for all $t \ge t_0$. Moreover, since $\dot{V} \le 0$,

$$V(t, x(t)) \le V(t_0, x(t_0)), \quad \forall\ t \ge t_0$$

By Lemma 4.3, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \le W_1(x) \le V(t, x) \le W_2(x) \le \alpha_2(\|x\|)$$

Combining the preceding two inequalities, we see that

$$\|x(t)\| \le \alpha_1^{-1}(V(t, x(t))) \le \alpha_1^{-1}(V(t_0, x(t_0))) \le \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

Since $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function (by Lemma 4.2), the inequality $\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$ shows that the origin is uniformly stable. □

Theorem 4.9 *Suppose the assumptions of Theorem 4.8 are satisfied with inequality (4.23) strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x) \tag{4.24}$$

$\forall\ t \ge 0$ *and* $\forall\ x \in D$, *where* $W_3(x)$ *is a continuous positive definite function on* D. *Then,* $x = 0$ *is uniformly asymptotically stable. Moreover, if* r *and* c *are chosen such that* $B_r = \{\|x\| \le r\} \subset D$ *and* $c < \min_{\|x\|=r} W_1(x)$, *then every trajectory starting in* $\{x \in B_r \mid W_2(x) \le c\}$ *satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ *function* β. *Finally, if* $D = R^n$ *and* $W_1(x)$ *is radially unbounded, then* $x = 0$ *is globally uniformly asymptotically stable.* ◇

Proof: Continuing with the proof of Theorem 4.8, we know that trajectories starting in $\{x \in B_r \mid W_2(x) \leq c\}$ stay in $\{x \in B_r \mid W_1(x) \leq c\}$ for all $t \geq t_0$. By Lemma 4.3, there exists a class $\mathcal{K}$ function α_3, defined on $[0, r]$, such that

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|)$$

Using the inequality

$$V \leq \alpha_2(\|x\|) \ \Leftrightarrow \ \alpha_2^{-1}(V) \leq \|x\| \ \Leftrightarrow \ \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$

we see that V satisfies the differential inequality

$$\dot{V} \leq \ -\alpha_3(\alpha_2^{-1}(V)) \overset{\text{def}}{=} -\alpha(V)$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0, r]$. (See Lemma 4.2.) Assume, without loss of generality,[20] that α is locally Lipschitz. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\dot{y} = -\alpha(y), \quad y(t_0) = V(t_0, x(t_0)) \geq 0$$

By (the comparison) Lemma 3.4,

$$V(t, x(t)) \leq y(t), \quad \forall\ t \geq t_0$$

By Lemma 4.4, there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, r] \times [0, \infty)$ such that

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall\ V(t_0, x(t_0)) \in [0, c]$$

Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies the inequality

$$\begin{aligned} \|x(t)\| \ &\leq \ \alpha_1^{-1}(V(t, x(t))) \ \leq \ \alpha_1^{-1}\left(\sigma(V(t_0, x(t_0)), t - t_0)\right) \\ &\leq \ \alpha_1^{-1}\left(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)\right) \ \overset{\text{def}}{=} \ \beta(\|x(t_0)\|, t - t_0) \end{aligned}$$

Lemma 4.2 shows that β is a class $\mathcal{KL}$ function. Thus, inequality (4.20) is satisfied, which implies that $x = 0$ is uniformly asymptotically stable. If $D = R^n$, the functions α_1, α_2, and α_3 are defined on $[0, \infty)$. Hence, α, and consequently β, are independent of c. As $W_1(x)$ is radially unbounded, c can be chosen arbitrarily large to include any initial state in $\{W_2(x) \leq c\}$. Thus, (4.20) holds for any initial state, showing that the origin is globally uniformly asymptotically stable. □

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[20] If α is not locally Lipschitz, we can choose a locally Lipschitz class $\mathcal{K}$ function β such that $\alpha(r) \geq \beta(r)$ over the domain of interest. Then, $\dot{V} \leq -\beta(V)$, and we can continue the proof with β instead of α. For example, suppose $\alpha(r) = \sqrt{r}$. The function $\sqrt{r}$ is a class $\mathcal{K}$ function, but not locally Lipschitz at $r = 0$. Define β as $\beta(r) = r$, for $r < 1$ and $\beta(r) = \sqrt{r}$, for $r \geq 1$. The function β is class $\mathcal{K}$ and locally Lipschitz. Moreover, $\alpha(r) \geq \beta(r)$ for all $r \geq 0$.

A function $V(t,x)$ is said to be *positive semidefinite* if $V(t,x) \geq 0$. It is said to be *positive definite* if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$, *radially unbounded* if $W_1(x)$ is so, and *decrescent* if $V(t,x) \leq W_2(x)$. A function $V(t,x)$ is said to be *negative definite* (*semidefinite*) if $-V(t,x)$ is *positive definite* (*semidefinite*). Therefore, Theorems 4.8 and 4.9 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decrescent function* $V(t,x)$, *whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded* $V(t,x)$.

Theorem 4.10 *Let* $x = 0$ *be an equilibrium point for* (4.15) *and* $D \subset R^n$ *be a domain containing* $x = 0$. *Let* $V : [0,\infty) \times D \rightarrow R$ *be a continuously differentiable function such that*

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a \tag{4.25}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a \tag{4.26}$$

$\forall\ t \geq 0$ *and* $\forall\ x \in D$, *where* k_1, k_2, k_3, *and* a *are positive constants. Then,* $x = 0$ *is exponentially stable. If the assumptions hold globally, then* $x = 0$ *is globally exponentially stable.* ◇

Proof: With the help of Figure 4.7, it can be seen that trajectories starting in $\{k_2\|x\|^a \leq c\}$, for sufficiently small c, remain bounded for all $t \geq t_0$. Inequalities (4.25) and (4.26) show that V satisfies the differential inequality

$$\dot{V} \leq -\frac{k_3}{k_2} V$$

By (the comparison) Lemma 3.4,

$$V(t,x(t)) \leq V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}$$

Hence,

$$\begin{aligned} \|x(t)\| &\leq \left[\frac{V(t,x(t))}{k_1}\right]^{1/a} \leq \left[\frac{V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} \\ &\leq \left[\frac{k_2\|x(t_0)\|^a e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} = \left(\frac{k_2}{k_1}\right)^{1/a} \|x(t_0)\| e^{-(k_3/k_2 a)(t-t_0)} \end{aligned}$$

Thus, the origin is exponentially stable. If all the assumptions hold globally, c can be chosen arbitrarily large and the foregoing inequality holds for all $x(t_0) \in R^n$. □

Example 4.19 Consider the scalar system

$$\dot{x} = -[1 + g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$. Using the Lyapunov function candidate $V(x) = x^2/2$, we obtain

$$\dot{V}(t,x) = -[1+g(t)]x^4 \leq -x^4, \quad \forall\, x \in R,\ \forall\, t \geq 0$$

The assumptions of Theorem 4.9 are satisfied globally with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$. Hence, the origin is globally uniformly asymptotically stable. △

Example 4.20 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \quad \forall\, t \geq 0$$

Taking $V(t,x) = x_1^2 + [1+g(t)]x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

Hence, $V(t,x)$ is positive definite, decrescent, and radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2+2g(t)-\dot{g}(t)]x_2^2$$

Using the inequality

$$2+2g(t)-\dot{g}(t) \geq 2+2g(t)-g(t) \geq 2$$

we obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{\text{def}}{=} -x^TQx$$

where Q is positive definite; therefore, $\dot{V}(t,x)$ is negative definite. Thus, all the assumptions of Theorem 4.9 are satisfied globally with positive definite quadratic functions W_1, W_2, and W_3. Recalling that a positive definite quadratic function x^TPx satisfies

$$\lambda_{min}(P)x^Tx \leq x^TPx \leq \lambda_{max}(P)x^Tx$$

we see that the conditions of Theorem 4.10 are satisfied globally with $a = 2$. Hence, the origin is globally exponentially stable. △

Example 4.21 The linear time-varying system

$$\dot{x} = A(t)x \tag{4.27}$$

has an equilibrium point at $x = 0$. Let $A(t)$ be continuous for all $t \geq 0$. Suppose there is a continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$; that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall\, t \geq 0$$

which satisfies the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \tag{4.28}$$

where $Q(t)$ is continuous, symmetric, and positive definite; that is,

$$Q(t) \geq c_3 I > 0, \quad \forall\, t \geq 0$$

The Lyapunov function candidate

$$V(t, x) = x^T P(t) x$$

satisfies

$$c_1 \|x\|_2^2 \leq V(t, x) \leq c_2 \|x\|_2^2$$

and its derivative along the trajectories of the system (4.27) is given by

$$\begin{aligned} \dot{V}(t, x) &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\ &= x^T [\dot{P}(t) + P(t)A(t) + A^T(t)P(t)] x \;=\; -x^T Q(t) x \;\leq\; -c_3 \|x\|_2^2 \end{aligned}$$

Thus, all the assumptions of Theorem 4.10 are satisfied globally with $a = 2$, and we conclude that the origin is globally exponentially stable. △

4.6 Linear Time-Varying Systems and Linearization

The stability behavior of the origin as an equilibrium point for the linear time-varying system

$$\dot{x}(t) = A(t)x \tag{4.29}$$

can be completely characterized in terms of the state transition matrix of the system. From linear system theory,[21] we know that the solution of (4.29) is given by

$$x(t) = \Phi(t, t_0) x(t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix. The next theorem characterizes uniform asymptotic stability in terms of $\Phi(t, t_0)$.

Theorem 4.11 *The equilibrium point $x = 0$ of (4.29) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality*

$$\|\Phi(t, t_0)\| \leq k e^{-\lambda(t - t_0)}, \quad \forall\, t \geq t_0 \geq 0 \tag{4.30}$$

for some positive constants k and λ. ◇

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[21] See, for example, [9], [35], [94], or [158].

Proof: Due to the linear dependence of $x(t)$ on $x(t_0)$, if the origin is uniformly asymptotically stable, it is globally so. Sufficiency of (4.30) is obvious since

$$\|x(t)\| \leq \|\Phi(t,t_0)\| \, \|x(t_0)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}$$

To prove necessity, suppose the origin is uniformly asymptotically stable. Then, there is a class $\mathcal{KL}$ function β such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall\ t \geq t_0, \ \forall\ x(t_0) \in R^n$$

From the definition of an induced matrix norm (Appendix A), we have

$$\|\Phi(t,t_0)\| = \max_{\|x\|=1} \|\Phi(t,t_0)x\| \leq \max_{\|x\|=1} \beta(\|x\|, t-t_0) = \beta(1, t-t_0)$$

Since

$$\beta(1,s) \to 0 \ \text{ as } \ s \to \infty$$

there exists $T > 0$ such that $\beta(1,T) \leq 1/e$. For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each. Using the transition property of $\Phi(t,t_0)$, we can write

$$\Phi(t,t_0) = \Phi(t, t_0 + (N-1)T)\Phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \Phi(t_0 + T, t_0)$$

Hence,

$$\begin{aligned}
\|\Phi(t,t_0)\| &\leq \|\Phi(t, t_0 + (N-1)T)\| \prod_{k=1}^{k=N-1} \|\Phi(t_0 + kT, t_0 + (k-1)T)\| \\
&\leq \beta(1,0) \prod_{k=1}^{k=N-1} \frac{1}{e} = e\beta(1,0)e^{-N} \\
&\leq e\beta(1,0)e^{-(t-t_0)/T} = ke^{-\lambda(t-t_0)}
\end{aligned}$$

where $k = e\beta(1,0)$ and $\lambda = 1/T$. □

Theorem 4.11 shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability. Although inequality (4.30) characterizes uniform asymptotic stability of the origin without the need to search for a Lyapunov function, it is not as useful as the eigenvalue criterion we have for linear time-invariant systems, because knowledge of the state transition matrix $\Phi(t,t_0)$ requires solving the state equation (4.29). Note that, for linear time-varying systems, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix A[22] as the following example shows.

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[22] There are special cases where uniform asymptotic stability of the origin as an equilibrium point for (4.29) is equivalent to an eigenvalue condition. One case is periodic systems. (See Exercise 4.40 and Example 10.8.) Another case is slowly-varying systems. (See Example 9.9.)

Example 4.22 Consider a second-order linear system with

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are given by $-0.25 \pm 0.25\sqrt{7}j$. Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. It can be verified that

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

which shows that there are initial states $x(0)$, arbitrarily close to the origin, for which the solution is unbounded and escapes to infinity. △

Although Theorem 4.11 may not be very helpful as a stability test, we will see that it guarantees the existence of a Lyapunov function for the linear system (4.29). We saw in Example 4.21 that if we can find a positive definite, bounded matrix $P(t)$ that satisfies the differential equation (4.28) for some positive definite $Q(t)$, then $V(t,x) = x^T P(t)x$ is a Lyapunov function for the system. If the matrix $Q(t)$ is chosen to be bounded in addition to being positive definite, that is,

$$0 < c_3 I \le Q(t) \le c_4 I, \quad \forall\, t \ge 0$$

and if $A(t)$ is continuous and bounded, then it can be shown that when the origin is exponentially stable, there is a solution of (4.28) that possesses the desired properties.

Theorem 4.12 *Let* $x = 0$ *be the exponentially stable equilibrium point of* (4.29). *Suppose* $A(t)$ *is continuous and bounded. Let* $Q(t)$ *be a continuous, bounded, positive definite, symmetric matrix. Then, there is a continuously differentiable, bounded, positive definite, symmetric matrix* $P(t)$ *that satisfies* (4.28). *Hence,* $V(t,x) = x^T P(t)x$ *is a Lyapunov function for the system that satisfies the conditions of Theorem* 4.10. ◇

Proof: Let

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\; d\tau$$

and $\phi(\tau;t,x)$ be the solution of (4.29) that starts at (t,x). Due to linearity, $\phi(\tau;t,x) = \Phi(\tau,t)x$. In view of the definition of $P(t)$, we have

$$x^T P(t)x = \int_t^\infty \phi^T(\tau;t,x)Q(\tau)\phi(\tau;t,x)\; d\tau$$

The use of (4.30) yields

$$\begin{aligned} x^T P(t) x &\leq \int_t^\infty c_4 \|\Phi(\tau,t)\|_2^2 \; \|x\|_2^2 d\tau \\ &\leq \int_t^\infty k^2 e^{-2\lambda(\tau-t)} \; d\tau \; c_4 \|x\|_2^2 \;=\; \frac{k^2 c_4}{2\lambda} \|x\|_2^2 \;\stackrel{\text{def}}{=}\; c_2 \|x\|_2^2 \end{aligned}$$

On the other hand, since

$$\|A(t)\|_2 \leq L, \quad \forall \; t \geq 0$$

the solution $\phi(\tau; t, x)$ satisfies the lower bound[23]

$$\|\phi(\tau; t, x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$\begin{aligned} x^T P(t) x &\geq \int_t^\infty c_3 \|\phi(\tau; t, x)\|_2^2 \; d\tau \\ &\geq \int_t^\infty e^{-2L(\tau-t)} \; d\tau \; c_3 \|x\|_2^2 \;=\; \frac{c_3}{2L} \|x\|_2^2 \;\stackrel{\text{def}}{=}\; c_1 \|x\|_2^2 \end{aligned}$$

Thus,

$$c_1 \|x\|_2^2 \leq x^T P(t) x \leq c_2 \|x\|_2^2$$

which shows that $P(t)$ is positive definite and bounded. The definition of $P(t)$ shows that it is symmetric and continuously differentiable. The fact that $P(t)$ satisfies (4.28) can be shown by differentiating $P(t)$ and using the property

$$\frac{\partial}{\partial t} \Phi(\tau, t) = -\Phi(\tau, t) A(t)$$

In particular,

$$\begin{aligned} \dot{P}(t) &= \int_t^\infty \Phi^T(\tau,t) Q(\tau) \frac{\partial}{\partial t} \Phi(\tau,t) \; d\tau \\ &\quad + \int_t^\infty \left[\frac{\partial}{\partial t} \Phi^T(\tau,t) \right] Q(\tau) \Phi(\tau,t) \; d\tau \; - \; Q(t) \\ &= -\int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) \; d\tau \; A(t) \\ &\quad -A^T(t) \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) \; d\tau \; - \; Q(t) \\ &= -P(t) A(t) - A^T(t) P(t) - Q(t) \end{aligned}$$

The fact that $V(t, x) = x^T P(t) x$ is a Lyapunov function is shown in Example 4.21.

□

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[23] See Exercise 3.17.

When the linear system (4.29) is time invariant, that is, when A is constant, the Lyapunov function $V(t,x)$ of Theorem 4.12 can be chosen to be independent of t. Recall that, for linear time-invariant systems,

$$\Phi(\tau,t) = \exp[(\tau - t)A]$$

which satisfies (4.30) when A is Hurwitz. Choosing Q to be a positive definite, symmetric (constant) matrix, the matrix $P(t)$ is given by

$$P = \int_t^\infty \exp[(\tau - t)A^T] Q \exp[(\tau - t)A] \; d\tau = \int_0^\infty \exp[A^T s] Q \exp[As] \; ds$$

which is independent of t. Comparing this expression for P with (4.13) shows that P is the unique solution of the Lyapunov equation (4.12). Thus, the Lyapunov function of Theorem 4.12 reduces to the one we used in Section 4.3.

The existence of Lyapunov functions for linear systems per Theorem 4.12 will now be used to prove a linearization result that extends Theorem 4.7 to the nonautonomous case. Consider the nonlinear nonautonomous system

$$\dot{x} = f(t,x) \tag{4.31}$$

where $f : [0,\infty) \times D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\|_2 < r\}$. Suppose the origin $x = 0$ is an equilibrium point for the system at $t = 0$; that is, $f(t,0) = 0$ for all $t \geq 0$. Furthermore, suppose the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t; thus,

$$\left\| \frac{\partial f_i}{\partial x}(t,x_1) - \frac{\partial f_i}{\partial x}(t,x_2) \right\|_2 \leq L_1 \|x_1 - x_2\|_2, \quad \forall\; x_1, x_2 \in D, \; \forall\; t \geq 0$$

for all $1 \leq i \leq n$. By the mean value theorem,

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t,z_i)\; x$$

where z_i is a point on the line segment connecting x to the origin. Since $f(t,0) = 0$, we can write $f_i(t,x)$ as

$$f_i(t,x) = \frac{\partial f_i}{\partial x}(t,z_i)\; x = \frac{\partial f_i}{\partial x}(t,0)\; x + \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0) \right] x$$

Hence,

$$f(t,x) = A(t)x + g(t,x)$$

where

$$A(t) = \frac{\partial f}{\partial x}(t,0) \;\; \text{and} \;\; g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0) \right] x$$

The function $g(t,x)$ satisfies

$$\|g(t,x)\|_2 \le \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0) \right\|_2^2 \right)^{1/2} \|x\|_2 \le L\|x\|_2^2$$

where $L = \sqrt{n}L_1$. Therefore, in a small neighborhood of the origin, we may approximate the nonlinear system (4.31) by its linearization about the origin. The next theorem states Lyapunov's indirect method for showing exponential stability of the origin in the nonautonomous case.

Theorem 4.13 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x)$$

where $f : [0,\infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t,x) \right|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: Since the linear system has an exponentially stable equilibrium point at the origin and $A(t)$ is continuous and bounded, Theorem 4.12 ensures the existence of a continuously differentiable, bounded, positive definite symmetric matrix $P(t)$ that satisfies (4.28), where $Q(t)$ is continuous, positive definite, and symmetric. We use $V(t,x) = x^T P(t) x$ as a Lyapunov function candidate for the nonlinear system. The derivative of $V(t,x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(t,x) &= x^T P(t) f(t,x) + f^T(t,x) P(t) x + x^T \dot{P}(t) x \\ &= x^T [P(t)A(t) + A^T(t)P(t) + \dot{P}(t)]x + 2x^T P(t) g(t,x) \\ &= -x^T Q(t) x + 2x^T P(t) g(t,x) \\ &\le -c_3\|x\|_2^2 + 2c_2 L \|x\|_2^3 \\ &\le -(c_3 - 2c_2 L\rho)\|x\|_2^2, \ \ \forall \ \|x\|_2 < \rho \end{aligned}$$

Choosing $\rho < \min\{r, c_3/(2c_2L)\}$ ensures that $\dot{V}(t,x)$ is negative definite in $\|x\|_2 < \rho$. Therefore, all the conditions of Theorem 4.10 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is exponentially stable. □

4.7 Converse Theorems

Theorems 4.9 and 4.10 establish uniform asymptotic stability or exponential stability of the origin by requiring the existence of a Lyapunov function $V(t,x)$ that satisfies certain conditions. Requiring the existence of an auxiliary function $V(t,x)$ that satisfies certain conditions is typical in many theorems of Lyapunov's method. The conditions of these theorems cannot be checked directly on the data of the problem. Instead, one has to search for the auxiliary function. Faced with this searching problem, two questions come to mind. First, is there a function that satisfies the conditions of the theorem? Second, how can we search for such a function? In many cases, Lyapunov theory provides an affirmative answer to the first question. The answer takes the form of a converse Lyapunov theorem, which is the inverse of one of Lyapunov's theorems. For example, a converse theorem for uniform asymptotic stability would confirm that if the origin is uniformly asymptotically stable, then there is a Lyapunov function that satisfies the conditions of Theorem 4.9. Most of these converse theorems are proven by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, this construction almost always assumes the knowledge of the solution of the differential equation. Therefore, the theorems do not help in the practical search for an auxiliary function. The mere knowledge that a function exists is, however, better than nothing. At least, we know that our search is not hopeless. The theorems are also useful in using Lyapunov theory to draw conceptual conclusions about the behavior of dynamical systems. Theorem 4.15 is an example of such use. Other examples will appear in the following chapters. In this section, we give three converse Lyapunov theorems.[24] The first one is a converse Lyapunov theorem when the origin is exponentially stable and, the second, when it is uniformly asymptotically stable. The third theorem applies to autonomous systems and defines the converse Lyapunov function for the whole region of attraction of an asymptotically stable equilibrium point.

The idea of constructing a converse Lyapunov function is not new to us. We have done it for linear systems in the proof of Theorem 4.12. A careful reading of that proof shows that linearity of the system does not play a crucial role in the proof, except for showing that $V(t,x)$ is quadratic in x. This observation leads to the first of our three converse theorems, whose proof is a simple extension of the proof of Theorem 4.12.

Theorem 4.14 *Let $x=0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x)$$

where $f : [0,\infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let k, λ, and r_0

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[24] See [72] or [107] for a comprehensive treatment of converse Lyapunov theorems and [118] and [193] for more recent results.

be positive constants with $r_0 < r/k$. *Let* $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. *Assume that the trajectories of the system satisfy*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall\ x(t_0) \in D_0,\ \forall\ t \geq t_0 \geq 0$$

Then, there is a function $V : [0, \infty) \times D_0 \to R$ *that satisfies the inequalities*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

for some positive constants c_1, c_2, c_3, *and* c_4. *Moreover, if* $r = \infty$ *and the origin is globally exponentially stable, then* $V(t,x)$ *is defined and satisfies the aforementioned inequalities on* R^n. *Furthermore, if the system is autonomous,* V *can be chosen independent of* t. ◇

Proof: Due to the equivalence of norms, it is sufficient to prove the theorem for the 2-norm. Let $\phi(\tau; t, x)$ denote the solution of the system that starts at (t, x); that is, $\phi(t; t, x) = x$. For all $x \in D_0$, $\phi(\tau; t, x) \in D$ for all $\tau \geq t$. Let

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau; t, x)\phi(\tau; t, x)\ d\tau$$

where δ is a positive constant to be chosen. Due to the exponentially decaying bound on the trajectories, we have

$$\begin{aligned} V(t,x) &= \int_t^{t+\delta} \|\phi(\tau; t, x)\|_2^2\ d\tau \\ &\leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)}\ d\tau\ \|x\|_2^2 = \frac{k^2}{2\lambda}(1 - e^{-2\lambda\delta})\|x\|_2^2 \end{aligned}$$

On the other hand, the Jacobian matrix $[\partial f/\partial x]$ is bounded on D. Let

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|_2 \leq L, \quad \forall\ x \in D$$

Then, $\|f(t,x)\|_2 \leq L\|x\|_2$ and $\phi(\tau; t, x)$ satisfies the lower bound[25]

$$\|\phi(\tau; t, x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$V(t,x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)}\ d\tau\ \|x\|_2^2 = \frac{1}{2L}(1 - e^{-2L\delta})\|x\|_2^2$$

---

[25] See Exercise 3.17.

Thus, $V(t,x)$ satisfies the first inequality of the theorem with

$$c_1 = \frac{(1-e^{-2L\delta})}{2L} \quad \text{and} \quad c_2 = \frac{k^2(1-e^{-2\lambda\delta})}{2\lambda}$$

To calculate the derivative of V along the trajectories of the system, define the sensitivity functions

$$\phi_t(\tau;t,x) = \frac{\partial}{\partial t}\phi(\tau;t,x); \quad \phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x)$$

Then,

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \phi^T(t;t,x)\phi(t;t,x) \\ &\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_t(\tau;t,x)\ d\tau \\ &\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)\ d\tau f(t,x) \\ &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \|x\|_2^2 \\ &\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)[\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x)]\ d\tau\end{aligned}$$

It is not difficult to show that[26]

$$\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x) \equiv 0, \quad \forall\ \tau \geq t$$

Therefore,

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \|x\|_2^2 \\ &\leq -(1-k^2e^{-2\lambda\delta})\|x\|_2^2\end{aligned}$$

By choosing $\delta = \ln(2k^2)/(2\lambda)$, the second inequality of the theorem is satisfied with $c_3 = 1/2$. To show the last inequality, let us note that $\phi_x(\tau;t,x)$ satisfies the sensitivity equation

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau;t,x))\ \phi_x, \quad \phi_x(t;t,x) = I$$

Since

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|_2 \leq L$$

---

[26] See Exercise 3.30.

on D, ϕ_x satisfies the bound[27]

$$\|\phi_x(\tau; t, x)\|_2 \leq e^{L(\tau - t)}$$

Therefore,

$$\begin{aligned}
\left\|\frac{\partial V}{\partial x}\right\|_2 &= \left\|\int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x)\ d\tau\right\|_2 \\
&\leq \int_t^{t+\delta} 2\|\phi(\tau; t, x)\|_2\ \|\phi_x(\tau; t, x)\|_2\ d\tau \\
&\leq \int_t^{t+\delta} 2ke^{-\lambda(\tau - t)}\ e^{L(\tau - t)}\ d\tau\ \|x\|_2 \\
&= \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]\|x\|_2
\end{aligned}$$

Thus, the last inequality of the theorem is satisfied with

$$c_4 = \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]$$

If all the assumptions hold globally, then clearly r_0 can be chosen arbitrarily large. If the system is autonomous, then $\phi(\tau; t, x)$ depends only on $(\tau - t)$; that is,

$$\phi(\tau; t, x) = \psi(\tau - t; x)$$

Then,

$$V(t, x) = \int_t^{t+\delta} \psi^T(\tau - t; x)\psi(\tau - t; x)\ d\tau = \int_0^{\delta} \psi^T(s; x)\psi(s; x)\ ds$$

which is independent of t. □

In Theorem 4.13, we saw that if the linearization of a nonlinear system about the origin has an exponentially stable equilibrium point, then the origin is an exponentially stable equilibrium point for the nonlinear system. We will use Theorem 4.14 to prove that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin.

Theorem 4.15 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left.\frac{\partial f}{\partial x}(t, x)\right|_{x=0}$$

---

[27] See Exercise 3.17.

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: The "if" part follows from Theorem 4.13. To prove the "only if" part, write the linear system as

$$\dot{x} = f(t,x) - [f(t,x) - A(t)x] = f(t,x) - g(t,x)$$

Recalling the argument preceding Theorem 4.13, we know that

$$\|g(t,x)\|_2 \le L\|x\|_2^2, \quad \forall\, x \in D, \ \forall\, t \ge 0$$

Since the origin is an exponentially stable equilibrium of the nonlinear system, there are positive constants k, λ, and c such that

$$\|x(t)\|_2 \le k\|x(t_0)\|_2 e^{-\lambda(t-t_0)}, \quad \forall\, t \ge t_0 \ge 0, \ \forall\ \|x(t_0)\|_2 < c$$

Choosing $r_0 < \min\{c, r/k\}$, all the conditions of Theorem 4.14 are satisfied. Let $V(t,x)$ be the function provided by Theorem 4.14 and use it as a Lyapunov function candidate for the linear system. Then,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) - \frac{\partial V}{\partial x}g(t,x) \\ &\le -c_3\|x\|_2^2 + c_4 L\|x\|_2^3 \\ &< -(c_3 - c_4 L\rho)\|x\|_2^2, \quad \forall\ \|x\|_2 < \rho \end{aligned}$$

The choice $\rho < \min\{r_0, c_3/(c_4 L)\}$ ensures that $\dot{V}(t,x)$ is negative definite in $\|x\|_2 < \rho$. Consequently, all the conditions of Theorem 4.10 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is an exponentially stable equilibrium point for the linear system. □

Corollary 4.3 *Let $x = 0$ be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let $A = [\partial f/\partial x](0)$. Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if A is Hurwitz.* ◇

Example 4.23 Consider the first-order system $\dot{x} = -x^3$. We saw in Example 4.14 that the origin is asymptotically stable, but linearization about the origin results in the linear system $\dot{x} = 0$, whose A matrix is not Hurwitz. Using Corollary 4.3, we conclude that the origin is not exponentially stable. △

The following converse Lyapunov theorems (Theorem 4.16 and 4.17) extend Theorem 4.15 in two different directions, but their proofs are more involved. Theorem 4.16 applies to the more general case of uniform asymptotic stability.[28] Theorem 4.17 applies to autonomous systems and produces a Lyapunov function that is defined on the whole region of attraction.

Theorem 4.16 *Let* $x = 0$ *be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ *is continuously differentiable,* $D = \{x \in R^n \mid \|x\| < r\}$*, and the Jacobian matrix* $[\partial f/\partial x]$ *is bounded on* D*, uniformly in* t*. Let* β *be a class* $\mathcal{KL}$ *function and* r_0 *be a positive constant such that* $\beta(r_0, 0) < r$*. Let* $D_0 = \{x \in R^n \mid \|x\| < r_0\}$*. Assume that the trajectory of the system satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ x(t_0) \in D_0, \ \forall\ t \geq t_0 \geq 0$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ *that satisfies the inequalities*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|)$$

where α_1*,* α_2*,* α_3*, and* α_4 *are class* $\mathcal{K}$ *functions defined on* $[0, r_0]$*. If the system is autonomous,* V *can be chosen independent of* t*.* ◇

Proof: See Appendix C.7.

Theorem 4.17 *Let* $x = 0$ *be an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ *is locally Lipschitz and* $D \subset R^n$ *is a domain that contains the origin. Let* $R_A \subset D$ *be the region of attraction of* $x = 0$*. Then, there is a smooth, positive definite function* $V(x)$ *and a continuous, positive definite function* $W(x)$*, both defined for all* $x \in R_A$*, such that*

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall\ x \in R_A$$

and for any $c > 0$*,* $\{V(x) \leq c\}$ *is a compact subset of* R_A*. When* $R_A = R^n$*,* $V(x)$ *is radially unbounded.* ◇

---

[28] Theorem 4.16 can be stated for a function $f(t, x)$ that is only locally Lipschitz, rather than continuously differentiable [125, Theorem 14]. It is also possible to state the theorem for the case of global uniform asymptotic stability [125, Theorem 23]

Proof: See Appendix C.8.

An interesting feature of Theorem 4.17 is that any bounded subset S of the region of attraction can be included in a compact set of the form $\{V(x) \leq c\}$ for some constant $c > 0$. This feature is useful because quite often we have to limit our analysis to a positively invariant, compact set of the form $\{V(x) \leq c\}$. With the property $S \subset \{V(x) \leq c\}$, our analysis will be valid for the whole set S. If, on the other hand, all we know is the existence of a Lyapunov function $V_1(x)$ on S, we will have to choose a constant c_1 such that $\{V_1(x) \leq c_1\}$ is compact and included in S; then our analysis will be limited to $\{V_1(x) \leq c_1\}$, which is only a subset of S.

4.8 Boundedness and Ultimate Boundedness

Lyapunov analysis can be used to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin. To motivate the idea, consider the scalar equation

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no equilibrium points and whose solution is given by

$$x(t) = e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \, d\tau$$

The solution satisfies the bound

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \, d\tau = e^{-(t-t_0)} a + \delta \left[1 - e^{-(t-t_0)}\right] \\ &\leq a, \quad \forall \, t \geq t_0 \end{aligned}$$

which shows that the solution is bounded for all $t \geq t_0$, uniformly in t_0, that is, with a bound independent of t_0. While this bound is valid for all $t \geq t_0$, it becomes a conservative estimate of the solution as time progresses, because it does not take into consideration the exponentially decaying term. If, on the other hand, we pick any number b such that $\delta < b < a$, it can be easily seen that

$$|x(t)| \leq b, \quad \forall \, t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be uniformly ultimately bounded and b is called the ultimate bound. Showing that the solution of $\dot{x} = -x + \delta \sin t$ has the uniform boundedness and ultimate boundedness properties can be done via Lyapunov analysis without using the explicit solution of the state equation. Starting with $V(x) = x^2/2$, we calculate the derivative of V along the trajectories of the system, to obtain

$$\dot{V} = x\dot{x} = -x^2 + x\delta \sin t \leq -x^2 + \delta|x|$$

The right-hand side of the foregoing inequality is not negative definite because, near the origin, the positive linear term $\delta|x|$ dominates the negative quadratic term $-x^2$. However, $\dot{V}$ is negative outside the set $\{|x| \leq \delta\}$. With $c > \delta^2/2$, solutions starting in the set $\{V(x) \leq c\}$ will remain therein for all future time since $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions are uniformly bounded. Moreover, if we pick any number ε such that $(\delta^2/2) < \varepsilon < c$, then $\dot{V}$ will be negative in the set $\{\varepsilon \leq V \leq c\}$, which shows that, in this set, V will decrease monotonically until the solution enters the set $\{V \leq \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \leq \varepsilon\}$ because $\dot{V}$ is negative on the boundary $V = \varepsilon$. Thus, we can conclude that the solution is uniformly ultimately bounded with the ultimate bound $|x| \leq \sqrt{2\varepsilon}$.

The purpose of this section is to show how Lyapunov analysis can be used to draw similar conclusions for the system

$$\dot{x} = f(t, x) \tag{4.32}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin.

Definition 4.6 *The solutions of* (4.32) *are*

- *uniformly bounded if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq \beta, \quad \forall\, t \geq t_0 \tag{4.33}$$

- *globally uniformly bounded if* (4.33) *holds for arbitrarily large a.*

- *uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq b, \quad \forall\, t \geq t_0 + T \tag{4.34}$$

- *globally uniformly ultimately bounded if* (4.34) *holds for arbitrarily large a.*

In the case of autonomous systems, we may drop the word "uniformly" since the solution depends only on $t - t_0$.

To see how Lyapunov analysis can be used to study boundedness and ultimate boundedness, consider a continuously differentiable, positive definite function $V(x)$ and suppose that the set $\{V(x) \leq c\}$ is compact, for some $c > 0$. Let

$$\Lambda = \{\varepsilon \leq V(x) \leq c\}$$

for some positive constant $\varepsilon < c$. Suppose the derivative of V along the trajectories of the system $\dot{x} = f(t,x)$ satisfies

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0 \tag{4.35}$$

where $W_3(x)$ is a continuous positive definite function. Inequality (4.35) implies that the sets $\Omega_c = \{V(x) \le c\}$ and $\Omega_\varepsilon = \{V(x) \le \varepsilon\}$ are positively invariant since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\varepsilon$, the derivative $\dot{V}$ is negative. A sketch of the sets Λ, Ω_c, and Ω_ε is shown in Figure 4.8. Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in a direction of decreasing $V(x(t))$. In fact, while in Λ, V satisfies inequalities (4.22) and (4.24) of Theorem 4.9. Therefore, the trajectory behaves as if the origin was uniformly asymptotically stable and satisfies an inequality of the form

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)$$

for some class $\mathcal{KL}$ function β. The function $V(x(t))$ will continue decreasing until the trajectory enters the set Ω_ε in finite time and stays therein for all future time. The fact that the trajectory enters Ω_ε in finite time can be shown as follows: Let $k = \min_{x\in\Lambda} W_3(x) > 0$. The minimum exists because $W_3(x)$ is continuous and Λ is compact. It is positive since $W_3(x)$ is positive definite. Hence,

$$W_3(x) \ge k, \quad \forall\, x \in \Lambda \tag{4.36}$$

Inequalities (4.35) and (4.36) imply that

$$\dot{V}(t,x) \le -k, \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0$$

Therefore,

$$V(x(t)) \le V(x(t_0)) - k(t - t_0) \le c - k(t - t_0)$$

which shows that $V(x(t))$ reduces to ε within the time interval $[t_0, t_0 + (c - \varepsilon)/k]$.

In many problems, the inequality $\dot{V} \le -W_3$ is obtained by using norm inequalities. In such cases, it is more likely that we arrive at

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, \mu \le \|x\| \le r,\ \forall\, t \ge t_0 \tag{4.37}$$

If r is sufficiently larger than μ, we can choose c and ε such that the set Λ is nonempty and contained in $\{\mu \le \|x\| \le r\}$. In particular, let α_1 and α_2 be class $\mathcal{K}$ functions such that [29]

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{4.38}$$

From the left inequality of (4.38), we have

$$V(x) \le c \;\Rightarrow\; \alpha_1(\|x\|) \le c \;\Leftrightarrow\; \|x\| \le \alpha_1^{-1}(c)$$

---

[29] By Lemma 4.3, it is always possible to find such class $\mathcal{K}$ functions.

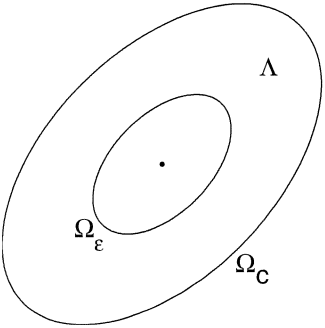

Figure 4.8: Representation of the set Λ, Ω_ε and Ω_c.

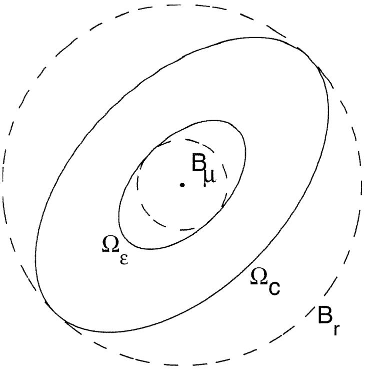

Figure 4.9: Representation of the sets Ω_ε, Ω_c (solid) and B_μ, B_r (dashed).

Therefore, taking $c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$. On the other hand, from the right inequality of (4.38), we have

$$\|x\| \le \mu \;\Rightarrow\; V(x) \le \alpha_2(\mu)$$

Consequently, taking $\varepsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\varepsilon$. To obtain $\varepsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$. A sketch of the sets Ω_c, Ω_ε, B_r, and B_μ is shown in Figure 4.9.

The foregoing argument shows that all trajectories starting in Ω_c enter Ω_ε within a finite time T.[30] To calculate the ultimate bound on $x(t)$, we use the left inequality of (4.38) to write

$$V(x) \le \varepsilon \Rightarrow \alpha_1(\|x\|) \le \varepsilon \Leftrightarrow \|x\| \le \alpha_1^{-1}(\varepsilon)$$

---

[30] If the trajectory starts in Ω_ε, $T = 0$.

Recalling that $\varepsilon = \alpha_2(\mu)$, we see that

$$x \in \Omega_\varepsilon \Rightarrow \|x\| \le \alpha_1^{-1}(\alpha_2(\mu))$$

Therefore, the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.

The ideas just presented for a continuously differentiable function $V(x)$ can be extended to a continuously differentiable function $V(t, x)$, as long as $V(t, x)$ satisfies inequality (4.38), which leads to the following Lyapunov-like theorem for showing uniform boundedness and ultimate boundedness.

Theorem 4.18 *Let $D \subset R^n$ be a domain that contains the origin and $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|) \tag{4.39}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x), \quad \underline{\forall\ \|x\| \ge \mu > 0} \tag{4.40}$$

$\forall\ t \ge 0$ and $\forall\ x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\boxed{\mu < \alpha_2^{-1}(\alpha_1(r))} \tag{4.41}$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(r))$, there is $T \ge 0$ (dependent on $x(t_0)$ and μ) such that the solution of (4.32) *satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t_0 \le t \le t_0 + T \tag{4.42}$$

$$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall\ t \ge t_0 + T \tag{4.43}$$

Moreover, if $D = R^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (4.42) and (4.43) *hold for any initial state $x(t_0)$, with no restriction on how large μ is.* ◇

Proof: See Appendix C.9.

Inequalities (4.42) and (4.43) show that $x(t)$ is uniformly bounded for all $t \ge t_0$ and uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$. The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero.

The main application of Theorem 4.18 arises in studying the stability of perturbed systems.[31] The next example illustrates the basic idea of that application.

Example 4.24 In Section 1.2.3, we saw that a mass–spring system with a hardening spring, linear viscous damping, and a periodic external force can be represented by the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

---

[31] See Section 9.2.

Taking $x_1 = y$, $x_2 = \dot{y}$ and assuming certain numerical values for the various constants, the system is represented by the state model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M\cos\omega t \end{aligned}$$

where $M \geq 0$ is proportional to the amplitude of the periodic external force. When $M = 0$, the system has an equilibrium point at the origin. It is shown in Example 4.6 that the origin is globally asymptotically stable and a Lyapunov function can be taken as[32]

$$\begin{aligned} V(x) &= x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3)\, dy = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + x_1^2 + \tfrac{1}{2}x_1^4 \\ &= x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \tfrac{1}{2}x_1^4 \overset{\text{def}}{=} x^T P x + \tfrac{1}{2}x_1^4 \end{aligned}$$

When $M > 0$, we apply Theorem 4.18 with $V(x)$ as a candidate function. The function $V(x)$ is positive definite and radially unbounded; hence, by Lemma 4.3, there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 that satisfy (4.39) globally. The derivative of V along the trajectories of the system is given by

$$\dot{V} = -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \leq -\|x\|_2^2 - x_1^4 + M\sqrt{5}\|x\|_2$$

where we wrote $(x_1 + 2x_2)$ as $y^T x$ and used the inequality $y^T x \leq \|x\|_2\|y\|_2$. To satisfy (4.40), we want to use part of $-\|x\|_2^2$ to dominate $M\sqrt{5}\|x\|_2$ for large $\|x\|$. Towards that end, we rewrite the foregoing inequality as

$$\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4 - \theta\|x\|_2^2 + M\sqrt{5}\|x\|_2$$

where $0 < \theta < 1$. Then,

μ

$$\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4, \quad \forall\ \|x\|_2 \geq \frac{M\sqrt{5}}{\theta}$$

which shows that inequality (4.40) is satisfied globally with $\mu = M\sqrt{5}/\theta$. We conclude that the solutions are globally uniformly ultimately bounded. Suppose we want to go the extra step of calculating the ultimate bound. In this case, we have to find the functions α_1 and α_2. From the inequalities

$$V(x) \geq x^T P x \geq \lambda_{min}(P)\|x\|_2^2$$

$$V(x) \leq x^T P x + \tfrac{1}{2}\|x\|_2^4 \leq \lambda_{max}(P)\|x\|_2^2 + \tfrac{1}{2}\|x\|_2^4$$

---

[32] The constants δ and k of Example 4.6 are taken as $\delta = 2$ and $k = 1/2$.

we see that α_1 and α_2 can be taken as

$$\alpha_1(r) = \lambda_{min}(P)r^2 \quad \text{and} \quad \alpha_2(r) = \lambda_{max}(P)r^2 + \tfrac{1}{2}r^4$$

Thus, the ultimate bound is given by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{\lambda_{min}(P)}} = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$

△

4.9 Input-to-State Stability

Consider the system

$$\dot{x} = f(t, x, u) \tag{4.44}$$

where $f : [0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$. Suppose the unforced system

$$\dot{x} = f(t, x, 0) \tag{4.45}$$

has a globally uniformly asymptotically stable equilibrium point at the origin $x = 0$. What can we say about the behavior of the system (4.44) in the presence of a bounded input $u(t)$? For the linear time-invariant system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A, we can write the solution as

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)\, d\tau$$

and use the bound $\|e^{(t-t_0)A}\| \leq ke^{-\lambda(t-t_0)}$ to estimate the solution by

$$\begin{aligned} \|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|\, d\tau \\ &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \end{aligned}$$

This estimate shows that the zero-input response decays to zero exponentially fast, while the zero-state response is bounded for every bounded input. In fact, the estimate shows more than a bounded-input–bounded-state property. It shows that the bound on the zero-state response is proportional to the bound on the input. How much of this behavior should we expect for the nonlinear system (4.44)? For a general nonlinear system, it should not be surprising that these properties may not hold even when the origin of the unforced system is globally uniformly asymptotically stable. Consider, for example, the scalar system

$$\dot{x} = -3x + (1 + 2x^2)u$$

which has a globally exponentially stable origin when $u = 0$. Yet, when $x(0) = 2$ and $u(t) \equiv 1$, the solution $x(t) = (3 - e^t)/(3 - 2e^t)$ is unbounded; it even has a finite escape time.

Let us view the system (4.44) as a perturbation of the unforced system (4.45). Suppose we have a Lyapunov function $V(t, x)$ for the unforced system and let us calculate the derivative of V in the presence of u. Due to the boundedness of u, it is plausible that in some cases it should be possible to show that $\dot{V}$ is negative outside a ball of radius μ, where μ depends on $\sup \|u\|$. This would be expected, for example, when the function $f(t, x, u)$ satisfies the Lipschitz condition

$$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\| \tag{4.46}$$

Showing that $\dot{V}$ is negative outside a ball of radius μ would enable us to apply Theorem 4.18 of the previous section to show that $x(t)$ satisfies (4.42) and (4.43). These inequalities show that $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ function $\beta(\|x(t_0)\|, t - t_0)$ over $[t_0, t_0+T]$ and by a class $\mathcal{K}$ function $\alpha_1^{-1}(\alpha_2(\mu))$ for $t \geq t_0+T$. Consequently,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu))$$

is valid for all $t \geq t_0$, which motivates the next definition of *input-to-state stability.*

Definition 4.7 *The system* (4.44) *is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{4.47}$$

Inequality (4.47) guarantees that for any bounded input $u(t)$, the state $x(t)$ will be bounded. Furthermore, as t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} \|u(t)\|$. We leave it to the reader (Exercise 4.58) to use inequality (4.47) to show that if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$.[33] Since, with $u(t) \equiv 0$, (4.47) reduces to

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

input-to-state stability implies that the origin of the unforced system (4.45) is globally uniformly asymptotically stable. The notion of input-to-state stability is defined for the global case where the initial state and the input can be arbitrarily large. A local version of this notion is presented in Exercise 4.60.

---

[33] Another interesting use of inequality (4.47) will be given shortly in Lemma 4.7.

The Lyapunov-like theorem that follows gives a sufficient condition for input-to-state stability.[34]

Theorem 4.19 *Let $V : [0, \infty) \times R^n \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{4.48}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W_3(x), \quad \forall\ \|x\| \geq \rho(\|u\|) > 0 \tag{4.49}$$

$\forall\ (t,x,u) \in [0,\infty) \times R^n \times R^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W_3(x)$ is a continuous positive definite function on R^n. Then, the system (4.44) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. ◇

Proof: By applying the global version of Theorem 4.18, we find that the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right), \quad \forall\ t \geq t_0 \tag{4.50}$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \leq \tau \leq t$, the supremum on the right-hand side of (4.50) can be taken over $[t_0, t]$, which yields (4.47).[35] □

The next lemma is an immediate consequences of the converse Lyapunov theorem for global exponential stability (Theorem 4.14).

Lemma 4.6 *Suppose $f(t,x,u)$ is continuously differentiable and globally Lipschitz in (x,u), uniformly in t. If the unforced system (4.45) has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system (4.44) is input-to-state stable.* ◇

Proof: View the system (4.44) as a perturbation of the unforced system (4.45). (The converse Lyapunov) Theorem 4.14 shows that the unforced system (4.45) has a Lyapunov function $V(t,x)$ that satisfies the inequality of the theorem globally. Due to the uniform global Lipschitz property of f, the perturbation term satisfies (4.46) for all $t \geq t_0$ and all (x,u). The derivative of V with respect to (4.44) satisfies

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\leq -c_3\|x\|^2 + c_4\|x\|L\|u\|\end{aligned}$$

---

[34] For autonomous systems, it is shown in [183] that the conditions of Theorem 4.19 are also necessary. In the literature, it is common to abbreviate input-to-state stability as ISS and to call the function V of Theorem 4.19 an ISS-Lyapunov function.

[35] In particular, repeat the aforementioned argument over the period $[0, T]$ to show that

$$\|x(\sigma)\| \leq \beta(\|x(t_0)\|, \sigma - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq T} \|u(\tau)\|\right), \quad \forall\ t_0 \leq \sigma \leq T$$

Then, set $\sigma = T = t$.

To use the term $-c_3\|x\|^2$ to dominate $c_4L\|x\|\|u\|$ for large $\|x\|$, we rewrite the foregoing inequality as

$$\dot{V} \leq -c_3(1-\theta)\|x\|^2 - c_3\theta\|x\|^2 + c_4L\|x\|\|u\|$$

where $0 < \theta < 1$. Then,

$$\dot{V} \leq -c_3(1-\theta)\|x\|^2, \quad \forall\ \|x\| \geq \frac{c_4L\|u\|}{c_3\theta}$$

for all (t, x, u). Hence, the conditions of Theorem 4.19 are satisfied with $\alpha_1(r) = c_1r^2$, $\alpha_2(r) = c_2r^2$, and $\rho(r) = (c_4L/c_3\theta)r$, and we conclude that the system is input-to-state stable with $\gamma(r) = \sqrt{c_2/c_1}(c_4L/c_3\theta)r$. □

Lemma 4.6 requires a globally Lipschitz function f and global exponential stability of the origin of the unforced system to conclude input-to-state stability. It is easy to construct examples where the lemma does not hold in the absence of one of these two conditions. The system $\dot{x} = -3x + (1+x^2)u$, which we discussed earlier in the section, does not satisfy the global Lipschitz condition. The system

$$\dot{x} = -\frac{x}{1+x^2} + u \overset{\text{def}}{=} f(x, u)$$

has a globally Lipschitz f since the partial derivatives of f with respect to x and u are globally bounded. The origin of $\dot{x} = -x/(1+x^2)$ is globally asymptotically stable, as it can be seen by the Lyapunov function $V(x) = x^2/2$, whose derivative $\dot{V}(x) = -x^2/(1+x^2)$ is negative definite for all x. It is locally exponentially stable because the linearization at the origin is $\dot{x} = -x$. However, it is not globally exponentially stable. This is easiest seen through the fact that the system is not input-to-state stable. Notice that with $u(t) \equiv 1$, $f(x, u) \geq 1/2$. Hence, $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$, which shows that the solution is unbounded.

In the absence of global exponential stability or globally Lipschitz functions, we may still be able to show input-to-state stability by applying Theorem 4.19. This process is illustrated by the three examples that follow.

Example 4.25 The system

$$\dot{x} = -x^3 + u$$

has a globally asymptotically stable origin when $u = 0$. Taking $V = x^2/2$, the derivative of V along the trajectories of the system is given by

$$\dot{V} = -x^4 + xu = -(1-\theta)x^4 - \theta x^4 + xu \leq -(1-\theta)x^4, \quad \forall\ |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}$$

where $0 < \theta < 1$. Thus, the system is input-to-state stable with $\gamma(r) = (r/\theta)^{1/3}$. △

Example 4.26 The system

$$\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$$

has a globally exponentially stable origin when $u = 0$, but Lemma 4.6 does not apply since f is not globally Lipschitz. Taking $V = x^2/2$, we obtain

$$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \le -x^4, \quad \forall\ |x| \ge u^2$$

Thus, the system is input-to-state stable with $\gamma(r) = r^2$. △

In Examples 4.25 and 4.26, the function $V(x) = x^2/2$ satisfies (4.48) with $\alpha_1(r) = \alpha_2(r) = r^2/2$. Hence, $\alpha_1^{-1}(\alpha_2(r)) = r$ and $\gamma(r)$ reduces to $\rho(r)$. In higher-dimensional systems, the calculation of γ is more involved.

Example 4.27 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

We start by setting $u = 0$ and investigate global asymptotic stability of the origin of the unforced system. Using

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}ax_2^4, \quad a > 0$$

as a Lyapunov function candidate, we obtain

$$\dot{V} = -x_1^2 + x_1x_2^2 - ax_2^4 = -(x_1 - \tfrac{1}{2}x_2^2)^2 - \left(a - \tfrac{1}{4}\right)x_2^4$$

Choosing $a > 1/4$ shows that the origin is globally asymptotically stable. Now we allow $u \neq 0$ and use $V(x)$ with $a = 1$ as a candidate function for Theorem 4.19. The derivative $\dot{V}$ is given by

$$\dot{V} = -\tfrac{1}{2}(x_1 - x_2^2)^2 - \tfrac{1}{2}(x_1^2 + x_2^4) + x_2^3u \le -\tfrac{1}{2}(x_1^2 + x_2^4) + |x_2|^3|u|$$

To use the term $-(x_1^2 + x_2^4)/2$ to dominate $|x_2|^3|u|$, we rewrite the foregoing inequality as

$$\dot{V} \le -\tfrac{1}{2}(1-\theta)(x_1^2 + x_2^4) - \tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u|$$

where $0 < \theta < 1$. The term

$$-\tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u|$$

will be ≤ 0 if $|x_2| \ge 2|u|/\theta$ or $|x_2| \le 2|u|/\theta$ and $|x_1| \ge (2|u|/\theta)^2$. This condition is implied by

$$\max\{|x_1|, |x_2|\} \ge \max\left\{\frac{2|u|}{\theta}, \left(\frac{2|u|}{\theta}\right)^2\right\}$$

Using the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ and defining the class $\mathcal{K}$ function ρ by

$$\rho(r) = \max\left\{\frac{2r}{\theta}, \left(\frac{2r}{\theta}\right)^2\right\}$$

we see that inequality (4.49) is satisfied as

$$\dot{V} \le -\tfrac{1}{2}(1-\theta)(x_1^2 + x_2^4), \quad \forall\ \|x\|_\infty \ge \rho(|u|)$$

Inequality (4.48) follows from Lemma 4.3 since $V(x)$ is positive definite and radially unbounded. Hence, the system is input-to-state stable. Suppose we want to find the class $\mathcal{K}$ function γ. In this case, we need to find α_1 and α_2. It is not hard to see that

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4 \le \tfrac{1}{2}\|x\|_\infty^2 + \tfrac{1}{4}\|x\|_\infty^4$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4 \ge \begin{cases} \tfrac{1}{2}|x_1|^2 = \tfrac{1}{2}\|x\|_\infty^2, & \text{if } |x_2| \le |x_1| \\ \tfrac{1}{4}|x_2|^4 = \tfrac{1}{4}\|x\|_\infty^4, & \text{if } |x_2| \ge |x_1| \end{cases}$$

Inequality (4.48) is satisfied with the class $\mathcal{K}_\infty$ functions

$$\alpha_1(r) = \min\left\{\tfrac{1}{2}r^2, \tfrac{1}{4}r^4\right\} \quad \text{and} \quad \alpha_2(r) = \tfrac{1}{2}r^2 + \tfrac{1}{4}r^4$$

Thus, $\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r)))$, where

$$\alpha_1^{-1}(s) = \begin{cases} (4s)^{\frac{1}{4}}, & \text{if } s \le 1 \\ \sqrt{2s}, & \text{if } s \ge 1 \end{cases}$$

The function γ depends on the choice of $\|x\|$. Had we chosen another p-norm, we could have ended up with a different γ. △

An interesting application of the concept of input-to-state stability arises in the stability analysis of the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2) \tag{4.51}$$

$$\dot{x}_2 = f_2(t, x_2) \tag{4.52}$$

where $f_1 : [0,\infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and $f_2 : [0,\infty) \times R^{n_2} \to R^{n_2}$ are piecewise continuous in t and locally Lipschitz in $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Suppose both

$$\dot{x}_1 = f_1(t, x_1, 0)$$

and (4.52) have globally uniformly asymptotically stable equilibrium points at their respective origins. Under what condition will the origin $x = 0$ of the cascade system possess the same property? The next lemma shows that this will be the case if (4.51), with x_2 viewed as input, is input-to-state stable.

Lemma 4.7 *Under the stated assumptions, if the system* (4.51), *with* x_2 *as input, is input-to-state stable and the origin of* (4.52) *is globally uniformly asymptotically stable, then the origin of the cascade system* (4.51) and (4.52) *is globally uniformly asymptotically stable.* ◇

Proof: Let $t_0 \geq 0$ be the initial time. The solutions of (4.51) and (4.52) satisfy

$$\begin{aligned} \|x_1(t)\| &\leq \beta_1(\|x_1(s)\|, t-s) + \gamma_1\left(\sup_{s\leq\tau\leq t} \|x_2(\tau)\|\right) && (4.53) \\ \|x_2(t)\| &\leq \beta_2(\|x_2(s)\|, t-s) && (4.54) \end{aligned}$$

globally, where $t \geq s \geq t_0$, β_1, β_2 are class $\mathcal{KL}$ functions and γ_1 is a class $\mathcal{K}$ function. Apply (4.53) with $s = (t+t_0)/2$ to obtain

$$\|x_1(t)\| \leq \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{\frac{t+t_0}{2}\leq\tau\leq t} \|x_2(\tau)\|\right) \tag{4.55}$$

To estimate $x_1((t+t_0)/2)$, apply (4.53) with $s = t_0$ and t replaced by $(t+t_0)/2$ to obtain

$$\left\|x_1\left(\frac{t+t_0}{2}\right)\right\| \leq \beta_1\left(\|x_1(t_0)\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{t_0\leq\tau\leq\frac{t+t_0}{2}} \|x_2(\tau)\|\right) \tag{4.56}$$

Using (4.54), we obtain

$$\begin{aligned} \sup_{t_0\leq\tau\leq\frac{t+t_0}{2}} \|x_2(\tau)\| &\leq \beta_2(\|x_2(t_0)\|, 0) && (4.57) \\ \sup_{\frac{t+t_0}{2}\leq\tau\leq t} \|x_2(\tau)\| &\leq \beta_2\left(\|x_2(t_0)\|, \frac{t-t_0}{2}\right) && (4.58) \end{aligned}$$

Substituting (4.56) through (4.58) into (4.55) and using the inequalities

$$\|x_1(t_0)\| \leq \|x(t_0)\|, \quad \|x_2(t_0)\| \leq \|x(t_0)\|, \quad \|x(t)\| \leq \|x_1(t)\| + \|x_2(t)\|$$

yield

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0)$$

where

$$\beta(r,s) = \beta_1\left(\beta_1\left(r, \frac{s}{2}\right) + \gamma_1\left(\beta_2(r,0)\right), \frac{s}{2}\right) + \gamma_1\left(\beta_2\left(r, \frac{s}{2}\right)\right) + \beta_2(r,s)$$

It can be easily verified that β is a class $\mathcal{KL}$ function for all $r \geq 0$. Hence, the origin of (4.51) and (4.52) is globally uniformly asymptotically stable. □

4.10 Exercises

4.1 Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

(1) stable node	**(2)** unstable node	**(3)** stable focus
(4) unstable focus	**(5)** center	**(6)** saddle

Justify your answer using phase portraits.

4.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

4.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

$$
\begin{array}{llll}
(1) & \dot{x}_1 = -x_1 + x_1x_2, & & \dot{x}_2 = -x_2 \\
(2) & \dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), & & \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2) \\
(3) & \dot{x}_1 = x_2(1 - x_1^2), & & \dot{x}_2 = -(x_1 + x_2)(1 - x_1^2) \\
(4) & \dot{x}_1 = -x_1 - x_2, & & \dot{x}_2 = 2x_1 - x_2^3
\end{array}
$$

Investigate whether the origin is globally asymptotically stable.

4.4 ([**151**]) Euler equations for a rotating rigid spacecraft are given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}
$$

where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

4.5 Let $g(x)$ be a map from R^n into R^n. Show that $g(x)$ is the gradient vector of a scalar function $V : R^n \to R$ if and only if

$$
\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, 2, \ldots, n
$$

4.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \neq 0$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

4.7 Consider the system $\dot{x} = -Q\phi(x)$, where Q is a symmetric positive definite matrix and $\phi(x)$ is a continuously differentiable function for which the ith component ϕ_i depends only on x_i, that is, $\phi_i(x) = \phi_i(x_i)$. Assume that $\phi_i(0) = 0$ and $y\phi_i(y) > 0$ in some neighborhood of $y = 0$, for all $1 \leq i \leq n$.

(a) Using the variable gradient method, find a Lyapunov function that shows that the origin is asymptotically stable.

(b) Under what conditions will it be globally asymptotically stable?

(c) Apply to the case

$$n = 2, \ \phi_1(x_1) = x_1 - x_1^2, \ \phi_2(x_2) = x_2 + x_2^3, \ Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4.8 (**[72]**) Consider the second-order system

$$\dot{x}_1 = \frac{-6x_1}{u^2} + 2x_2, \qquad \dot{x}_2 = \frac{-2(x_1 + x_2)}{u^2}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, and compare with the slope of the tangents to the hyperbola.

4.9 In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $\|x\| \rightarrow \infty$ along the principal axes. This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\; d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem:
For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem:
For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$\textbf{(1)} \qquad \dot{x}_1 = x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$\textbf{(2)} \qquad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y)\; dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\, dy + x_2^2/2 + \int_0^{x_3} h_2(y)\, dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

4.17 ([77]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\, dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (1/2)\left[x_2 + \int_0^{x_1} h(y)\, dy\right]^2 + \int_0^{x_1} g(y)\, dy$.

4.18 The mass–spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

4.19 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin $(q = 0,\ \dot{q} = 0)$ is stable.

(b) With $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point $(q = q^*,\ \dot{q} = 0)$ is an asymptotically stable equilibrium point.

4.20 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

4.21 (**[81]**) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \le 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Take $D = R^n$. Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \ge 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

4.22 Consider the Lyapunov equation $PA + A^TP = -C^TC$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall\ t$ if and only if $x = 0$.

4.23 Consider the linear system $\dot{x} = (A - BR^{-1}B^TP)x$, where $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - \underline{PBR^{-1}B^TP} = 0$$

$R = R^T > 0$, and $Q = Q^T \ge 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

(1) $Q > 0$.

(2) $Q = C^TC$ and (A, C) is observable; see the hint of Exercise 4.22.

4.24 Consider the system[36]

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

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[36] This is a closed-loop system under optimal stabilizing control. See [172].

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x} f(x) + q(x) - \tfrac{1}{4} \frac{\partial V}{\partial x} G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a nonsingular matrix, and k is a positive constant. Using $V(x)$ as a Lyapunov function candidate, show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

When will the origin be globally asymptotically stable?

4.25 Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At} B B^T e^{-A^T t} \, dt$ for some $\tau > 0$. Show that W is positive definite and let $K = B^T W^{-1}$. Use $V(x) = x^T W^{-1} x$ as a Lyapunov function candidate for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz.

4.26 Let $\dot{x} = f(x)$, where $f : R^n \rightarrow R^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : R^n \rightarrow R^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left. \frac{\partial T}{\partial x} f(x) \right|_{x = T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable or unstable) if and only if $z = 0$ is stable (asymptotically stable or unstable).

4.27 Consider the system

$$\dot{x}_1 = -x_2 x_3 + 1, \qquad \dot{x}_2 = x_1 x_3 - x_2, \qquad \dot{x}_3 = x_3^2(1 - x_3)$$

(a) Show that the system has a unique equilibrium point.

(b) Using linearization, show that the equilibrium point asymptotically stable. Is it globally asymptotically stable?

4.28 Consider the system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = (x_1 x_2 - 1) x_2^3 + (x_1 x_2 - 1 + x_1^2) x_2$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show, by using linearization, that $x = 0$ is asymptotically stable.

(c) Show that $\Gamma = \{x \in R^2 \mid x_1 x_2 \geq 2\}$ is a positively invariant set.

(d) Is $x = 0$ globally asymptotically stable?

4.29 Consider the system

$$\dot{x}_1 = x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = 3x_1 - x_2$$

(a) Find all equilibrium point of the system.

(b) Using linearization, study the stability of each equilibrium point.

(c) Using quadratic Lyapunov functions, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimate as large as possible.

(d) Construct the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

4.30 Repeat the previous exercise for the system

$$\dot{x}_1 = -\tfrac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2, \qquad \dot{x}_2 = x_1 - \tfrac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)$$

4.31 For each of the systems of Exercise 4.3, use linearization to show that the origin is asymptotically stable.

4.32 For each for the following systems, investigate whether the origin is stable, asymptotically stable, or unstable:

$$(1)\quad \begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 \\ \dot{x}_2 &= -x_2 + x_3^2 \\ \dot{x}_3 &= x_3 - x_1^2 \end{aligned} \qquad (2)\quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1[-2x_3 - \text{sat}(y)]^2 \\ \dot{x}_3 &= -2x_3 - \text{sat}(y) \\ &\ \text{where } y = -2x_1 - 5x_2 + 2x_3 \end{aligned}$$

$$(3)\quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_3 \end{aligned} \qquad (4)\quad \begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_1 x_3 \\ \dot{x}_3 &= (x_1 + 1)x_2 \end{aligned}$$

4.33 Consider the second-order system $\dot{x} = f(x)$, where $f(0) = 0$ and $f(x)$ is twice continuously differentiable in some neighborhood of the origin. Suppose $[\partial f/\partial x](0) = -B$, where B be Hurwitz. Let P be the positive definite solution of the Lyapunov equation $PB + B^T P = -I$ and take $V(x) = x^T P x$. Show that there exists $c^* > 0$ such that, for every $0 < c < c^*$, the surface $V(x) = c$ is closed and $[\partial V/\partial x]f(x) > 0$ for all $x \in \{V(x) = c\}$.

4.34 Prove Lemma 4.2.

4.35 Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \le \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a/2)$$

4.36 Is the origin of the scalar system $\dot{x} = -x/(t+1)$, $t \ge 0$, uniformly asymptotically stable?

4.37 For each of the following linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable:

$$(1)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x,\ |\alpha(t)| \le 1 \qquad (2)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

$$(3)\ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x,\ \alpha(t) \ge 2 \qquad (4)\ \dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

In all cases, $\alpha(t)$ is continuous and bounded for all $t \ge 0$.

4.38 ([95]) An *RLC* circuit with time-varying elements is represented by

$$\dot{x}_1 = \frac{1}{L(t)} x_2, \qquad \dot{x}_2 = -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \le L(t) \le k_2$, $k_3 \le C(t) \le k_4$, and $k_5 \le R(t) \le k_6$ for all $t \ge 0$, where k_1, k_3, and k_5 are positive. Consider a Lyapunov function candidate

$$V(t,x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)}\right] x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

4.39 ([154]) A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - g(t)x_2$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \le g(t) \le \beta < \infty \quad \text{and} \quad \dot{g}(t) \le \gamma < 2$$

for all $t \ge 0$. Consider the Lyapunov function candidate

$$V(t,x) = \tfrac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t, x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2 - \gamma)(1 - \cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

4.40 (Floquet theory) Consider the linear system $\dot{x} = A(t)x$, where $A(t) = A(t + T)$.[37] Let $\Phi(\cdot, \cdot)$ be the state transition matrix. Define a constant matrix B via the equation $\exp(BT) = \Phi(T, 0)$, and let $P(t) = \exp(Bt)\Phi(0, t)$. Show that

(a) $P(t + T) = P(t)$.

(b) $\Phi(t, \tau) = P^{-1}(t)\exp[(t - \tau)B]P(\tau)$.

(c) the origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

4.41 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t + 1)x_2$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

4.42 Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x$$

where a is a positive constant, I_n is the $n \times n$ identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Show that the origin is globally exponentially stable.

4.43 Consider the system $\dot{x} = f(x) + G(x)u$. Suppose there exist a positive definite symmetric matrix P, a positive semidefinite function $W(x)$, and positive constants γ and σ such that

$$2x^TPf(x) + \gamma x^TPx + W(x) - 2\sigma x^TPG(x)G^T(x)Px \leq 0, \quad \forall\ x \in R^n$$

Show that with $u = -\sigma G^T(x)Px$ the closed-loop system has a globally exponentially stable equilibrium point at the origin.

4.44 Consider the system

$$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin t, \qquad \dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos t$$

Show that the origin is exponentially stable and estimate the region of attraction.

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[37] See [158] for a comprehensive treatment of Floquet theory.

4.45 Consider the system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3, \qquad \dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$.

(a) Is the equilibrium point $x = 0$ uniformly asymptotically stable?

(b) Is it exponentially stable?

(c) Is it globally uniformly asymptotically stable?

(d) Is it globally exponentially stable?

4.46 Show that the origin of the system

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

is asymptotically stable. Is it exponentially stable?

4.47 Consider the system

$$\dot{x}_1 = -\phi(t)x_1 + a\phi(t)x_2, \qquad \dot{x}_2 = b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3$$

where a, b, and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall\, t \geq 0$$

Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

4.48 Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : R^n \rightarrow R^n$ and $h : R^n \rightarrow R$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.

4.49 Show that the system

$$\dot{x}_1 = -ax_1 + b, \qquad \dot{x}_2 = -cx_2 + x_1(\alpha - \beta x_1 x_2)$$

where all coefficients are positive, has a globally exponentially stable equilibrium point.
Hint: Shift the equilibrium point to the origin and use V of the form $V = k_1Y_1^2 + k_2y_2^2 + k_3y_1^4$, where (y_1, y_2) are the new coordinates.

4.50 Consider the system

$$\dot{x} = f(t, x); \quad f(t, 0) = 0$$

where $[\partial f/\partial x]$ is bounded and Lipschitz in x in a neighborhood of the origin, uniformly in t for all $t \geq t_0 \geq 0$. Suppose that the origin of the linearization at $x = 0$ is exponentially stable, and the solutions of the system satisfy

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \geq t_0 \geq 0, \quad \forall\ \|x(t_0)\| < c \tag{4.59}$$

for some class $\mathcal{KL}$ function β and some positive constant c.

(a) Show that there is a class $\mathcal{K}$ function α and a positive constant γ such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \exp[-\gamma(t - t_0)], \quad \forall\ t \geq t_0, \quad \forall\ \|x(t_0)\| < c$$

(b) Show that there is a positive constant M, possibly dependent on c, such that

$$\|x(t)\| \leq M\|x(t_0)\| \exp[-\gamma(t - t_0)], \quad \forall\ t \geq t_0, \quad \forall\ \|x(t_0)\| < c \tag{4.60}$$

(c) If inequality (4.59) holds globally, can you state inequality (4.60) globally?

4.51 Suppose the assumptions of Theorem 4.18 are satisfied with $\alpha_1(r) = k_1 r^a$, $\alpha_2(r) = k_2 r^a$, and $W(x) \geq k_3\|x\|^a$, for some positive constants k_1, k_2, k_3, and a. Show that (4.42) and (4.43) are satisfied with $\beta(r, s) = kr\exp(-\gamma s)$ and $\alpha_1^{-1}(\alpha_2(\mu)) = k\mu$, where $k = (k_2/k_1)^{1/a}$ and $\gamma = k_3/(k_2 a)$.

4.52 Consider Theorem 4.18 when $V(t, x) = V(x)$ and suppose inequality (4.40) is replaced by

$$\frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall\ W_4(x) \geq \mu > 0$$

for some continuous positive definite functions $W_3(x)$ and $W_4(x)$. Show that (4.42) and (4.43) hold for every initial state $x(t_0) \in \{V(x) \leq c\} \subset D$, provided $\{V(x) \leq c\}$ is compact and $\max_{W_4(x) \leq \mu} V(x) < c$.

4.53 ([72]) Consider the system $\dot{x} = f(t, x)$ and suppose there is a function $V(t, x)$ that satisfies

$$W_1(x) \leq V(t, x) \leq W_2(x), \quad \forall\ \|x\| \geq r > 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) < 0, \quad \forall\ \|x\| \geq r_1 \geq r$$

where $W_1(x)$ and $W_2(x)$ are continuous, positive definite functions. Show that the solutions of the system are uniformly bounded.
Hint: Notice that $V(t, x)$ is not necessarily positive definite.

4.54 For each of the following scalar systems, investigate input-to-state stability:

$$(1)\ \dot{x} = -(1+u)x^3 \qquad (2)\ \dot{x} = -(1+u)x^3 - x^5$$

$$(3)\ \dot{x} = -x + x^2 u \qquad (4)\ \dot{x} = x - x^3 + u$$

4.55 For each of the following systems, investigate input-to-state stability:

$$\begin{array}{llllll}
(1) & \dot{x}_1 & = & -x_1 + x_1^2 x_2, & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(2) & \dot{x}_1 & = & -x_1 + x_2, & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(3) & \dot{x}_1 & = & x_2, & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(4) & \dot{x}_1 & = & (x_1 - x_2 + u)(x_1^2 - 1), & \dot{x}_2 & = & (x_1 + x_2 + u)(x_1^2 - 1) \\
(5) & \dot{x}_1 & = & -x_1 + x_1^2 x_2, & \dot{x}_2 & = & -x_2 + x_1 + u \\
(6) & \dot{x}_1 & = & -x_1 - x_2 + u_1, & \dot{x}_2 & = & x_1 - x_2^3 + u_2 \\
(7) & \dot{x}_1 & = & -x_1 + x_2, & \dot{x}_2 & = & -x_1 - \sigma(x_1) - x_2 + u
\end{array}$$

where σ is a locally Lipschitz function, $\sigma(0) = 0$, and $y\sigma(y) \geq 0$ for all $y \neq 0$.

4.56 Using Lemma 4.7, show that the origin of the system

$$\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_2^3$$

is globally asymptotically stable.

4.57 Prove another version of Theorem 4.19, where all the assumptions are the same except that inequality (4.49) is replaced by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_3(\|x\|) + \psi(u)$$

where α_3 is a class $\mathcal{K}_\infty$ function and $\psi(u)$ is a continuous function of u with $\psi(0) = 0$.

4.58 Use inequality (4.47) to show that if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$.

4.59 Consider the scalar system $\dot{x} = -x^3 + e^{-t}$. Show that $x(t) \to 0$ as $t \to \infty$.

4.60 Suppose the assumptions of Theorem 4.19 are satisfied for $\|x\| < r$ and $\|u\| < r_u$ with class $\mathcal{K}$ functions α_1 and α_2 that are not necessarily class $\mathcal{K}_\infty$. Show that there exist positive constants k_1 and k_2 such that inequality (4.47) is satisfied for $\|x(t_0)\| < k_1$ and $\sup_{t \geq t_0} \|u(t)\| < k_2$. In this case, the system is said to be *locally input-to-state stable.*

4.61 Consider the system

$$\dot{x}_1 = x_1 \left\{ \left[\sin\left(\frac{\pi x_2}{2} \right) \right]^2 - 1 \right\}, \qquad \dot{x}_2 = -x_2 + u$$

(a) With $u = 0$, show that the origin is globally asymptotically stable.

(b) Show that for any bounded input $u(t)$, the state $x(t)$ is bounded.

(c) With $u(t) \equiv 1$, $x_1(0) = a$, and $x_2(0) = 1$, show that the solution is $x_1(t) \equiv a$, $x_2(t) \equiv 1$.

(d) Is the system input-to-state stable?

In the next seven exercises, we deal with the discrete-time dynamical system[38]

$$x(k+1) = f(x(k)), \quad f(0) = 0 \tag{4.61}$$

The rate of change of a scalar function $V(x)$ along the motion of (4.61) is defined by

$$\Delta V(x) = V(f(x)) - V(x)$$

4.62 Restate Definition 4.1 for the origin of the discrete-time system (4.61).

4.63 Show that the origin of (4.61) is stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite. Show that it is asymptotically stable if, in addition, $\Delta V(x)$ is negative definite. Finally, show that the origin is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.

4.64 Show that the origin of (4.61) is exponentially stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ such that

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \qquad \Delta V(x) \leq -c_3\|x\|^2$$

for some positive constants c_1, c_2, and c_3.
Hint: For discrete-time systems, exponential stability is defined by the inequality $\|x(k)\| \leq \alpha\|x(0)\|\gamma^k$ for all $k \geq 0$, where $\alpha \geq 1$ and $0 < \gamma < 1$.

4.65 Show that the origin of (4.61) is asymptotically stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite and $\Delta V(x)$ does not vanish identically for any $x \neq 0$.

4.66 Consider the linear system $x(k+1) = Ax(k)$. Show that the following statements are equivalent:

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[38] See [95] for a detailed treatment of Lyapunov stability for discrete-time dynamical systems.

(1) $x = 0$ is asymptotically stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) Given any $Q = Q^T > 0$, there exists $P = P^T > 0$, which is the unique solution of the linear equation $A^T PA - P = -Q$.

4.67 Let A be the linearization of (4.61) at the origin; that is, $A = [\partial f/\partial x](0)$. Show that the origin is asymptotically stable if all the eigenvalues of A have magnitudes less than one.

4.68 Let $x = 0$ be an equilibrium point for the nonlinear discrete-time system $x(k+1) = f(x(k))$, where $f : D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\| < r\}$. Let C, $\gamma < 1$, and r_0 be positive constants with $r_0 < r/C$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the solutions of the system satisfy

$$\|x(k)\| \le C\|x(0)\|\gamma^k, \quad \forall\ x(0) \in D_0, \ \forall\ k \ge 0$$

Show that there is a function $V : D_0 \to R$ that satisfies

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\Delta V(x) = V(f(x)) - V(x) \le -c_3\|x\|^2$$

$$|V(x) - V(y)| \le c_4\|x - y\| \left(\|x\| + \|y\|\right)$$

for all $x, y \in D_0$ for some positive constants c_1, c_2, c_3, and c_4.

Chapter 5

Input–Output Stability

In most of this book, we use the state–space approach to model nonlinear dynamical systems and place a lot of emphasis on the behavior of the state variables. An alternative approach to the mathematical modeling of dynamical systems is the input–output approach.[1] An input–output model relates the output of the system directly to the input, with no knowledge of the internal structure that is represented by the state equation. The system is viewed as a black box that can be accessed only through its input and output terminals. In Section 5.1, we introduce input–output mathematical models and define $\mathcal{L}$ stability, a concept of stability in the input–output sense. In Section 5.2, we study $\mathcal{L}$ stability of nonlinear systems represented by state models. In Section 5.3, we discuss the calculation of the $\mathcal{L}_2$ gain for a class of time-invariant systems. Finally, in Section 5.4, we present a version of the small-gain theorem.

5.1 $\mathcal{L}$ Stability

We consider a system whose input–output relation is represented by

$$y = Hu$$

where H is some mapping or operator that specifies y in terms of u. The input u belongs to a space of signals that map the time interval $[0, \infty)$ into the Euclidean space R^m; that is, $u : [0, \infty) \rightarrow R^m$. Examples are the space of piecewise continuous, bounded functions; that is, $\sup_{t\geq 0} \|u(t)\| < \infty$, and the space of piecewise continuous, square-integrable functions; that is, $\int_0^\infty u^T(t)u(t)\ dt < \infty$. To measure the size of a signal, we introduce the norm function $\|u\|$, which satisfies three properties:

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[1] In this chapter, we give just enough of a glimpse of the input–output approach to enable the reader to understand the relationship of Lyapunov stability to input–output stability and to introduce the terminology needed to state the small-gain theorem. For a comprehensive treatment of the subject, the reader may consult [53], [208], or [162]. The foundation of the input–output approach to nonlinear systems can be found in the 1960's work of Sandberg and Zames. (See, for example, [164], [217], and [218].)

- The norm of a signal is zero if and only if the signal is identically zero and is strictly positive otherwise.

- Scaling a signal results in a corresponding scaling of the norm; that is, $\|au\| = a\|u\|$ for any positive constant a and every signal u.

- The norm satisfies the triangle inequality $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$ for any signals u_1 and u_2.

For the space of piecewise continuous, bounded functions, the norm is defined as

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\| < \infty$$

and the space is denoted by $\mathcal{L}_\infty^m$. For the space of piecewise continuous, square-integrable functions, the norm is defined by

$$\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\ dt} < \infty$$

and the space is denoted by $\mathcal{L}_2^m$. More generally, the space $\mathcal{L}_p^m$ for $1 \leq p < \infty$ is defined as the set of all piecewise continuous functions $u : [0,\infty) \to R^m$ such that

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\ dt\right)^{1/p} < \infty$$

The subscript p in $\mathcal{L}_p^m$ refers to the type of p-norm used to define the space, while the superscript m is the dimension of the signal u. If they are clear from the context, we may drop one or both of them and refer to the space simply as $\mathcal{L}_p$, $\mathcal{L}^m$, or $\mathcal{L}$. To distinguish the norm of u as a vector in the space $\mathcal{L}$ from the norm of $u(t)$ as a vector in R^m, we write the first norm as $\|\cdot\|_{\mathcal{L}}$.[2]

If we think of $u \in \mathcal{L}^m$ as a "well-behaved" input, the question to ask is whether the output y will be "well behaved" in the sense that $y \in \mathcal{L}^q$, where $\mathcal{L}^q$ is the same space as $\mathcal{L}^m$, except that the number of output variables q is, in general, different from the number of input variables m. A system having the property that any "well-behaved" input will generate a "well-behaved" output will be defined as a stable system. However, we cannot define H as a mapping from $\mathcal{L}^m$ to $\mathcal{L}^q$, because we have to deal with systems which are unstable, in that an input $u \in \mathcal{L}^m$ may generate an output y that does not belong to $\mathcal{L}^q$. Therefore, H is usually defined as a mapping from an extended space $\mathcal{L}_e^m$ to an extended space $\mathcal{L}_e^q$, where $\mathcal{L}_e^m$ is defined by

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[2] Note that the norm $\|\cdot\|$ used in the definition of $\|\cdot\|_{\mathcal{L}_p}$, for any $p \in [1,\infty]$, can be any p-norm in R^m; the number p is not necessarily the same in the two norms. For example, we may define the $\mathcal{L}_\infty$ space with $\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\|_1$, $\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\|_2$, or $\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\|_\infty$. However, it is common to define the $\mathcal{L}_2$ space with the 2-norm in R^m.

$$\mathcal{L}_e^m = \{u \mid u_\tau \in \mathcal{L}^m, \forall\ \tau \in [0, \infty)\}$$

and u_τ is a truncation of u defined by

$$u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

The extended space $\mathcal{L}_e^m$ is a linear space that contains the unextended space $\mathcal{L}^m$ as a subset. It allows us to deal with unbounded "ever-growing" signals. For example, the signal $u(t) = t$ does not belong to the space $\mathcal{L}_\infty$, but its truncation

$$u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

belongs to $\mathcal{L}_\infty$ for every finite τ. Hence, $u(t) = t$ belongs to the extended space $\mathcal{L}_{\infty e}$.

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is said to be causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t. This is equivalent to

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Causality is an intrinsic property of dynamical systems represented by state models.

With the space of input and output signals defined, we can now define input–output stability.

Definition 5.1 *A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a class $\mathcal{K}$ function α, defined on $[0, \infty)$, and a nonnegative constant β such that*

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \alpha\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta \tag{5.1}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma\|u_\tau\|_{\mathcal{L}} + \beta \tag{5.2}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

The constant β in (5.1) or (5.2) is called the bias term. It is included in the definition to allow for systems where Hu does not vanish at $u = 0$.[3] When inequality (5.2) is satisfied, we are usually interested in the smallest possible γ for which there is β such that (5.2) is satisfied. When this value of γ is well defined, we will call it the gain of the system. When inequality (5.2) is satisfied with some $\gamma \ge 0$, we say that the system has an $\mathcal{L}$ gain less than or equal to γ.

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[3] See Exercise 5.3 for a different role of the bias term.

For causal, $\mathcal{L}$ stable systems, it can be shown by a simple argument that

$$u \in \mathcal{L}^m \Rightarrow Hu \in \mathcal{L}^q$$

and

$$\|Hu\|_{\mathcal{L}} \leq \alpha\left(\|u\|_{\mathcal{L}}\right) + \beta, \quad \forall\ u \in \mathcal{L}^m$$

For causal, finite-gain $\mathcal{L}$ stable systems, the foregoing inequality takes the form

$$\|Hu\|_{\mathcal{L}} \leq \gamma \|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\ u \in \mathcal{L}^m$$

The definition of $\mathcal{L}_\infty$ stability is the familiar notion of bounded-input–bounded-output stability; namely, if the system is $\mathcal{L}_\infty$ stable, then for every bounded input $u(t)$, the output $Hu(t)$ is bounded.

Example 5.1 A memoryless, possibly time-varying, function $h : [0, \infty) \times R \to R$ can be viewed as an operator H that assigns to every input signal $u(t)$ the output signal $y(t) = h(t, u(t))$. We use this simple operator to illustrate the definition of $\mathcal{L}$ stability. Let

$$h(u) = a + b \tanh cu = a + b\ \frac{e^{cu} - e^{-cu}}{e^{cu} + e^{-cu}}$$

for some nonnegative constants a, b, and c. Using the fact

$$h'(u) = \frac{4bc}{\left(e^{cu} + e^{-cu}\right)^2} \leq bc, \quad \forall\ u \in R$$

we have

$$|h(u)| \leq a + bc|u|, \quad \forall\ u \in R$$

Hence, H is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = bc$ and $\beta = a$. Furthermore, if $a = 0$, then for each $p \in [1, \infty)$,

$$\int_0^\infty |h(u(t))|^p\ dt \leq (bc)^p \int_0^\infty |u(t)|^p\ dt$$

Thus, for each $p \in [1, \infty]$, the operator H is finite-gain $\mathcal{L}_p$ stable with zero bias and $\gamma = bc$. Let h be a time-varying function that satisfies

$$|h(t, u)| \leq a|u|, \quad \forall\ t \geq 0,\ \forall\ u \in R$$

for some positive constant a. For each $p \in [1, \infty]$, the operator H is finite-gain $\mathcal{L}_p$ stable with zero bias and $\gamma = a$. Finally, let

$$h(u) = u^2$$

Since

$$\sup_{t \geq 0} |h(u(t))| \leq \left(\sup_{t \geq 0} |u(t)|\right)^2$$

H is $\mathcal{L}_\infty$ stable with zero bias and $\alpha(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because the function $h(u) = u^2$ cannot be bounded by a straight line of the form $|h(u)| \leq \gamma|u| + \beta$, for all $u \in R$. △

Example 5.2 Consider a single-input–single-output system defined by the causal convolution operator

$$y(t) = \int_0^t h(t-\sigma)u(\sigma)\, d\sigma$$

where $h(t) = 0$ for $t < 0$. Suppose $h \in \mathcal{L}_{1e}$; that is, for every $\tau \in [0, \infty)$,

$$\|h_\tau\|_{\mathcal{L}_1} = \int_0^\infty |h_\tau(\sigma)|\, d\sigma = \int_0^\tau |h(\sigma)|\, d\sigma < \infty$$

If $u \in \mathcal{L}_{\infty e}$ and $\tau \geq t$, then

$$\begin{aligned} |y(t)| &\leq \int_0^t |h(t-\sigma)|\, |u(\sigma)|\, d\sigma \\ &\leq \int_0^t |h(t-\sigma)|\, d\sigma \sup_{0\leq\sigma\leq\tau} |u(\sigma)| = \int_0^t |h(s)|\, ds \sup_{0\leq\sigma\leq\tau} |u(\sigma)| \end{aligned}$$

Consequently,

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h_\tau\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \quad \forall\ \tau \in [0, \infty)$$

This inequality resembles (5.2), but it is not the same as (5.2) because the constant γ in (5.2) is required to be independent of τ. While $\|h_\tau\|_{\mathcal{L}_1}$ is finite for every finite τ, it may not be bounded uniformly in τ. For example, $h(t) = e^t$ has $\|h_\tau\|_{\mathcal{L}_1} = (e^\tau - 1)$, which is finite for all $\tau \in [0, \infty)$ but not uniformly bounded in τ. Inequality (5.2) will be satisfied if $h \in \mathcal{L}_1$; that is,

$$\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)|\, d\sigma < \infty$$

Then, the inequality

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \quad \forall\ \tau \in [0, \infty)$$

shows that the system is finite-gain $\mathcal{L}_\infty$ stable. The condition $\|h\|_{\mathcal{L}_1} < \infty$ actually guarantees finite-gain $\mathcal{L}_p$ stability for each $p \in [1, \infty]$. Consider first the case $p = 1$. For $t \leq \tau < \infty$, we have

$$\int_0^\tau |y(t)|\, dt = \int_0^\tau \left| \int_0^t h(t-\sigma)u(\sigma)\, d\sigma \right| dt \leq \int_0^\tau \int_0^t |h(t-\sigma)|\, |u(\sigma)|\, d\sigma\, dt$$

Reversing the order of integration yields

$$\int_0^\tau |y(t)|\, dt \leq \int_0^\tau |u(\sigma)| \int_\sigma^\tau |h(t-\sigma)|\, dt\, d\sigma \leq \int_0^\tau |u(\sigma)|\, \|h\|_{\mathcal{L}_1}\, d\sigma \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_1}$$

Thus,

$$\|y_\tau\|_{\mathcal{L}_1} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_1}, \quad \forall\ \tau \in [0, \infty)$$

Consider now the case $p \in (1,\infty)$ and let $q \in (1,\infty)$ be defined by $1/p + 1/q = 1$. For $t \le \tau < \infty$, we have

$$\begin{aligned}
|y(t)| &\le \int_0^t |h(t-\sigma)|\ |u(\sigma)|\ d\sigma \\
&= \int_0^t |h(t-\sigma)|^{1/q} |h(t-\sigma)|^{1/p} |u(\sigma)|\ d\sigma \\
&\le \left(\int_0^t |h(t-\sigma)|\ d\sigma\right)^{1/q} \left(\int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma\right)^{1/p} \\
&\le (\|h_\tau\|_{\mathcal{L}_1})^{1/q} \left(\int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma\right)^{1/p}
\end{aligned}$$

where the second inequality is obtained by applying Hölder's inequality.[4] Thus,

$$\begin{aligned}
\left(\|y_\tau\|_{\mathcal{L}_p}\right)^p &= \int_0^\tau |y(t)|^p\ dt \\
&\le \int_0^\tau (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \left(\int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma\right)\ dt \\
&= (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \int_0^\tau \int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma\ dt
\end{aligned}$$

By reversing the order of integration, we obtain

$$\begin{aligned}
\left(\|y_\tau\|_{\mathcal{L}_p}\right)^p &\le (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \int_0^\tau |u(\sigma)|^p \int_\sigma^\tau |h(t-\sigma)|\ dt\ d\sigma \\
&\le (\|h_\tau\|_{\mathcal{L}_1})^{p/q}\ \|h_\tau\|_{\mathcal{L}_1} \left(\|u_\tau\|_{\mathcal{L}_p}\right)^p\ =\ (\|h_\tau\|_{\mathcal{L}_1})^p \left(\|u_\tau\|_{\mathcal{L}_p}\right)^p
\end{aligned}$$

Hence,

$$\|y_\tau\|_{\mathcal{L}_p} \le \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_p}$$

In summary, if $\|h\|_{\mathcal{L}_1} < \infty$, then for each $p \in [1,\infty]$, the causal convolution operator is finite-gain $\mathcal{L}_p$ stable and (5.2) is satisfied with $\gamma = \|h\|_{\mathcal{L}_1}$ and $\beta = 0$. △

One drawback of Definition 5.1 is the implicit requirement that inequality (5.1) or inequality (5.2) be satisfied for all signals in the input space $\mathcal{L}^m$. This excludes systems where the input–output relation may be defined only for a subset of the input space. The next example explores the point and motivates the definition of small-signal $\mathcal{L}$ stability that follows the example.

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[4] Hölder's inequality states that if $f \in \mathcal{L}_{pe}$ and $g \in \mathcal{L}_{qe}$, where $p \in (1,\infty)$ and $1/p + 1/q = 1$, then

$$\int_0^\tau |f(t)g(t)|\ dt \le \left(\int_0^\tau |f(t)|^p\ dt\right)^{1/p} \left(\int_0^\tau |g(t)|^q\ dt\right)^{1/q}$$

for every $\tau \in [0,\infty)$. (See [14].)

Example 5.3 Consider a single-input–single-output system defined by the nonlinearity

$$y = \tan u$$

The output $y(t)$ is defined only when the input signal is restricted to

$$|u(t)| < \frac{\pi}{2}, \quad \forall\, t \geq 0$$

Thus, the system is not $\mathcal{L}_\infty$ stable in the sense of Definition 5.1. However, if we restrict $u(t)$ to the set

$$|u| \leq r < \frac{\pi}{2}$$

then

$$|y| \leq \left(\frac{\tan r}{r}\right) |u|$$

and the system will satisfy the inequality

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\tan r}{r}\right) \|u\|_{\mathcal{L}_p}$$

for every $u \in \mathcal{L}_p$ such that $|u(t)| \leq r$ for all $t \geq 0$, where p could be any number in $[1, \infty]$. In the space $\mathcal{L}_\infty$, the requirement $|u(t)| \leq r$ implies that $\|u\|_{\mathcal{L}_\infty} \leq r$, showing that the foregoing inequality holds only for input signals of small norm. However, for other $\mathcal{L}_p$ spaces with $p < \infty$ the instantaneous bound on $|u(t)|$ does not necessarily restrict the norm of the input signal. For example, the signal

$$u(t) = re^{-rt/a}, \quad a > 0$$

which belongs to $\mathcal{L}_p$ for each $p \in [1, \infty]$, satisfies the instantaneous bound $|u(t)| \leq r$ while its $\mathcal{L}_p$ norm

$$\|u\|_{\mathcal{L}_p} = r\left(\frac{a}{rp}\right)^{1/p}, \quad 1 \leq p < \infty$$

can be arbitrarily large. △

Definition 5.2 *A mapping* $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ *is small-signal* $\mathcal{L}$ *stable (respectively, small-signal finite-gain* $\mathcal{L}$ *stable) if there is a positive constant* r *such that inequality* (5.1) *[respectively, inequality* (5.2)*] is satisfied for all* $u \in \mathcal{L}_e^m$ *with* $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$.

5.2 $\mathcal{L}$ Stability of State Models

The notion of input–output stability is intuitively appealing. This is probably why most of us were introduced to dynamical system stability in the framework of bounded-input–bounded-output stability. Since, in Lyapunov stability, we put a lot of emphasis on studying the stability of equilibrium points and the asymptotic behavior of state variables, one may wonder: What can we see about input–output stability starting from the formalism of Lyapunov stability? In this section, we show how Lyapunov stability tools can be used to establish $\mathcal{L}$ stability of nonlinear systems represented by state models.

Consider the system

$$\begin{aligned} \dot{x} &= f(t,x,u), \quad x(0)=x_0 && (5.3) \\ y &= h(t,x,u) && (5.4) \end{aligned}$$

where $x \in R^n$, $u \in R^m$, $y \in R^q$, $f : [0,\infty) \times D \times D_u \rightarrow R^n$ is piecewise continuous in t and locally Lipschitz in (x,u), $h : [0,\infty) \times D \times D_u \rightarrow R^q$ is piecewise continuous in t and continuous in (x,u), $D \subset R^n$ is a domain that contains $x=0$, and $D_u \subset R^m$ is a domain that contains $u=0$. For each fixed $x_0 \in D$, the state model given by (5.3) and (5.4) defines an operator H that assigns to each input signal $u(t)$ the corresponding output signal $y(t)$. Suppose $x=0$ is an equilibrium point of the unforced system

$$\dot{x} = f(t,x,0) \tag{5.5}$$

The theme of this section is that if the origin of (5.5) is uniformly asymptotically stable (or exponentially stable), then, under some assumptions on f and h, the system (5.3) and (5.4) will be $\mathcal{L}$ stable or small-signal $\mathcal{L}$ stable for a certain signal space $\mathcal{L}$. We pursue this idea first in the case of exponentially stability, and then for the more general case of uniform asymptotic stability.

Theorem 5.1 *Consider the system* (5.3)–(5.4) *and take* $r > 0$ *and* $r_u > 0$ *such that* $\{\|x\| \leq r\} \subset D$ *and* $\{\|u\| \leq r_u\} \subset D_u$. *Suppose that*

- $x = 0$ *is an exponentially stable equilibrium point of* (5.5), *and there is a Lyapunov function V(t,x) that satisfies*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \tag{5.6}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \leq -c_3\|x\|^2 \tag{5.7}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| \tag{5.8}$$

for all $(t,x) \in [0,\infty) \times D$ *for some positive constants* c_1, c_2, c_3, *and* c_4.

- f *and* h *satisfy the inequalities*

$$\|f(t,x,u) - f(t,x,0)\| \leq L\|u\| \tag{5.9}$$

$$\|h(t,x,u)\| \leq \eta_1\|x\| + \eta_2\|u\| \tag{5.10}$$

for all $(t,x,u) \in [0,\infty) \times D \times D_u$ *for some nonnegative constants* L, η_1, *and* η_2.

Then, for each x_0 *with* $\|x_0\| \le r\sqrt{c_1/c_2}$, *the system* (5.3)–(5.4) *is small-signal finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. *In particular, for each* $u \in \mathcal{L}_{pe}$ *with* $\sup_{0\le t\le \tau} \|u(t)\| \le \min\{r_u, c_1c_3r/(c_2c_4L)\}$, *the output* $y(t)$ *satisfies*

$$\|y_\tau\|_{\mathcal{L}_p} \le \gamma\|u_\tau\|_{\mathcal{L}_p} + \beta \tag{5.11}$$

for all $\tau \in [0, \infty)$, *with*

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \quad \beta = \eta_1\|x_0\|\sqrt{\frac{c_2}{c_1}}\rho, \text{ where } \rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{2c_2}{c_3 p}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}$$

Furthermore, if the origin is globally exponentially stable and all the assumptions hold globally (*with* $D = R^n$ *and* $D_u = R^m$), *then, for each* $x_0 \in R^n$, *the system* (5.3)–(5.4) *is finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. ◇

Proof: The derivative of V along the trajectories of (5.3) satisfies

$$\begin{aligned} \dot{V}(t,x,u) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\le -c_3\|x\|^2 + c_4L\|x\|\,\|u\| \end{aligned}$$

Take $W(t) = \sqrt{V(t,x(t))}$. When $V(t,x(t)) \ne 0$, use $\dot{W} = \dot{V}/(2\sqrt{V})$ and (5.6) to obtain

$$\dot{W} \le -\frac{1}{2}\left(\frac{c_3}{c_2}\right)W + \frac{c_4L}{2\sqrt{c_1}}\|u(t)\|$$

When $V(t,x(t)) = 0$, it can be verified[5] that

$$D^+W(t) \le \frac{c_4L}{2\sqrt{c_1}}\|u(t)\| \tag{5.12}$$

Hence,

$$D^+W(t) \le -\frac{1}{2}\left(\frac{c_3}{c_2}\right)W + \frac{c_4L}{2\sqrt{c_1}}\|u(t)\|$$

for all values of $V(t,x(t))$. By (the comparison) Lemma 3.4, $W(t)$ satisfies the inequality

$$W(t) \le e^{-tc_3/2c_2}W(0) + \frac{c_4L}{2\sqrt{c_1}}\int_0^t e^{-(t-\tau)c_3/2c_2}\|u(\tau)\|\,d\tau$$

Using (5.6), we obtain

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x_0\|e^{-tc_3/2c_2} + \frac{c_4L}{2c_1}\int_0^t e^{-(t-\tau)c_3/2c_2}\|u(\tau)\|\,d\tau$$

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[5] See Exercise 5.6.

It can be easily verified that

$$\|x_0\| \leq r\sqrt{\frac{c_1}{c_2}} \quad \text{and} \quad \sup_{0\leq\sigma\leq t} \|u(\sigma)\| \leq \frac{c_1 c_3 r}{c_2 c_4 L}$$

ensure that $\|x(t)\| \leq r$; hence, $x(t)$ stays within the domain of validity of the assumptions. Using (5.10), we have

$$\|y(t)\| \leq k_1 e^{-at} + k_2 \int_0^t e^{-a(t-\tau)} \|u(\tau)\| \, d\tau + k_3 \; \|u(t)\| \tag{5.13}$$

where

$$k_1 = \sqrt{\frac{c_2}{c_1}}\|x_0\|\eta_1, \quad k_2 = \frac{c_4 L \eta_1}{2c_1}, \quad k_3 = \eta_2, \quad a = \frac{c_3}{2c_2}$$

Set

$$y_1(t) = k_1 e^{-at}, \quad y_2(t) = k_2 \int_0^t e^{-a(t-\tau)} \|u(\tau)\| \, d\tau, \quad y_3(t) = k_3 \; \|u(t)\|$$

Suppose now that $u \in \mathcal{L}_{pe}^m$ for some $p \in [1, \infty]$. Using the results of Example 5.2, it can be easily verified that

$$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \frac{k_2}{a} \|u_\tau\|_{\mathcal{L}_p}$$

It is also straightforward to see that

$$\|y_{3\tau}\|_{\mathcal{L}_p} \leq k_3 \|u_\tau\|_{\mathcal{L}_p}$$

As for the first term, $y_1(t)$, it can be verified that

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq k_1 \rho, \quad \text{where} \quad \rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{1}{ap}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}$$

Thus, by the triangle inequality, (5.11) is satisfied with

$$\gamma = k_3 + \frac{k_2}{a}, \quad \beta = k_1 \rho$$

When all the assumptions hold globally, there is no need to restrict $\|x_0\|$ or the instantaneous values of $\|u(t)\|$. Therefore, (5.11) is satisfied for each $x_0 \in R^n$ and $u \in \mathcal{L}_{pe}$. □

The use of (the converse Lyapunov) Theorem 4.14 shows the existence of a Lyapunov function satisfying (5.6) through (5.8). Consequently, we have the following corollary.

Corollary 5.1 *Suppose that, in some neighborhood of* $(x = 0, u = 0)$, *the function* $f(t, x, u)$ *is continuously differentiable, the Jacobian matrices* $[\partial f/\partial x]$ *and* $[\partial f/\partial u]$ *are bounded, uniformly in* t, *and* $h(t, x, u)$ *satisfies* (5.10). *If the origin* $x = 0$ *is an exponentially stable equilibrium point of* (5.5), *then there is a constant* $r_0 > 0$ *such that for each* x_0 *with* $\|x_0\| < r_0$, *the system* (5.3)–(5.4) *is small-signal finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. *Furthermore, if all the assumptions hold globally and the origin* $x = 0$ *is a globally exponentially stable equilibrium point of* (5.5), *then for each* $x_0 \in R^n$, *the system* (5.3)–(5.4) *is finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. $\diamond$

For the linear time-invariant system

$$\dot{x} = Ax + Bu, \tag{5.14}$$

$$y = Cx + Du \tag{5.15}$$

the global exponential stability condition of Theorem 5.1 is equivalent to the condition that A is Hurwitz. Thus, we have the following result for linear systems:

Corollary 5.2 *The linear time-invariant system* (5.14)–(5.15) *is finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$ *if* A *is Hurwitz. Moreover,* (5.11) *is satisfied with*

$$\gamma = \|D\|_2 + \frac{2\lambda^2_{max}(P)\|B\|_2\|C\|_2}{\lambda_{min}(P)}, \quad \beta = \rho\|C\|_2\|x_0\|\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$$

where

$$\rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{2\lambda_{max}(P)}{p}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}$$

and P *is the solution of the Lyapunov equation* $PA + A^TP = -I$. $\diamond$

We leave it for the reader to derive the foregoing expressions for γ and β.

Example 5.4 Consider the single-input–single-output, first-order system

$$\begin{aligned} \dot{x} &= -x - x^3 + u, \quad x(0) = x_0 \\ y &= \tanh x + u \end{aligned}$$

The origin of $\dot{x} = -x - x^3$ is globally exponentially stable, as can be seen by the Lyapunov function $V(x) = x^2/2$. The function V satisfies (5.6) through (5.8) globally with $c_1 = c_2 = 1/2$, $c_3 = c_4 = 1$. The functions f and h satisfy (5.9) and (5.10) globally with $L = \eta_1 = \eta_2 = 1$. Hence, for each $x_0 \in R$ and each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable. $\triangle$

Example 5.5 Consider the single-input–single-output second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - a\tanh x_1 + u \\ y &= x_1 \end{aligned}$$

where a is a nonnegative constant. Use

$$V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

as a Lyapunov function candidate for the unforced system:

$$\dot{V} = -2p_{12}(x_1^2 + a x_1 \tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2ap_{22}x_2\tanh x_1 - 2(p_{22} - p_{12})x_2^2$$

Choose $p_{11} = p_{12} + p_{22}$ to cancel the cross-product term x_1x_2. Then, taking $p_{22} = 2p_{12} = 1$ makes P positive definite and results in

$$\dot{V} = -x_1^2 - x_2^2 - ax_1\tanh x_1 - 2ax_2\tanh x_1$$

Using the fact that $x_1 \tanh x_1 \geq 0$ for all $x_1 \in R$, we obtain

$$\dot{V} \leq -\|x\|_2^2 + 2a|x_1|\;|x_2| \leq -(1-a)\|x\|_2^2$$

Thus, for all $a < 1$, V satisfies (5.6) through (5.8) globally with $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, $c_3 = 1 - a$, and $c_4 = 2\|P\|_2 = 2\lambda_{\max}(P)$. The functions f and h satisfy (5.9) and (5.10) globally with $L = \eta_1 = 1$, $\eta_2 = 0$. Hence, for each $x_0 \in R$ and each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable. △

We turn now to the more general case when the origin of (5.5) is uniformly asymptotically stable and restrict our attention to the study of $\mathcal{L}_\infty$ stability. The next two theorems give conditions for small-signal $\mathcal{L}_\infty$ stability and $\mathcal{L}_\infty$ stability, respectively

Theorem 5.2 *Consider the system* (5.3)–(5.4) *and take* $r > 0$ *such that* $\{\|x\| \leq r\} \subset D$. *Suppose that*

- $x = 0$ *is a uniformly asymptotically stable equilibrium point of* (5.5), *and there is a Lyapunov function V(t,x) that satisfies*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{5.16}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \leq -\alpha_3(\|x\|) \tag{5.17}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|) \tag{5.18}$$

for all $(t,x) \in [0,\infty) \times D$ *for some class* $\mathcal{K}$ *functions* α_1 *to* α_4.

- *f and h satisfy the inequalities*

$$\|f(t,x,u) - f(t,x,0)\| \le \alpha_5(\|u\|) \tag{5.19}$$

$$\|h(t,x,u)\| \le \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta \tag{5.20}$$

for all $(t,x,u) \in [0,\infty) \times D \times D_u$ for some class $\mathcal{K}$ functions α_5 to α_7, and a nonnegative constant η.

Then, for each x_0 with $\|x_0\| \le \alpha_2^{-1}(\alpha_1(r))$, the system (5.3)–(5.4) is small-signal $\mathcal{L}_\infty$ stable. ◇

Proof: The derivative of V along the trajectories of (5.3) satisfies

$$\begin{aligned}\dot{V}(t,x,u) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\alpha_5(\|u\|) \\ &\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \alpha_4(r)\alpha_5\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\end{aligned}$$

where $0 < \theta < 1$. Set

$$\mu = \alpha_3^{-1}\left(\frac{\alpha_4(r)\alpha_5\left(\sup_{0\le t\le\tau}\|u(t)\|\right)}{\theta}\right)$$

and choose $r_u > 0$ small enough that $\{\|u\| \le r_u\} \subset D_u$ and $\mu < \alpha_2^{-1}(\alpha_1(r))$ for $\sup_{0\le t\le\tau}\|u(t)\| \le r_u$. Then,

$$\dot{V} \le -(1-\theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \ge \mu$$

By applying Theorem 4.18, we conclude from (4.42) and (4.43) that $\|x(t)\|$ satisfies the inequality

$$\|x(t)\| \le \beta(\|x_0\|, t) + \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right) \tag{5.21}$$

for all $0 \le t \le \tau$, where β and γ are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively. Using (5.20), we obtain

$$\begin{aligned}\|y(t)\| &\le \alpha_6\left(\beta(\|x_0\|,t) + \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right) + \alpha_7(\|u(t)\|) + \eta \\ &\le \alpha_6\left(2\beta(\|x_0\|,t)\right) + \alpha_6\left(2\gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right) + \alpha_7(\|u(t)\|) + \eta\end{aligned}$$

where we used the general property of class $\mathcal{K}$ functions [6]

$$\alpha(a+b) \le \alpha(2a) + \alpha(2b)$$

---

[6] See Exercise 4.35.

Thus,

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \gamma_0 \left(\|u_\tau\|_{\mathcal{L}_\infty}\right) + \beta_0 \tag{5.22}$$

where

$$\gamma_0 = \alpha_6 \circ 2\gamma + \alpha_7 \quad \text{and} \quad \beta_0 = \alpha_6(2\beta(\|x_0\|, 0)) + \eta$$

□

The use of (the converse Lyapunov) Theorem 4.16 shows the existence of a Lyapunov function satisfying (5.16) through (5.18). Consequently, we have the following corollary:

Corollary 5.3 *Suppose that, in some neighborhood of* $(x = 0, u = 0)$, *the function* $f(t, x, u)$ *is continuously differentiable, the Jacobian matrices* $[\partial f/\partial x]$ *and* $[\partial f/\partial u]$ *are bounded, uniformly in* t, *and* $h(t, x, u)$ *satisfies* (5.20). *If the unforced system* (5.5) *has a uniformly asymptotically stable equilibrium point at the origin* $x = 0$, *then the system* (5.3)–(5.4) *is small-signal* $\mathcal{L}_\infty$ *stable.* ◇

To extend the proof of Theorem 5.2 to show $\mathcal{L}_\infty$ stability, we need to demonstrate that (5.21) holds for any initial state $x_0 \in R^n$ and any bounded input. As we discussed in Section 4.9, such inequality will not automatically hold when the conditions of Theorem 5.2 are satisfied globally, even when the origin of (5.5) is globally uniformly asymptotically stable. However, it will follow from input-to-state stability of the system (5.3), which can be checked using Theorem 4.19.

Theorem 5.3 *Consider the system* (5.3)–(5.4) *with* $D = R^n$ *and* $D_u = R^m$. *Suppose that*

- *The system* (5.3) *is input-to-state stable.*
- h *satisfies the inequality*

  $$\|h(t, x, u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta \tag{5.23}$$

  for all $(t, x, u) \in [0, \infty) \times R^n \times R^m$ *for some class* $\mathcal{K}$ *functions* α_1, α_2, *and a nonnegative constant* η.

Then, for each $x_0 \in R^n$, *the system* (5.3)–(5.4) *is* $\mathcal{L}_\infty$ *stable.* ◇

Proof: Input-to-state stability shows that an inequality similar to (5.21) holds for any $x_0 \in R^n$ and any $u \in \mathcal{L}_{\infty e}$. The rest of the proof is the same as that of Theorem 5.2. □

Example 5.6 Consider the single-input–single-output first-order system

$$\begin{aligned} \dot{x} &= -x - 2x^3 + (1 + x^2)u^2 \\ y &= x^2 + u \end{aligned}$$

We saw in Example 4.26 that the state equation is input-to-state stable. The output function h satisfies (5.23) globally with $\alpha_1(r) = r^2$, $\alpha_2(r) = r$, and $\eta = 0$. Thus, the system is $\mathcal{L}_\infty$ stable. $\triangle$

Example 5.7 Consider the single-input–single-output second-order system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + g(t)x_2 \\ \dot{x}_2 &= -g(t)x_1 - x_2^3 + u \\ y &= x_1 + x_2 \end{aligned}$$

where $g(t)$ is continuous and bounded for all $t \geq 0$. Taking $V = (x_1^2 + x_2^2)$, we have

$$\dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$$

Using

$$x_1^4 + x_2^4 \geq \tfrac{1}{2}\|x\|_2^4$$

yields

$$\begin{aligned} \dot{V} &\leq -\|x\|_2^4 + 2\|x\|_2|u| \\ &= -(1-\theta)\|x\|_2^4 - \theta\|x\|_2^4 + 2\|x\|_2|u|, \quad 0 < \theta < 1 \\ &\leq -(1-\theta)\|x\|_2^4, \quad \forall \ \|x\|_2 \geq \left(\frac{2|u|}{\theta}\right)^{1/3} \end{aligned}$$

Thus, V satisfies inequalities (4.48) and (4.49) of Theorem 4.19 globally, with $\alpha_1(r) = \alpha_2(r) = r^2$, $W_3(x) = -(1-\theta)\|x\|_2^4$, and $\rho(r) = (2r/\theta)^{1/3}$. Hence, the state equation is input-to-state stable. Furthermore, the function $h = x_1 + x_2$ satisfies (5.23) globally with $\alpha_1(r) = \sqrt{2}r$, $\alpha_2 = 0$, and $\eta = 0$. Thus, the system is $\mathcal{L}_\infty$ stable. $\triangle$

5.3 $\mathcal{L}_2$ Gain

$\mathcal{L}_2$ stability plays a special role in systems analysis. It is natural to work with square-integrable signals, which can be viewed as finite-energy signals.[7] In many control problems,[8] the system is represented as an input–output map, from a disturbance input u to a controlled output y, which is required to be small. With $\mathcal{L}_2$ input signals, the control system is designed to make the input–output map finite-gain $\mathcal{L}_2$ stable and to minimize the $\mathcal{L}_2$ gain. In such problems, it is important not only to be able to find out that the system is finite-gain $\mathcal{L}_2$ stable, but also to calculate the $\mathcal{L}_2$ gain or an upper bound on it. In this section, we show how to calculate the $\mathcal{L}_2$ gain for a special class of time-invariant systems. We start with linear systems.

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[7] If you think of $u(t)$ as current or voltage, then $u^T(t)u(t)$ is proportional to the instantaneous power of the signal, and its integral over all time is a measure of the energy content of the signal.

[8] See the literature on H_∞ control; for example, [20], [54], [61], [90], [199], or [219].

Theorem 5.4 *Consider the linear time-invariant system*

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad (5.24) \\ y &= Cx + Du \qquad (5.25) \end{aligned}$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$. Then, the $\mathcal{L}_2$ gain of the system is $\sup_{\omega \in R} \|G(j\omega)\|_2$.[9] ◇

Proof: Due to linearity, we set $x(0) = 0$. From Fourier transform theory,[10] we know that for a causal signal $y \in \mathcal{L}_2$, the Fourier transform $Y(j\omega)$ is given by

$$Y(j\omega) = \int_0^\infty y(t)e^{-j\omega t}\, dt$$

and

$$Y(j\omega) = G(j\omega)U(j\omega)$$

Using Parseval's theorem,[11] we can write

$$\begin{aligned} \|y\|_{\mathcal{L}_2}^2 &= \int_0^\infty y^T(t)y(t)\, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty Y^*(j\omega)Y(j\omega)\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty U^*(j\omega)G^T(-j\omega)G(j\omega)U(j\omega)\, d\omega \\ &\le \left(\sup_{\omega\in R}\|G(j\omega)\|_2\right)^2 \frac{1}{2\pi}\int_{-\infty}^\infty U^*(j\omega)U(j\omega)\, d\omega \\ &= \left(\sup_{\omega\in R}\|G(j\omega)\|_2\right)^2 \|u\|_{\mathcal{L}_2}^2 \end{aligned}$$

which shows that the $\mathcal{L}_2$ gain is less than or equal to $\sup_{\omega\in R}\|G(j\omega)\|_2$. Showing that the $\mathcal{L}_2$ gain is equal to $\sup_{\omega\in R}\|G(j\omega)\|_2$ is done by a contradiction argument that is given in Appendix C.10. □

The case of linear time-invariant systems is exceptional in that we can actually find the exact $\mathcal{L}_2$ gain. In more general cases, like the case of the next theorem, we can only find an upper bound on the $\mathcal{L}_2$ gain.

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[9] This is the induced 2-norm of the complex matrix $G(j\omega)$, which is equal to $\sqrt{\lambda_{\max}[G^T(-j\omega)G(j\omega)]} = \sigma_{\max}[G(j\omega)]$. This quantity is known as the H_∞ norm of $G(j\omega)$, when $G(j\omega)$ is viewed as an element of the Hardy space H_∞. (See [61].)

[10] See [53].

[11] Parseval's theorem [53] states that for a causal signal $y \in \mathcal{L}_2$,

$$\int_0^\infty y^T(t)y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^\infty Y^*(j\omega)Y(j\omega)\, d\omega$$

Theorem 5.5 *Consider the time-invariant nonlinear system*

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \quad x(0) = x_0 && (5.26) \\ y &= h(x) && (5.27) \end{aligned}$$

where $f(x)$ is locally Lipschitz, and $G(x)$, $h(x)$ are continuous over R^n. The matrix G is $n \times m$ and $h : R^n \rightarrow R^q$. The functions f and h vanish at the origin; that is, $f(0) = 0$ and $h(0) = 0$. Let γ be a positive number and suppose there is a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the inequality

$$\mathcal{H}(V, f, G, h, \gamma) \overset{\text{def}}{=} \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \le 0 \quad (5.28)$$

for all $x \in R^n$. Then, for each $x_0 \in R^n$, the system (5.26)–(5.27) *is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.* ◇

Proof: By completing the squares, we have

$$\begin{aligned} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u &= -\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|_2^2 + \frac{\partial V}{\partial x} f(x) \\ &\quad + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}\gamma^2 \|u\|_2^2 \end{aligned}$$

Substituting (5.28) yields

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \le \frac{1}{2}\gamma^2 \|u\|_2^2 - \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|_2^2$$

Hence,

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \le \frac{1}{2}\gamma^2 \|u\|_2^2 - \frac{1}{2}\|y\|_2^2 \quad (5.29)$$

Note that the left-hand side of (5.29) is the derivative of V along the trajectories of the system (5.26). Integrating (5.29) yields

$$V(x(\tau)) - V(x_0) \le \frac{1}{2}\gamma^2 \int_0^\tau \|u(t)\|_2^2 \, dt - \frac{1}{2} \int_0^\tau \|y(t)\|_2^2 \, dt$$

where $x(t)$ is the solution of (5.26) for a given $u \in \mathcal{L}_{2e}$. Using $V(x) \ge 0$, we obtain

$$\int_0^\tau \|y(t)\|_2^2 \, dt \le \gamma^2 \int_0^\tau \|u(t)\|_2^2 \, dt + 2V(x_0)$$

Taking the square roots and using the inequality $\sqrt{a^2 + b^2} \le a + b$ for nonnegative numbers a and b, we obtain

$$\|y_\tau\|_{\mathcal{L}_2} \le \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{2V(x_0)} \quad (5.30)$$

which completes the proof. □

Inequality (5.28) is known as the *Hamilton–Jacobi inequality* (or the *Hamilton–Jacobi equation* when $\leq$ is replaced by $=$). The search for a function $V(x)$ that satisfies (5.28) requires basically the solution of a partial differential equation, which might be difficult to solve. If we succeed in finding $V(x)$, we obtain a finite-gain $\mathcal{L}_2$ stability result, which, unlike Theorem 5.1, does not require the origin of the unforced system to be exponentially stable. This point is illustrated by the next example.

Example 5.8 Consider the single-input–single-output system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + u \\ y &= x_2 \end{aligned}$$

where a and k are positive constants. The unforced system is a special case of the class of systems treated in Example 4.9. In that example, we used the energy-like Lyapunov function $V(x) = ax_1^4/4 + x_2^2/2$ to show that the origin is globally asymptotically stable. Using $V(x) = \alpha(ax_1^4/4 + x_2^2/2)$ with $\alpha > 0$ as a candidate for the solution of the Hamilton–Jacobi inequality (5.28), it can be shown that

$$\mathcal{H}(V, f, G, h, \gamma) = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right) x_2^2$$

To satisfy (5.28), we need to choose $\alpha > 0$ and $\gamma > 0$ such that

$$-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \leq 0 \tag{5.31}$$

By simple algebraic manipulation, we can rewrite this inequality as

$$\gamma^2 \geq \frac{\alpha^2}{2\alpha k - 1}$$

Since we are interested in the smallest possible γ, we choose α to minimize the right-hand side of the preceding inequality. The minimum value $1/k^2$ is achieved at $\alpha = 1/k$. Thus, choosing $\gamma = 1/k$, we conclude that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. We note that the conditions of Theorem 5.1 are not satisfied in this example because the origin of the unforced system is not exponentially stable. Linearization at the origin yields the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & -k \end{bmatrix}$$

which is not Hurwitz. △

The idea of the preceding example is generalized in the next one.

Example 5.9 Consider the nonlinear system (5.26)–(5.27), with $m = q$, and suppose there is a continuously differentiable positive semidefinite function $W(x)$ that satisfies[12]

$$\begin{aligned} \frac{\partial W}{\partial x} f(x) &\leq -kh^T(x)h(x), \quad k > 0 && (5.32) \\ \frac{\partial W}{\partial x} G(x) &= h^T(x) && (5.33) \end{aligned}$$

for all $x \in R^n$. Using $V(x) = \alpha W(x)$ with $\alpha > 0$ as a candidate for the solution of the Hamilton–Jacobi inequality (5.28), it can be shown that

$$\mathcal{H}(V, f, G, h, \gamma) = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \right) h^T(x)h(x)$$

To satisfy (5.28), we need to choose $\alpha > 0$ and $\gamma > 0$ such that

$$-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \leq 0$$

This inequality is the same as inequality (5.31) of Example 5.8. By repeating the argument used there, it can be shown that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. △

Example 5.10 Consider the nonlinear system (5.26)–(5.27), with $m = q$, and suppose there is a continuously differentiable positive semidefinite function $W(x)$ that satisfies[13]

$$\begin{aligned} \frac{\partial W}{\partial x} f(x) &\leq 0 && (5.34) \\ \frac{\partial W}{\partial x} G(x) &= h^T(x) && (5.35) \end{aligned}$$

for all $x \in R^n$. The output feedback control

$$u = -ky + v, \quad k > 0$$

results in the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) - kG(x)G^T(x) \left(\frac{\partial W}{\partial x} \right)^T + G(x)v \overset{\text{def}}{=} f_c(x) + G(x)v \\ y &= h(x) = G^T(x) \left(\frac{\partial W}{\partial x} \right)^T \end{aligned}$$

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[12] A system satisfying (5.32) and (5.33) will be defined in the next chapter as an output strictly passive system.

[13] A system satisfying (5.34) and (5.35) will be defined in the next chapter as a passive system. We will come back to this example in Section 6.5 and look at it as a feedback connection of two passive systems.

It can be easily verified that, for the closed-loop system, $W(x)$ satisfies (5.32) and (5.33) of the previous example. Hence, the input–output map from v to y is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. This shows, in essence, that the $\mathcal{L}_2$ gain can be made arbitrarily small by choosing the feedback gain k sufficiently large. △

Example 5.11 Consider the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Suppose there is a positive semidefinite solution P of the Riccati equation

$$PA + A^TP + \frac{1}{\gamma^2}PBB^TP + C^TC = 0 \tag{5.36}$$

for some $\gamma > 0$. Taking $V(x) = (1/2)x^TPx$ and using the expression $[\partial V/\partial x] = x^TP$, it can be easily seen that $V(x)$ satisfies the Hamilton–Jacobi equation

$$\mathcal{H}(V, Ax, B, Cx) = x^TPAx + \frac{1}{2\gamma^2}x^TPB^TBPx + \frac{1}{2}x^TC^TCx = 0$$

Hence, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ. This result gives an alternative method for computing an upper bound on the $\mathcal{L}_2$ gain, as opposed to the frequency-domain calculation of Theorem 5.4. It is interesting to note that the existence of a positive semidefinite solution of (5.36) is a necessary and sufficient condition for the $\mathcal{L}_2$ gain to be less than or equal to γ.[14] △

In Theorem 5.5, we assumed that the assumptions hold globally. It is clear from the proof of the theorem that if the assumptions hold only on a finite domain D, we will still arrive at inequality (5.30) as long as the solution of (5.26) stays in D.

Corollary 5.4 *Suppose the assumptions of Theorem 5.5 are satisfied on a domain $D \subset R^n$ that contains the origin. Then, for any $x_0 \in D$ and any $u \in \mathcal{L}_{2e}$ for which the solution of* (5.26) *satisfies $x(t) \in D$ for all $t \in [0, \tau]$, we have*

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{2V(x_0)}$$

◇

Ensuring that the solution $x(t)$ of (5.26) remains in some neighborhood of the origin, when both $\|x_0\|$ and $sup_{0\leq t\leq\tau}\|u(t)\|$ are sufficiently small, follows from asymptotic stability of the origin of $\dot{x} = f(x)$. This fact is used to show small-signal $\mathcal{L}_2$ stability in the next lemma.

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[14] See [54] for the proof of necessity.

Lemma 5.1 *Suppose the assumptions of Theorem 5.5 are satisfied on a domain* $D \subset R^n$ *that contains the origin,* $f(x)$ *is continuously differentiable, and* $x = 0$ *is an asymptotically stable equilibrium point of* $\dot{x} = f(x)$. *Then, there is* $k_1 > 0$ *such that for each* x_0 *with* $\|x_0\| \le k_1$, *the system* (5.26)–(5.27) *is small-signal finite-gain* $\mathcal{L}_2$ *stable with* $\mathcal{L}_2$ *gain less than or equal to* γ. ◇

Proof: Take $r > 0$ such that $\{\|x\| \le r\} \subset D$. By (the converse Lyapunov) Theorem 4.16, there exist $r_0 > 0$ and a continuously differentiable Lyapunov function $W(x)$ that satisfies

$$\alpha_1(\|x\|) \le W(x) \le \alpha_2(\|x\|)$$

$$\frac{\partial W}{\partial x} f(x) \le -\alpha_3(\|x\|)$$

for all $\|x\| < r_0$, for some class $\mathcal{K}$ functions α_1 to α_3. The derivative of W along the trajectories of (5.26) satisfies

$$\begin{aligned}\dot{W}(x,u) &= \frac{\partial W}{\partial x} f(x,0) + \frac{\partial W}{\partial x}[f(x,u) - f(x,0)] \;\le\; -\alpha_3(\|x\|) + kL\|u\| \\ &\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + kL \sup_{0\le t\le\tau} \|u(t)\| \\ &\le -(1-\theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \ge \alpha_3^{-1}\left(kL \sup_{0\le t\le\tau} \|u(t)\|/\theta\right)\end{aligned}$$

where k is an upper bound on $\|\partial W/\partial x\|$, L is a Lipschitz constant of f with respect to u, and $0 < \theta < 1$. Similar to the proof of Theorem 5.2, we can apply Theorem 4.18 to show that there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ_0, and positive constants k_1 and k_2 such that, for any initial state x_0 with $\|x_0\| \le k_1$ and any input $u(t)$ with $\sup_{0\le t\le\tau} \|u(t)\| \le k_2$, the solution $x(t)$ satisfies

$$\|x(t)\| \le \beta(\|x_0\|, t) + \gamma_0\left(\sup_{0\le t\le\tau} \|u(t)\|\right)$$

for all $0 \le t \le \tau$. Thus, by choosing k_1 and k_2 small enough, we can be sure that $\|x(t)\| \le r$ for all $0 \le t \le \tau$. The lemma follows then from Corollary 5.4. □

To apply Lemma 5.1, we need to check asymptotic stability of the origin of $\dot{x} = f(x)$. This task can be done by using linearization or searching for a Lyapunov function. The next lemma shows that, under certain conditions, we can use the same function V that satisfies the Hamilton–Jacobi inequality (5.28) as a Lyapunov function for showing asymptotic stability.

Lemma 5.2 *Suppose the assumptions of Theorem 5.5 are satisfied on a domain* $D \subset R^n$ *that contains the origin,* $f(x)$ *is continuously differentiable, and no solution of* $\dot{x} = f(x)$ *can stay identically in* $S = \{x \in D \mid h(x) = 0\}$ *other than the trivial solution* $x(t) \equiv 0$. *Then, the origin of* $\dot{x} = f(x)$ *is asymptotically stable and there is* $k_1 > 0$ *such that for each* x_0 *with* $\|x_0\| \le k_1$, *the system* (5.26)–(5.27) *is small-signal finite-gain* $\mathcal{L}_2$ *stable with* $\mathcal{L}_2$ *gain less than or equal to* γ. ◇

Proof: Take $u(t) \equiv 0$. By (5.28), we have

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\ \tfrac{1}{2} h^T(x) h(x), \quad \forall\ x \in D \tag{5.37}$$

Take $r > 0$ such that $B_r = \{\|x\| \leq r\} \subset D$. We will show that $V(x)$ is positive definite in B_r. Toward that end, let $\phi(t; x)$ be the solution of $\dot{x} = f(x)$ that starts at $\phi(0; x) = x \in B_r$. By existence and uniqueness of solutions (Theorem 3.1) and continuous dependence of the solution on initial states (Theorem 3.4), there exists $\delta > 0$ such that for each $x \in B_r$ the solution $\phi(t; x)$ stays in D for all $t \in [0, \delta]$. Integrating (5.37) over $[0, \tau]$ for $\tau \leq \delta$, we obtain

$$V(\phi(\tau; x)) - V(x) \leq -\ \tfrac{1}{2} \int_0^\tau \|h(\phi(t; x))\|_2^2\ dt$$

Using $V(\phi(\tau; x)) \geq 0$, we obtain

$$V(x) \geq \tfrac{1}{2} \int_0^\tau \|h(\phi(t; x))\|_2^2\ dt$$

Suppose now that there is $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. The foregoing inequality implies that

$$\int_0^\tau \|h(\phi(t; \bar{x}))\|_2^2\ dt = 0, \quad \forall\ \tau \in [0, \delta] \ \Rightarrow\ h(\phi(t; \bar{x})) \equiv 0, \quad \forall\ t \in [0, \delta]$$

Since during this interval the solution stays in S and, by assumption, the only solution that can stay identically in S is the trivial solution, we conclude that $\phi(t; \bar{x}) \equiv 0 \Rightarrow \bar{x} = 0$. Thus, $V(x)$ is positive definite in B_r. Using $V(x)$ as a Lyapunov function candidate for $\dot{x} = f(x)$, we conclude from (5.37) and LaSalle's invariance principle (Corollary 4.1) that the origin of $\dot{x} = f(x)$ is asymptotically stable. Application of Lemma 5.1 completes the proof. □

Example 5.12 As a variation on the theme of Examples 5.8 and 5.9, consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(x_1 - \tfrac{1}{3}x_1^3) - kx_2 + u \\ y &= x_2 \end{aligned}$$

where $a,\ k > 0$. The function $V(x) = \alpha\left[a\left(x_1^2/2 - x_1^4/12\right) + x_2^2/2\right]$, with $\alpha > 0$, is positive semidefinite in the set $\{|x_1| \leq \sqrt{6}\}$. Using $V(x)$ as a candidate for the solution of the Hamilton–Jacobi inequality (5.28), it can be shown that

$$\mathcal{H}(V, f, G, h, \gamma) = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right) x_2^2$$

Repeating the argument used in Example 5.8, it can be easily seen that by choosing $\alpha = \gamma = 1/k$, inequality (5.28) is satisfied for all $x \in R^2$. Since the conditions of Theorem 5.5 are not satisfied globally, we investigate small-signal finite-gain stability by using Lemma 5.1. We need to show that the origin of the unforced system is asymptotically stable. This can be shown by linearization at the origin, which results in a Hurwitz matrix. Alternatively, we can apply Lemma 5.2, whose conditions are satisfied in the domain $D = \{|x_1| < \sqrt{3}\}$, because

$$x_2(t) \equiv 0 \Rightarrow x_1(t)[3 - x_1^2(t)] \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Thus, we conclude that the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. △

5.4 Feedback Systems: The Small-Gain Theorem

The formalism of input–output stability is particularly useful in studying stability of interconnected systems, since the gain of a system allows us to track how the norm of a signal increases or decreases as it passes through the system. This is particularly so for the feedback connection of Figure 5.1. Here, we have two systems $H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q$ and $H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$. Suppose both systems are finite-gain $\mathcal{L}$ stable;[15] that is,

$$\begin{aligned} \|y_{1\tau}\|_{\mathcal{L}} &\leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \ \ \forall\ e_1 \in \mathcal{L}_e^m,\ \forall\ \tau \in [0, \infty) \qquad (5.38) \\ \|y_{2\tau}\|_{\mathcal{L}} &\leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \ \ \forall\ e_2 \in \mathcal{L}_e^q,\ \forall\ \tau \in [0, \infty) \qquad (5.39) \end{aligned}$$

Suppose further that the feedback system is *well defined* in the sense that for every pair of inputs $u_1 \in \mathcal{L}_e^m$ and $u_2 \in \mathcal{L}_e^q$, there exist unique outputs $e_1, y_2 \in \mathcal{L}_e^m$ and $e_2, y_1 \in \mathcal{L}_e^q$.[16] Define

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

The question of interest is whether the feedback connection, when viewed as a mapping from the input u to the output e or a mapping from input u to the output y, is finite-gain $\mathcal{L}$ stable.[17] It is not hard to see (Exercise 5.21) that the mapping from u to e is finite-gain $\mathcal{L}$ stable if and only if the mapping from u to y is finite-gain $\mathcal{L}$ stable. Therefore, we can simply say that the feedback connection is finite-gain

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[15] In this section, we present a version of the classical small-gain theorem that applies to finite-gain $\mathcal{L}$ stability. For more general versions which apply to $\mathcal{L}$ stability, see [93] and [123].

[16] Sufficient conditions for existence and uniqueness of solutions are available in the literature. The most common approach uses the contraction mapping principle. (See, for example, [53, Theorem III.3.1].) A more recent approach that makes use of existence and uniqueness of the solution of state equations can be found in [93].

[17] See Exercise 5.20 for an explanation of why we have to consider both inputs and outputs in studying the stability of the feedback connection.

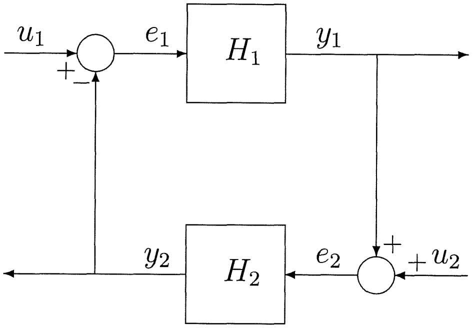

Figure 5.1: Feedback connection.

$\mathcal{L}$ stable if either mapping is finite-gain $\mathcal{L}$ stable. The following theorem, known as *the small-gain theorem*, gives a sufficient condition for finite-gain $\mathcal{L}$ stability of the feedback connection.

Theorem 5.6 *Under the preceding assumptions, the feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$.* ◇

Proof: Assuming existence of the solution, we can write

$$e_{1\tau} = u_{1\tau} - (H_2e_2)_\tau, \qquad e_{2\tau} = u_{2\tau} + (H_1e_1)_\tau$$

Then,

$$\begin{aligned} \|e_{1\tau}\|_\mathcal{L} &\le \|u_{1\tau}\|_\mathcal{L} + \|(H_2e_2)_\tau\|_\mathcal{L} \le \|u_{1\tau}\|_\mathcal{L} + \gamma_2\|e_{2\tau}\|_\mathcal{L} + \beta_2 \\ &\le \|u_{1\tau}\|_\mathcal{L} + \gamma_2\left(\|u_{2\tau}\|_\mathcal{L} + \gamma_1\|e_{1\tau}\|_\mathcal{L} + \beta_1\right) + \beta_2 \\ &= \gamma_1\gamma_2\|e_{1\tau}\|_\mathcal{L} + \left(\|u_{1\tau}\|_\mathcal{L} + \gamma_2\|u_{2\tau}\|_\mathcal{L} + \beta_2 + \gamma_2\beta_1\right) \end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

$$\|e_{1\tau}\|_\mathcal{L} \le \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{1\tau}\|_\mathcal{L} + \gamma_2\|u_{2\tau}\|_\mathcal{L} + \beta_2 + \gamma_2\beta_1\right) \tag{5.40}$$

for all $\tau \in [0,\infty)$. Similarly,

$$\|e_{2\tau}\|_\mathcal{L} \le \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{2\tau}\|_\mathcal{L} + \gamma_1\|u_{1\tau}\|_\mathcal{L} + \beta_1 + \gamma_1\beta_2\right) \tag{5.41}$$

for all $\tau \in [0,\infty)$. The proof is competed by noting that $\|e\|_\mathcal{L} \le \|e_1\|_\mathcal{L} + \|e_2\|_\mathcal{L}$, which follows from the triangle inequality. □

The feedback connection of Figure 5.1 provides a convenient setup for studying robustness issues in dynamical systems. Quite often, dynamical systems subject to model uncertainties can be represented in the form of a feedback connection with H_1, say, as a stable nominal system and H_2 as a stable perturbation. Then, the requirement $\gamma_1\gamma_2 < 1$ is satisfied whenever γ_2 is small enough. Therefore, the small-gain theorem provides a conceptual framework for understanding many of the robustness results that arise in the study of dynamical systems, especially when feedback is used. Many of the robustness results that we can derive by using Lyapunov stability techniques can be interpreted as special cases of the small-gain theorem.

Example 5.13 Consider the feedback connection of Figure 5.1. Let H_1 be a linear time-invariant system with a Hurwitz square transfer function matrix $G(s) = C(sI - A)^{-1}B$. Let H_2 be a memoryless function $y_2 = \psi(t, e_2)$ that satisfies

$$\|\psi(t,y)\|_2 \leq \gamma_2 \|y\|_2, \quad \forall\, t \geq 0,\ \forall\, y \in R^m$$

From Theorem 5.4, we know that H_1 is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is given by

$$\gamma_1 = \sup_{w \in R} \|G(j\omega)\|_2$$

We have seen in Example 5.1 that H_2 is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ_2. Assuming the feedback connection is well defined, we conclude by the small-gain theorem that it will be finite-gain $\mathcal{L}_2$ stable if $\gamma_1\gamma_2 < 1$. △

Example 5.14 Consider the system

$$\begin{aligned} \dot{x} &= f(t, x, v + d_1(t)) \\ \varepsilon \dot{z} &= Az + B[u + d_2(t)] \\ v &= Cz \end{aligned}$$

where f is a smooth function of its arguments, A is a Hurwitz matrix, $-CA^{-1}B = I$, ε is a small positive parameter, and d_1, d_2 are disturbance signals. The linear part of this model represents actuator dynamics that are, typically, much faster than the plant dynamics represented here by the nonlinear equation $\dot{x} = f$. The disturbance signals d_1 and d_2 enter the system at the input of the plant and the input of the actuator, respectively. Suppose the disturbance signals d_1 and d_2 belong to a signal space $\mathcal{L}$, where $\mathcal{L}$ could be any $\mathcal{L}_p$ space, and the control goal is to attenuate the effect of this disturbance on the state x. This goal can be met if feedback control can be designed so that the closed-loop input–output map from (d_1, d_2) to x is finite-gain $\mathcal{L}$ stable and the $\mathcal{L}$ gain is less than some given tolerance $\delta > 0$. To simplify the design problem, it is common to neglect the actuator dynamics by setting $\varepsilon = 0$ and substituting $v = -CA^{-1}B(u + d_2) = u + d_2$ in the plant equation to obtain the reduced-order model

$$\dot{x} = f(t, x, u + d)$$

where $d = d_1 + d_2$. Assuming that the state variables are available for measurement, we use this model to design a state feedback control law $u = \gamma(t, x)$ to meet the design objective. Suppose we have succeeded in designing a smooth state feedback control $u = \gamma(t, x)$ such that

$$\|x\|_{\mathcal{L}} \leq \gamma \|d\|_{\mathcal{L}} + \beta \tag{5.42}$$

for some $\gamma < \delta$. Will the control meet the design objective when applied to the actual system with the actuator dynamics included? This is a question of robustness of the controller with respect to the unmodeled actuator dynamics.[18] When the control is applied to the actual system, the closed-loop equation is given by

$$\begin{aligned} \dot{x} &= f(t, x, Cz + d_1(t)) \\ \varepsilon \dot{z} &= Az + B[\gamma(t, x) + d_2(t)] \end{aligned}$$

Let us assume that $d_2(t)$ is differentiable and $\dot{d}_2 \in \mathcal{L}$. The change of variables

$$\eta = z + A^{-1}B[\gamma(t, x) + d_2(t)]$$

brings the closed-loop system into the form

$$\begin{aligned} \dot{x} &= f(t, x, \gamma(t, x) + d(t) + C\eta) \\ \varepsilon \dot{\eta} &= A\eta + \varepsilon A^{-1}B[\dot{\gamma} + \dot{d}_2(t)] \end{aligned}$$

where

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial x} f(t, x, \gamma(t, x) + d(t) + C\eta)$$

It is not difficult to see that the closed-loop system can be represented in the form of Figure 5.1 with H_1 defined by

$$\begin{aligned} \dot{x} &= f(t, x, \gamma(t, x) + e_1) \\ y_1 &= \dot{\gamma} = \frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial x} f(t, x, \gamma(t, x) + e_1) \end{aligned}$$

H_2 defined by

$$\begin{aligned} \dot{\eta} &= \frac{1}{\varepsilon} A\eta + A^{-1}Be_2 \\ y_2 &= -C\eta \end{aligned}$$

and

$$u_1 = d_1 + d_2 = d, \quad u_2 = \dot{d}_2$$

In this representation, the system H_1 is the nominal reduced-order closed-loop system, while H_2 represents the effect of the unmodeled dynamics. Setting $\varepsilon = 0$

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[18] In Example 11.14, we investigate a similar robustness problem, in the context of stabilization, using singular perturbation theory.

opens the loop and the overall closed-loop system reduces to the nominal one. Let us assume that the feedback function $\gamma(t,x)$ satisfies the inequality

$$\left\| \frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial x} f(t,x,\gamma(t,x)+e_1) \right\| \le c_1\|x\| + c_2\|e_1\| \tag{5.43}$$

for all (t,x,e_1), where c_1 and c_2 are nonnegative constants. Using (5.42) and (5.43), it can be shown that

$$\|y_1\|_{\mathcal{L}} \le \gamma_1\|e_1\|_{\mathcal{L}} + \beta_1$$

where

$$\gamma_1 = c_1\gamma + c_2, \quad \beta_1 = c_1\beta$$

Since H_2 is a linear time-invariant system and A is Hurwitz, we apply Corollary 5.2 to show that H_2 is finite-gain $\mathcal{L}_p$ stable for any $p \in [1,\infty]$ and

$$\|y_2\|_{\mathcal{L}} \le \gamma_2\|e_2\|_{\mathcal{L}} + \beta_2 \overset{\text{def}}{=} \varepsilon\gamma_f\|e_2\|_{\mathcal{L}} + \beta_2$$

where

$$\gamma_f = \frac{2\lambda_{max}^2(Q)\|A^{-1}B\|_2\|C\|_2}{\lambda_{min}(Q)}, \quad \beta_2 = \rho\|C\|_2\|\eta(0)\|\sqrt{\frac{\lambda_{max}(Q)}{\lambda_{min}(Q)}}$$

$$\rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{2\varepsilon\lambda_{max}(Q)}{p}\right)^{1/p}, & \text{if } p \in [1,\infty) \end{cases}$$

and Q is the solution of the Lyapunov equation $QA + A^TQ = -I$.[19] Thus, assuming the feedback connection is well defined, we conclude from the small-gain theorem that the input–output map from u to e is $\mathcal{L}$ stable. From (5.40), we have

$$\|e_1\|_{\mathcal{L}} \le \frac{1}{1-\varepsilon\gamma_1\gamma_f}[\|u_1\|_{\mathcal{L}} + \varepsilon\gamma_f\|u_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2]$$

Using

$$\|x\|_{\mathcal{L}} \le \gamma\|e_1\|_{\mathcal{L}} + \beta$$

which follows from (5.42), and the definition of u_1 and u_2, we obtain

$$\|x\|_{\mathcal{L}} \le \frac{\gamma}{1-\varepsilon\gamma_1\gamma_f}[\|d\|_{\mathcal{L}} + \varepsilon\gamma_f\|\dot{d}_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2] + \beta \tag{5.44}$$

It is interesting to note that the right-hand side of (5.44) approaches

$$\gamma\|d\|_{\mathcal{L}} + \beta + \gamma\beta_2$$

as $\varepsilon \to 0$, which shows that for sufficiently small ε the upper bound on the $\mathcal{L}$ gain of the map from d to x, under the actual closed-loop system, will be close to the corresponding quantity under the nominal closed-loop system. △

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[19] P of Corollary 5.2 is taken as εQ so that $(\varepsilon Q)(A/\varepsilon) + (A/\varepsilon)^T(\varepsilon Q) = -I$.

5.5 Exercises

5.1 Show that the series connection of two $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable) systems is $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable).

5.2 Show that the parallel connection of two $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable) systems is $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable).

5.3 Consider a system defined by the memoryless function $y = u^{1/3}$.

(a) Show that the system is $\mathcal{L}_\infty$ stable with zero bias.

(b) For any positive constant a, show that the system is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = a$ and $\beta = (1/a)^{1/2}$.

(c) Compare the two statements.

5.4 Consider a system defined by the memoryless function by $y = h(u)$, where $h : R^m \to R^q$ is globally Lipschitz. Investigate $\mathcal{L}_p$ stability for each $p \in [1, \infty]$ when

$$(1)\ h(0) = 0. \qquad (2)\ h(0) \neq 0.$$

5.5 For each of the relay characteristices shown in Figure 5.2, investigate $\mathcal{L}_\infty$ and $\mathcal{L}_2$ stability.

5.6 Verify that $D^+W(t)$ satisfies (5.12) when $V(t, x(t)) = 0$.
Hint: Using Exercise 3.24, show that $V(t+h, x(t+h)) \le c_4 h^2 L^2 \|u\|^2/2 + h\ o(h)$, where $o(h)/h \to 0$ as $h \to 0$. Then, use the fact that $c_4 \ge 2c_1$.

5.7 Suppose the assumptions of Theorem 5.1 are satisfied, except (5.10), which is replaced by

$$\|h(t, x, u)\| \le \eta_1 \|x\| + \eta_2 \|u\| + \eta_3, \quad \eta_3 > 0$$

Show that the system is small-signal finite-gain $\mathcal{L}_\infty$ stable (or finite-gain $\mathcal{L}_\infty$ stable, if the assumptions hold globally) and find the constants γ and β in (5.11).

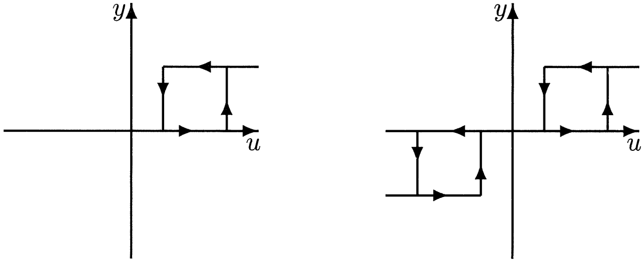
5.8 Suppose the assumptions of Theorem 5.1 are satisfied, except (5.10), which is replaced by (5.20). Show that the system is small-signal $\mathcal{L}_\infty$ stable (or $\mathcal{L}_\infty$ stable, if the assumptions hold globally).

5.9 Derive a result similar to Corollary 5.2 for linear time-varying systems.

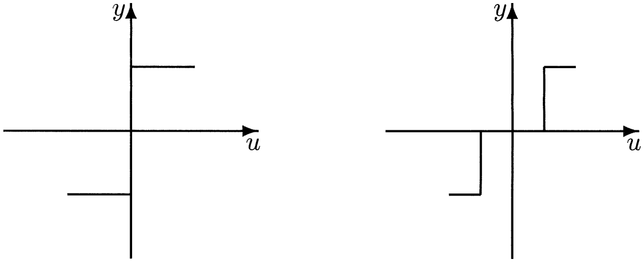
5.10 For each of the following systems, investigate $\mathcal{L}_\infty$ and finite-gain $\mathcal{L}_\infty$ stability:

$$(1)\quad \begin{aligned} \dot{x} &= -(1+u)x^3 \\ y &= x \end{aligned} \qquad\qquad (2)\quad \begin{aligned} \dot{x} &= -(1+u)x^3 - x^5 \\ y &= x + u \end{aligned}$$

$$(3)\quad \begin{aligned} \dot{x} &= -x/(1+x^2) + u \\ y &= x/(1+x^2) \end{aligned} \qquad\qquad (4)\quad \begin{aligned} \dot{x} &= -x - x^3 + x^2 u \\ y &= x \sin u \end{aligned}$$



(a) On–off with hysteresis (b) On–off with dead zone and hysteresis



(c) Ideal on–off (d) On–off with dead zone

Figure 5.2: Relay characteristics

5.11 For each of the following systems, investigate $\mathcal{L}_\infty$ and finite-gain $\mathcal{L}_\infty$ stability:

(1) $\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u, \qquad y = x_1$

(2) $\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u, \qquad y = x_2$

(3) $\dot{x}_1 = (x_1 + u)(\|x\|_2^2 - 1), \qquad \dot{x}_2 = x_2(\|x\|_2^2 - 1), \qquad y = x_1$

(4) $\dot{x}_1 = -x_1 - x_2 + u_1, \qquad \dot{x}_2 = x_1 - x_2^3 + u_2, \qquad y = x_1(x_2 + u_1)$

(5) $\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = x_1 - x_2 + u, \qquad y = x_1 + u$

(6) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u, \qquad y = x_2$

(7) $\dot{x}_1 = -x_1 - x_2, \qquad \dot{x}_2 = x_1 - x_3 + u, \qquad y(t) = x_1(t - T)$

where $T > 0$.

5.12 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -y - h(y) + u, \qquad y = x_1 + x_2$$

where h is continuously differentiable, $h(0) = 0$, and $zh(z) > az^2$ for all $z \in R$, for some $a > 0$. Show that the system is finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$.

5.13 ([192]) Consider the time-invariant system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

where f is locally Lipschitz, h is continuous, $f(0, 0) = 0$, and $h(0, 0) = 0$. Suppose there is a continuously differentiable, positive definite, radially unbounded function $V(x)$ such that

$$\frac{\partial V}{\partial x} f(x, u) \le -W(x) + \psi(u), \quad \forall \; (x, u)$$

where $W(x)$ is continuous, positive definite, and radially unbounded, $\psi(u)$ is continuous, and $\psi(0) = 0$. Show that the system is $\mathcal{L}_\infty$ stable.

5.14 Let $H(s)$ be a Hurwitz strictly proper transfer function, and $h(t) = \mathcal{L}^{-1}\{H(s)\}$ be the corresponding impulse response function. Show that

$$\sup_{\omega \in R} |H(j\omega)| \le \int_0^\infty |h(t)| \; dt$$

5.15 For each of the following systems, show that the system is finite-gain (or small-signal finite-gain) $\mathcal{L}_2$ stable and find an upper bound on the $\mathcal{L}_2$ gain:

$$\textbf{(1)} \quad \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -a \sin x_1 - k x_2 + u \\ y & = & x_2 \\ & & a > 0, \; k > 0 \end{array} \qquad\qquad \textbf{(2)} \quad \begin{array}{rcl} \dot{x}_1 & = & -x_2 \\ \dot{x}_2 & = & x_1 - x_2 \text{ sat}(x_2^2 - x_3^2) + x_2 u \\ \dot{x}_3 & = & x_3 \text{ sat}(x_2^2 - x_3^2) - x_3 u \\ y & = & x_2^2 - x_3^2 \end{array}$$

$$\textbf{(3)} \quad \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_1 - \text{sat}(2x_1 + x_2) + u \\ y & = & x_1 \end{array} \qquad\qquad \textbf{(4)} \quad \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -(1 + x_1^2) x_2 - x_1^3 + x_1 u \\ y & = & x_1 x_2 \end{array}$$

5.16 Consider the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1 - \sigma(x_1) - x_2 + u, \qquad y = x_2$$

where σ is locally Lipschitz, $\sigma(0) = 0$, and $z\sigma(z) \ge 0$ for all $z \in R$.

(a) Is the system $\mathcal{L}_\infty$ stable?

(b) Is it finite-gain $\mathcal{L}_2$ stable? If yes, find an upper bound on the $\mathcal{L}_2$ gain.

5.17 ([77]) Consider the system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x) + J(x)u$$

where f, G, h, and J are smooth functions of x. Suppose there is a positive constant γ such that $\gamma^2 I - J^T(x)J(x) > 0$ and

$$\mathcal{H} = \frac{\partial V}{\partial x} f + \frac{1}{2}\left[h^T J + \frac{\partial V}{\partial x} G\right]\left(\gamma^2 I - J^T J\right)^{-1}\left[h^T J + \frac{\partial V}{\partial x} G\right]^T + \frac{1}{2} h^T h \leq 0$$

$\forall$ x. Show that the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to γ.
Hint: Set

$$\gamma^2 I - J^T(x)J(x) \;=\; W^T(x)W(x), \quad L(x) = -\left[W^T(x)\right]^{-1}\left[h^T(x)J(x) + \frac{\partial V}{\partial x} G(x)\right]^T$$

and show that the following inequality holds $\forall$ u

$$\frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} G u = -\,\frac{1}{2}[L + Wu]^T[L + Wu] + \frac{\gamma^2}{2} u^T u - \frac{1}{2} y^T y + \mathcal{H}$$

5.18 ([199]) Consider the system

$$\dot{x} = f(x) + G(x)u + K(x)w, \qquad y = h(x)$$

where u is a control input and w is a disturbance input. The functions f, G, K, and h are smooth, and $f(0) = 0$, $h(0) = 0$. Let $\gamma > 0$. Suppose there is a smooth positive semidefinite function $V(x)$ that satisfies

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2}\frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2} K(x)K^T(x) - G(x)G^T(x)\right]\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x)h(x) \leq 0$$

$\forall$ x. Show that with the feedback control $u = -G^T(x)(\partial V/\partial x)^T$ the closed-loop map from w to $\begin{bmatrix} y \\ u \end{bmatrix}$ is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to γ.

5.19 ([200]) The purpose of this exercise is to show that the $\mathcal{L}_2$ gain of a linear time-invariant system of the form (5.24)–(5.25), with a Hurwitz matrix A, is the same, whether the space of functions is defined on $R_+ = [0, \infty)$ or on the whole real line $R = (-\infty, \infty)$. Let $\mathcal{L}_2$ be the space of square integrable functions on R_+ with the norm $\|u\|^2_{\mathcal{L}_2} = \int_0^\infty u^T(t)u(t)\,dt$ and $\mathcal{L}_{2R}$ be the space of square integrable functions on R with the norm $\|u\|^2_{\mathcal{L}_{2R}} = \int_{-\infty}^{\infty} u^T(t)u(t)\,dt$. Let γ_2 and γ_{2R} be the $\mathcal{L}_2$ gains on $\mathcal{L}_2$ and $\mathcal{L}_{2R}$, respectively. Since $\mathcal{L}_2$ is a subset of $\mathcal{L}_{2R}$, it is clear that $\gamma_2 \leq \gamma_{2R}$. We will show that $\gamma_2 = \gamma_{2R}$ by showing that, for every $\varepsilon > 0$, there is a signal $u \in \mathcal{L}_2$ such that $y \in \mathcal{L}_2$ and $\|y\|_{\mathcal{L}_2} \geq (1 - \varepsilon)\gamma_{2R}\|u\|_{\mathcal{L}_2}$.

(a) Given $\varepsilon > 0$ show that we can always choose $0 < \delta < 1$ such that

$$\frac{1 - \varepsilon/2 - \sqrt{\delta}}{\sqrt{1-\delta}} \geq 1 - \varepsilon$$

(b) Show that we can always select $u \in \mathcal{L}_{2R}$ and time $t_1 < \infty$ such that

$$\|u\|_{\mathcal{L}_{2R}} = 1, \quad \|y\|_{\mathcal{L}_{2R}} \geq \gamma_{2R}\left(1 - \frac{\varepsilon}{2}\right), \quad \int_{-\infty}^{t_1} u^T(t)u(t)\, dt = \delta$$

(c) Let $u(t) = u_1(t) + u_2(t)$, where u_1 vanishes for $t < t_1$ and u_2 vanishes for $t > t_1$. Let $y_1(t)$ be the output corresponding to the input $u_1(t)$. Show that

$$\frac{\|y_1\|_{\mathcal{L}_{2R}}}{\|u_1\|_{\mathcal{L}_{2R}}} \geq \frac{1 - \varepsilon/2 - \sqrt{\delta}}{\sqrt{1-\delta}}\gamma_{2R} \geq (1-\varepsilon)\gamma_{2R}$$

(d) For all $t \geq 0$, define $u(t)$ and $y(t)$ by $u(t) = u_1(t + t_1)$ and $y(t) = y_1(t + t_1)$. Show that both u and y belong to $\mathcal{L}_2$, $y(t)$ is the output corresponding to $u(t)$, and $\|y\|_{\mathcal{L}_2} \geq (1-\varepsilon)\gamma_{2R}\|u\|_{\mathcal{L}_2}$.

5.20 Consider the feedback connection of Figure 5.1, where H_1 and H_2 are linear time-invariant systems represented by the transfer functions $H_1(s) = (s-1)/(s+1)$ and $H_2(s) = 1/(s-1)$. Find the closed-loop transfer functions from (u_1, u_2) to (y_1, y_2) and from (u_1, u_2) to (e_1, e_2). Use these transfer functions to discuss why we have to consider both inputs (u_1, u_2) and both outputs (e_1, e_2) (or (y_1, y_2)) in studying the stability of the feedback connection.

5.21 Consider the feedback connection of Figure 5.1. Show that the mapping from (u_1, u_2) to (y_1, y_2) is finite-gain $\mathcal{L}$ stable if and only if the mapping from (u_1, u_2) to (e_1, e_2) is finite-gain $\mathcal{L}$ stable.

5.22 Let $d_2(t) = a \sin \omega t$ in Example 5.14, where a and ω are positive constants.

(a) Show that, for sufficiently small ε, the state of the closed-loop system is uniformly bounded.

(b) Investigate the effect of increasing ω.

5.23 Consider the feedback connection of Figure 5.1, where H_1 and H_2 are given by

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_2 \\ \dot{x}_2 & = & -x_1^3 - x_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \quad \text{and} \quad H_2 : \left\{ \begin{array}{rcl} \dot{x}_3 & = & -x_3^3 + e_2 \\ y_2 & = & (1/2)x_3^3 \end{array} \right.$$

Let $u_2 = 0$, $u = u_1$ be the input, and $y = y_1$ be the output.

(a) Using $x = [x_1,\ x_2,\ x_3]^T$ as the state vector, find the state model of the system.

(b) Is the system $\mathcal{L}_2$ stable?

Chapter 6

Passivity

Passivity provides us with a useful tool for the analysis of nonlinear systems, which relates nicely to Lyapunov and $\mathcal{L}_2$ stability. We start in Section 6.1 by defining passivity of memoryless nonlinearities. We extend the definition to dynamical systems, represented by state models, in Section 6.2. In both cases, we use electrical networks to motivate the definitions. In Section 6.3, we study positive real and strictly positive real transfer functions and show that they represent passive and strictly passive systems, respectively. The connection between passivity and both Lyapunov and $\mathcal{L}_2$ stability is established in Section 6.4. These four sections prepare us to address the main results of the chapter, namely, the passivity theorems of Section 6.5. The main passivity theorem states that *the (negative) feedback connection of two passive systems is passive.* Under additional observability conditions, the feedback connection is also asymptotically stable. The passivity theorems of Section 6.5 and the small-gain theorem of Section 5.4 provide a conceptually important generalization of the fact that the feedback connection of two stable linear systems will be stable if the loop gain is less than one or the loop phase is less than 180 degrees. The connection between passivity and the phase of a transfer function comes from the frequency-domain characterization of positive real transfer functions, given in Section 6.3. There we know that the phase of a positive real transfer function cannot exceed 90 degree. Hence, the loop phase cannot exceed 180 degrees. If one of the two transfer functions is strictly positive real, the loop phase will be strictly less than 180 degrees. Section 6.5 discusses also loop transformations, which allow us, in certain cases, to transform the feedback connection of two systems that may not be passive into the feedback connection of two passive systems, hence extending the utility of the passivity theorems.

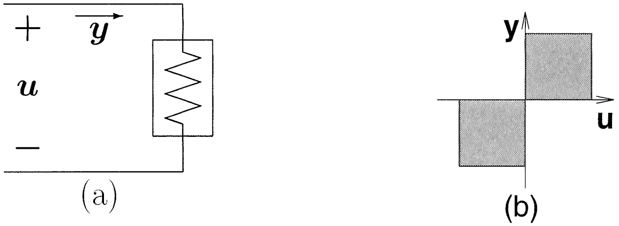


Figure 6.1: (a) A passive resistor; (b) u–y characteristic lies in the first–third quadrant.

6.1 Memoryless Functions

Our goal in this section is to define passivity of the memoryless function $y = h(t, u)$, where $h : [0, \infty) \times R^p \to R^p$. We use electric networks to motivate the definition. Figure 6.1(a) shows a one-port resistive element with voltage u and current y. We view this element as a system with input u and output y. The resistive element is passive if the inflow of power is always nonnegative; that is, if $uy \geq 0$ for all points (u, y) on its u–y characteristic. Geometrically, this means that the u–y curve must lie in the first and third quadrants, as shown in Figure 6.1(b). The simplest such resistive element is a linear resistor that obeys Ohm's law $u = Ry$ or $y = Gu$, where R is the resistance and $G = 1/R$ is the conductance. For positive resistance, the u–y characteristic is a straight line of slope G and the product $uy = Gu^2$ is always nonnegative. In fact, it is always positive except at the origin point $(0, 0)$. Nonlinear passive resistive elements have nonlinear u–y curves lying in the first and third quadrants; examples are shown in Figures 6.2(a) and (b). Notice that the tunnel-diode characteristic of Figure 6.2(b) is still passive even though the curve has negative slope in some region. As an example of an element that is not passive, Figure 6.2(c) shows the u–y characteristic of a negative resistance that was used in Section 1.2.4 to construct the negative resistance oscillator. Such characteristic can be only realized using active devices such as the twin tunnel-diode circuit of Figure 1.7. For a multiport network where u and y are vectors, the power flow into the network is the inner product $u^T y = \Sigma_{i=1}^{p} u_i y_i = \Sigma_{i=1}^{p} u_i h_i(u)$. The network is passive if $u^T y \geq 0$ for all u. This concept of passivity is now abstracted and assigned to any function $y = h(t, u)$ irrespective of its physical origin. We think of $u^T y$ as the power flow into the system and say that the system is passive if $u^T y \geq 0$ for all u. For the scalar case, the graph of the input–output relation must lie in the first and third quadrants. We also say that the graph belongs to the sector $[0, \infty]$, where zero and infinity are the slopes of the boundaries of the first–third quadrant region. The graphical representation is valid even when h is time varying. In this case, the u–y curve will be changing with time, but will always belong to the sector $[0, \infty]$. For a vector function, we can give a graphical representation in the special case when $h(t, u)$ is decoupled in the sense that $h_i(t, u)$ depends only on u_i; that is,

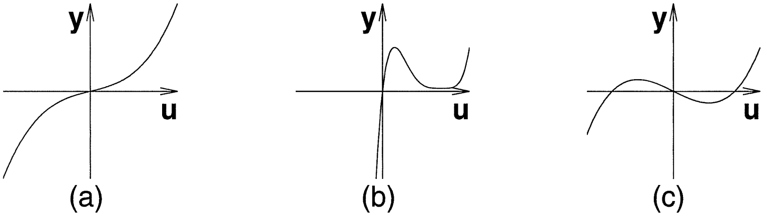


Figure 6.2: (a) and (b) are examples of nonlinear passive resistor characteristics; (c) is an example of a nonpassive resistor.

$$h(t,u) = \begin{bmatrix} h_1(t,u_1) \\ h_2(t,u_2) \\ \vdots \\ h_p(t,u_p) \end{bmatrix} \tag{6.1}$$

In this case, the graph of each component belongs to the sector $[0,\infty]$. In the general case, such graphical representation is not possible, but we will continue to use the sector terminology by saying that h belongs to the sector $[0,\infty]$ if $u^T h(t,u) \geq 0$ for all (t,u).

An extreme case of passivity happens when $u^T y = 0$. In this case, we say that the system is lossless. An example of a lossless system is the ideal transformer shown in Figure 6.3. Here $y = Su$, where

$$u = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}, \quad y = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -N \\ N & 0 \end{bmatrix}$$

The matrix S is skew-symmetric; that is, $S + S^T = 0$. Hence, $u^T y = u^T S u = (1/2) u^T (S + S^T) u = 0$.

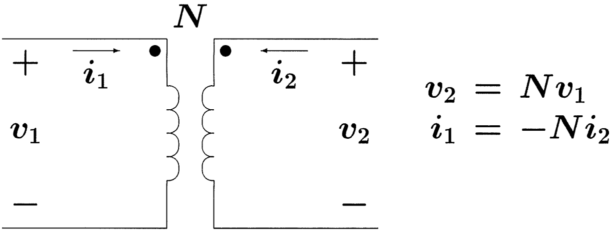


Figure 6.3: Ideal transformer

Consider now a function h satisfying $u^T y \geq u^T \varphi(u)$ for some function $\varphi(u)$. When $u^T \varphi(u) > 0$ for all $u \neq 0$, h is called *input strictly passive* because passivity is strict in the sense that $u^T y = 0$ only if $u = 0$. Equivalently, in the scalar case,

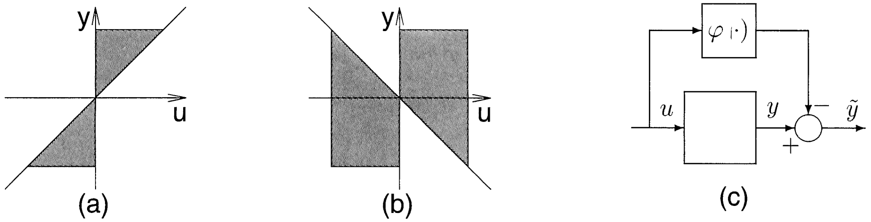


Figure 6.4: A graphical representation of $u^T y \geq \varepsilon u^T u$ for (a) $\varepsilon > 0$ (excess of passivity); (b) $\varepsilon < 0$ (shortage of passivity); (c) removal of excess or shortage of passivity by input-feedforward operation.

the u–y graph does not touch the u-axis, except at the origin. The term $u^T\varphi(u)$ represents the "excess" of passivity. On the other hand, if $u^T\varphi(u)$ is negative for some values of u, then the function h is not necessarily passive. The term $u^T\varphi(u)$ represents the "shortage" of passivity. Excess and shortage of passivity are more transparent when h is scalar and $\varphi(u) = \varepsilon u$. In this case, h belongs to the sector $[\varepsilon, \infty]$, shown in Figure 6.4, with excess of passivity when $\varepsilon > 0$ and shortage of passivity when $\varepsilon < 0$. Excess or shortage of passivity can be removed by the input-feedforward operation shown in Figure 6.4(c). With the new output defined as $\tilde{y} = y - \varphi(u)$, we have

$$u^T\tilde{y} = u^T[y - \varphi(u)] \geq u^T\varphi(u) - u^T\varphi(u) = 0$$

Thus, any function satisfying $u^T y \geq u^T\varphi(u)$ can be transformed into a function that belongs to the sector $[0, \infty]$ via input feedforward. Such a function is called *input-feedforward passive.* On the other hand, suppose $u^T y \geq y^T\rho(y)$ for some function $\rho(y)$. Similar to the foregoing case, there is excess of passivity when $y^T\rho(y) > 0$ for all $y \neq 0$, and shortage of passivity when $y^T\rho(y)$ is negative for some values of y. A graphical representation of the scalar case with $\rho(y) = \delta y$ is shown in Figure 6.5. There is "excess" of passivity when $\delta > 0$ and shortage of passivity when $\delta < 0$. Excess or shortage of passivity can be removed by the output-feedback operation shown in Figure 6.5(c). With the new input defined as $\tilde{u} = u - \rho(y)$, we have

$$\tilde{u}^T y = [u - \rho(y)]^T y \geq y^T\rho(y) - y^T\rho(y) = 0$$

Hence, any function satisfying $u^T y \geq y^T\rho(y)$ can be transformed into a function that belongs to the sector $[0, \infty]$ via output feedback. Such a function is called *output-feedback passive.* When $y^T\rho(y) > 0$ for all $y \neq 0$, the function is called *output strictly passive* because passivity is strict in the sense that $u^T y = 0$ only if $y = 0$. Equivalently, in the scalar case, the u–y graph does not touch the y-axis, except at the origin. For convenience, we summarize the various notions of passivity in the next definition.

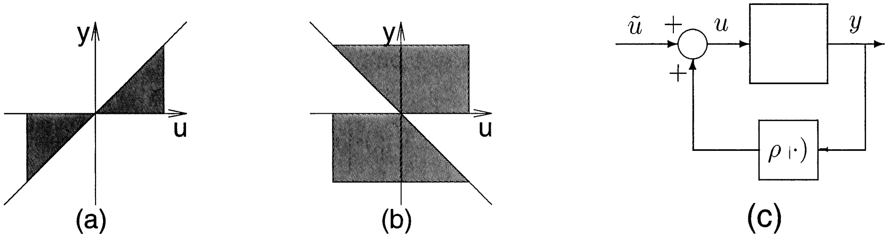


Figure 6.5: A graphical representation of $u^T y \geq \delta y^T y$ for (a) $\delta > 0$ (excess of passivity); (b) $\delta < 0$ (shortage of passivity); (c) removal of excess or shortage of passivity by output-feedback operation.

Definition 6.1 *The system* $y = h(t, u)$ *is*

- *passive if* $u^T y \geq 0$.
- *lossless if* $u^T y = 0$.
- *input-feedforward passive if* $u^T y \geq u^T \varphi(u)$ *for some function* φ.
- *input strictly passive if* $u^T y \geq u^T \varphi(u)$ *and* $u^T \varphi(u) > 0, \ \forall \ u \neq 0$.
- *output-feedback passive if* $u^T y \geq y^T \rho(y)$ *for some function* ρ.
- *output strictly passive if* $u^T y \geq y^T \rho(y)$ *and* $y^T \rho(y) > 0, \ \forall \ y \neq 0$.

In all cases, the inequality should hold for all (t, u).

Consider next a scalar function $y = h(t, u)$, which satisfies the inequality

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2 \tag{6.2}$$

for all (t, u), where α and β are real numbers with $\beta \geq \alpha$. The graph of this function belongs to a sector whose boundaries are the lines $y = \alpha u$ and $y = \beta u$. We say that h belongs to the sector $[\alpha, \beta]$. Figure 6.6 shows the sector $[\alpha, \beta]$ for $\beta > 0$ and different signs of α. If strict inequality is satisfied on either side of (6.2), we say that h belongs to a sector $(\alpha, \beta]$, $[\alpha, \beta)$, or (α, β), with obvious implications. Comparing the sectors of Figure 6.6 with those of Figures 6.4 and 6.5 shows that a function in the sector $[\alpha, \beta]$ combines input-feedforward passivity with output strict passivity since the sector $[\alpha, \beta]$ is the intersection of the sectors $[\alpha, \infty]$ and $[0, \beta]$. Indeed, we can show that such a function can be transformed into a function that belongs to the sector $[0, \infty]$ by a sequence of input-feedforward and output-feedback operations. Before we do that, we extend the sector definition to the vector case. Toward that end, note that (6.2) is equivalent to

$$[h(t, u) - \alpha u][h(t, u) - \beta u] \leq 0 \tag{6.3}$$

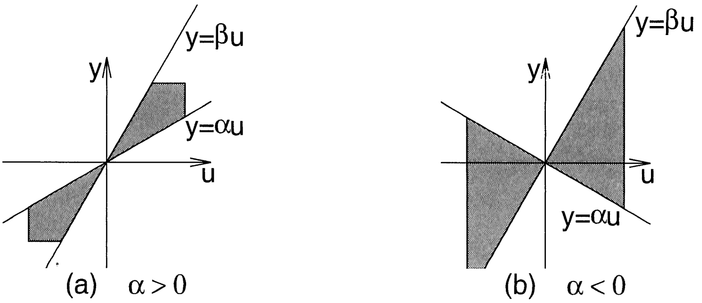


Figure 6.6: The sector $[\alpha, \beta]$ for $\beta > 0$ and (a) $\alpha > 0$; (b) $\alpha < 0$.

for all (t, u). For the vector case, let us consider first a function $h(t, u)$ that is decoupled as in (6.1). Suppose each component h_i satisfies the sector condition (6.2) with constants α_i and $\beta_i > \alpha_i$. Taking

$$K_1 = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_p), \quad K_2 = \text{diag}(\beta_1, \beta_2, \ldots, \beta_p)$$

it can be easily seen that

$$[h(t, u) - K_1 u]^T [h(t, u) - K_2 u] \leq 0 \tag{6.4}$$

for all (t, u). Note that $K = K_2 - K_1$ is a positive definite symmetric (diagonal) matrix. Inequality (6.4) may hold for more general vector functions. For example, suppose $h(t, u)$ satisfies the inequality

$$\|h(t, u) - Lu\|_2 \leq \gamma \|u\|_2$$

for all (t, u). Taking $K_1 = L - \gamma I$ and $K_2 = L + \gamma I$, we can write

$$[h(t, u) - K_1 u]^T [h(t, u) - K_2 u] = \|h(t, u) - Lu\|_2^2 - \gamma^2 \|u\|_2^2 \leq 0$$

Once again, $K = K_2 - K_1$ is a positive definite symmetric (diagonal) matrix. We use inequality (6.4) with a positive definite symmetric matrix $K = K_2 - K_1$ as a definition of the sector $[K_1, K_2]$ in the vector case. The next definition summaries the sector terminology.

Definition 6.2 *A memoryless function* $h : [0, \infty) \times R^p \to R^p$ *is said to belong to the sector*

- $[0, \infty]$ *if* $u^T h(t, u) \geq 0$.
- $[K_1, \infty]$ *if* $u^T [h(t, u) - K_1 u] \geq 0$.
- $[0, K_2]$ *with* $K_2 = K_2^T > 0$ *if* $h^T(t, u)[h(t, u) - K_2 u] \leq 0$.

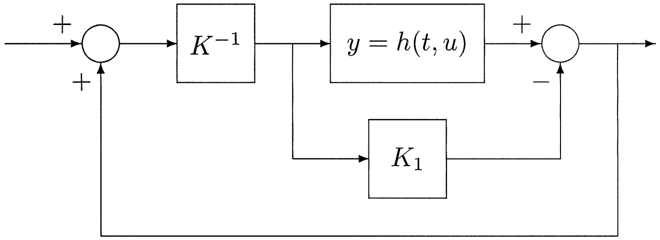


Figure 6.7: A function in the sector $[K_1, K_2]$, where $K = K_2 - K_1 = K^T > 0$, can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback.

- $[K_1, K_2]$ *with* $K = K_2 - K_1 = K^T > 0$ *if*

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \le 0 \tag{6.5}$$

In all cases, the inequality should hold for all (t, u). *If in any case the inequality is strict, we write the sector as* $(0, \infty)$, (K_1, ∞), $(0, K_2)$, *or* (K_1, K_2). *In the scalar case, we write* $(\alpha, \beta]$, $[\alpha, \beta)$, *or* (α, β) *to indicate that one or both sides of (6.2) is satisfied as a strict inequality.*

The sector $[0, \infty]$ corresponds to passivity. The sector $[K_1, \infty]$ corresponds to input-feedforward passivity with $\varphi(u) = K_1 u$. The sector $[0, K_2]$ with $K_2 = (1/\delta)I > 0$ corresponds to output strict passivity with $\rho(y) = \delta y$. We leave it to the reader (Exercise 6.1) to verify that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 6.7.

6.2 State Models

Let us now define passivity for a dynamical system represented by the state model

$$\dot{x} = f(x, u) \tag{6.6}$$

$$y = h(x, u) \tag{6.7}$$

where $f : R^n \times R^p \to R^n$ is locally Lipschitz, $h : R^n \times R^p \to R^p$ is continuous, $f(0,0) = 0$, and $h(0,0) = 0$. The system has the same number of inputs and outputs. The following *RLC* circuit motivates the definition.

Example 6.1 The *RLC* circuit of Figure 6.8 features a voltage source connected to an *RLC* network with linear inductor and capacitor and nonlinear resistors. The nonlinear resistors 1 and 3 are represented by their v–i characteristics $i_1 = h_1(v_1)$ and $i_3 = h_3(v_3)$, while resistor 2 is represented by its i–v characteristic $v_2 = h_2(i_2)$.

We take the voltage u as the input and the current y as the output. The product uy is the power flow into the network. Taking the current x_1 through the inductor and the voltage x_2 across the capacitor as the state variables, we can write the state model as

$$\begin{aligned} L\dot{x}_1 &= u - h_2(x_1) - x_2 \\ C\dot{x}_2 &= x_1 - h_3(x_2) \\ y &= x_1 + h_1(u) \end{aligned}$$

The new feature of an *RLC* network over a resistive network is the presence of the energy-storing elements L and C. The system is passive if the energy absorbed by the network over any period of time $[0, t]$ is greater than or equal to the increase in the energy stored in the network over the same period; that is,

$$\int_0^t u(s)y(s)\, ds \geq V(x(t)) - V(x(0)) \tag{6.8}$$

where $V(x) = (1/2)Lx_1^2 + (1/2)Cx_2^2$ is the energy stored in network. If (6.8) holds with strict inequality, then the difference between the absorbed energy and the increase in the stored energy must be the energy dissipated in the resistors. Since (6.8) must hold for every $t \geq 0$, the instantaneous power inequality

$$u(t)y(t) \geq \dot{V}(x(t), u(t)) \tag{6.9}$$

must hold for all t; that is, the power flow into the network must be greater than or equal to the rate of change of the energy stored in the network. We can investigate inequality (6.9) by calculating the derivative of V along the trajectories of the system. We have

$$\begin{aligned} \dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 = x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\ &= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\ &= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\ &= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \end{aligned}$$

Thus,

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive. Other possibilities are illustrated by four different special cases of the network.

Case 1: If $h_1 = h_2 = h_3 = 0$, $uy = \dot{V}$ and there is no energy dissipation in the network; that is, the system is lossless.

Case 2: If h_2 and h_3 belong to the sector $[0, \infty]$,

$$uy \geq \dot{V} + uh_1(u)$$

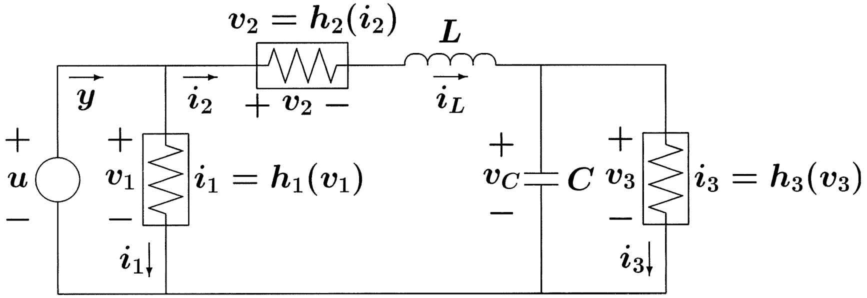


Figure 6.8: RLC circuit of Example 6.1.

The term $uh_1(u)$ could represent excess or shortage of passivity. If $uh_1(u) > 0$ for all $u \neq 0$, there is excess of passivity since the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the input $u(t)$ is identically zero. This is a case of input strict passivity. On the other hand, if $uh_1(u)$ is negative for some values of u, there is shortage of passivity. As we saw with memoryless functions, this type of excess or shortage of passivity can be removed by input feedforward shown in Figure 6.4(c).

Case 3: If $h_1 = 0$ and $h_3 \in [0, \infty]$,

$$uy \geq \dot{V} + yh_2(y)$$

Excess or shortage of passivity of h_2 results in the same property for the network. Once again, as with memoryless functions, this type of excess or shortage of passivity can be removed by output feedback, as in Figure 6.5(c). When $yh_2(y) > 0$ for all $y \neq 0$, we have output strict passivity because the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the output $y(t)$ is identically zero.

Case 4: If $h_1 \in [0, \infty]$, $h_2 \in (0, \infty)$, and $h_3 \in (0, \infty)$,

$$uy \geq \dot{V} + x_1 h_2(x_1) + x_2 h_3(x_2)$$

where $x_1 h_2(x_1) + x_2 h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the state $x(t)$ is identically zero. A system having this property is called *state strictly passive* or, simply, *strictly passive.* Clearly, there is no counterpart for state strict passivity in memoryless functions since there is no state.

$\triangle$

Definition 6.3 *The system* (6.6)–(6.7) *is said to be passive if there exists a continuously differentiable positive semidefinite function* $V(x)$ (*called the storage function*) *such that*

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \quad \forall\ (x, u) \in R^n \times R^p \tag{6.10}$$

Moreover, it is said to be

- *lossless if* $u^T y = \dot{V}$.
- *input-feedforward passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *for some function* φ.
- *input strictly passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *and* $u^T \varphi(u) > 0,\ \forall\ u \neq 0$.
- *output-feedback passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *for some function* ρ.
- *output strictly passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *and* $y^T \rho(y) > 0,\ \forall\ y \neq 0$.
- *strictly passive if* $u^T y \geq \dot{V} + \psi(x)$ *for some positive definite function* ψ.

In all cases, the inequality should hold for all (x, u).

Definition 6.3 reads almost the same as Definition 6.1 for memoryless functions, except for the presence of a storage function $V(x)$. If we adopt the convention that $V(x) = 0$ for a memoryless function, Definition 6.3 can be used for both state models and memoryless functions.

Example 6.2 The integrator of Figure 6.9(a), represented by

$$\dot{x} = u, \qquad y = x$$

is a lossless system since, with $V(x) = (1/2)x^2$ as the storage function, we have $uy = \dot{V}$. When a memoryless function is connected in parallel with the integrator, as shown in Figure 6.9(b), the system is represented by

$$\dot{x} = u, \qquad y = x + h(u)$$

Clearly, the system is input-feedforward passive since the parallel path $h(u)$ can be cancelled by feedforward from the input. With $V(x) = (1/2)x^2$ as the storage function, we have $uy = \dot{V} + uh(u)$. If $h \in [0, \infty]$, the system is passive. If $uh(u) > 0$ for all $u \neq 0$, the system is input strictly passive. When the loop is closed around an integrator with a memoryless function, as in Figure 6.9(c), the system is represented by

$$\dot{x} = -h(x) + u, \qquad y = x$$

Plainly, the system is output-feedback passive since the feedback path can be cancelled by a feedback from the output. With $V(x) = (1/2)x^2$ as the storage function, we have $uy = \dot{V} + yh(y)$. If $h \in [0, \infty]$, the system is passive. If $yh(y) > 0$ for all $y \neq 0$, the system is output strictly passive. △

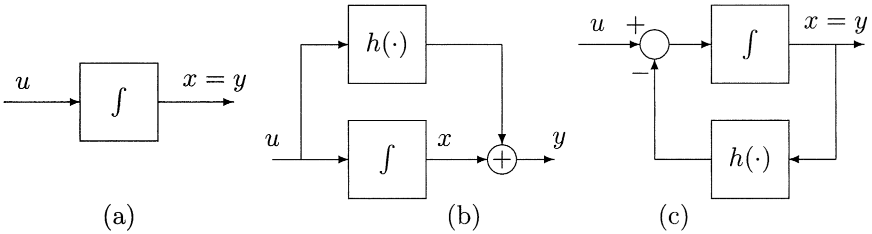


Figure 6.9: Example 6.2

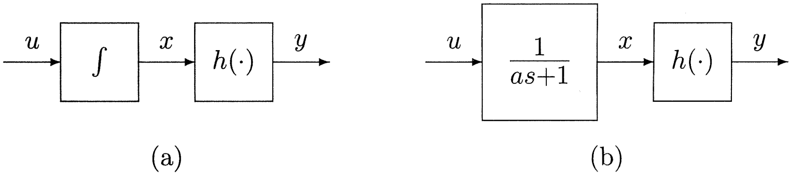


Figure 6.10: Example 6.3

Example 6.3 The cascade connection of an integrator and a passive memoryless function, shown in Figure 6.10(a), is represented by

$$\dot{x} = u, \qquad y = h(x)$$

Passivity of h guarantees that $\int_0^x h(\sigma)\, d\sigma \geq 0$ for all x. With $V(x) = \int_0^x h(\sigma)\, d\sigma$ as the storage function, we have $\dot{V} = h(x)\dot{x} = yu$. Hence, the system is lossless. Suppose now the integrator is replaced by the transfer function $1/(as+1)$ with $a > 0$, as shown in Figure 6.10(b). The system can be represented by the state model

$$a\dot{x} = -x + u, \qquad y = h(x)$$

With $V(x) = a \int_0^x h(\sigma)\, d\sigma$ as the storage function, we have

$$\dot{V} = h(x)(-x + u) = yu - xh(x) \leq yu$$

Hence, the system is passive. When $xh(x) > 0$ for all $x \neq 0$, the system is strictly passive. △

6.3 Positive Real Transfer Functions

Definition 6.4 *A $p \times p$ proper rational transfer function matrix $G(s)$ is called positive real if*

- *poles of all elements of $G(s)$ are in $Re[s] \leq 0$,*

- *for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite, and*

- *any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s\to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian.*

The transfer function $G(s)$ is called strictly positive real[1] *if $G(s-\varepsilon)$ is positive real for some $\varepsilon > 0$.*

When $p = 1$, the second condition of Definition 6.4 reduces to $Re[G(j\omega)] \geq 0$, $\forall\, \omega \in R$, which holds when the Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane. This is a condition that can be satisfied only if the relative degree of the transfer function is zero or one.[2]

The next lemma gives an equivalent characterization of strictly positive real transfer functions.

Lemma 6.1 *Let $G(s)$ be a $p \times p$ proper rational transfer function matrix, and suppose* $\det[G(s) + G^T(-s)]$ *is not identically zero.*[3] *Then, $G(s)$ is strictly positive real if and only if*

- *$G(s)$ is Hurwitz; that is, poles of all elements of $G(s)$ have negative real parts,*

- *$G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega \in R$, and*

- *either $G(\infty) + G^T(\infty)$ is positive definite or it is positive semidefinite and $\lim_{\omega\to\infty} \omega^2 M^T[G(j\omega)+G^T(-j\omega)]M$ is positive definite for any $p\times(p-q)$ full-rank matrix M such that $M^T[G(\infty)+G^T(\infty)]M = 0$, where $q = \operatorname{rank}[G(\infty)+G^T(\infty)]$.*

Proof: See Appendix C.11.

If $G(\infty) + G^T(\infty) = 0$, we can take $M = I$. In the scalar case ($p = 1$), the frequency-domain condition of the lemma reduces to $Re[G(j\omega)] > 0$ for all $\omega \in R$ and either $G(\infty) > 0$ or $G(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2 Re[G(j\omega)] > 0$.

Example 6.4 The transfer function $G(s) = 1/s$ is positive real since it has no poles in $Re[s] > 0$, has a simple pole at $s = 0$ whose residue is 1, and

$$Re[G(j\omega)] = Re\left[\frac{1}{j\omega}\right] = 0, \quad \forall\, \omega \neq 0$$

---

[1] The definition of strictly positive real transfer functions is not uniform in the literature. (See [206] for various definitions and the relationship between them.)

[2] The relative degree of a rational transfer function $G(s) = n(s)/d(s)$ is deg d - deg n. For a proper transfer function, the relative degree is a nonnegative integer.

[3] Equivalently, $G(s) + G^T(-s)$ has a normal rank p over the field of rational functions of s.

It is not strictly positive real since $1/(s-\varepsilon)$ has a pole in $Re[s] > 0$ for any $\varepsilon > 0$. The transfer function $G(s) = 1/(s+a)$ with $a > 0$ is positive real, since it has no poles in $Re[s] \geq 0$ and

$$Re[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \quad \forall\ \omega \in R$$

Since this is so for every $a > 0$, we see that for any $\varepsilon \in (0, a)$ the transfer function $G(s-\varepsilon) = 1/(s+a-\varepsilon)$ will be positive real. Hence, $G(s) = 1/(s+a)$ is strictly positive real. The same conclusion can be drawn from Lemma 6.1 by noting that

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0$$

The transfer function

$$G(s) = \frac{1}{s^2 + s + 1}$$

is not positive real because its relative degree is two. We can see it also by calculating

$$Re[G(j\omega)] = \frac{1 - \omega^2}{(1-\omega^2)^2 + \omega^2} < 0, \quad \forall\ |\omega| > 1$$

Consider the 2×2 transfer function matrix

$$G(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We cannot apply Lemma 6.1 because $\det[G(s) + G^T(-s)] \equiv 0\ \forall\ s$. However, $G(s)$ is strictly positive real, as can be seen by checking the conditions of Definition 6.4. Note that, for $\varepsilon < 1$, the poles of the elements of $G(s-\varepsilon)$ are in $Re[s] < 0$ and

$$G(j\omega - \varepsilon) + G^T(-j\omega - \varepsilon) = \frac{2(1-\varepsilon)}{\omega^2 + (1-\varepsilon)^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is positive semidefinite for all $\omega \in R$. Similarly, it can be seen that the 2×2 transfer function matrix

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s & 1 \\ -1 & 2s+1 \end{bmatrix}$$

is strictly positive real. This time, however, $\det[G(s) + G^T(-s)]$ is not identically zero, and we can apply Lemma 6.1 to arrive at the same conclusion by noting that $G(\infty) + G^T(\infty)$ is positive definite and

$$G(j\omega) + G^T(-j\omega) = \frac{2}{\omega^2 + 1} \begin{bmatrix} \omega^2 & -j\omega \\ j\omega & 2\omega^2 + 1 \end{bmatrix}$$

is positive definite for all $\omega \in R$. Finally, the 2×2 transfer function matrix

$$G(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix}$$

has

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

It can be verified that

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2\omega^2}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix}$$

is positive definite for all $\omega \in R$. Taking $M^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$, it can be verified that

$$\lim_{\omega\to\infty} \omega^2 M^T [G(j\omega) + G^T(-j\omega)] M = 4$$

Consequently, by Lemma 6.1, we conclude that $G(s)$ is strictly positive real. △

Passivity properties of positive real transfer functions can be shown by using the next two lemmas, which are known, respectively, as the *positive real lemma* and the *Kalman–Yakubovich–Popov lemma.* The lemmas give algebraic characterization of positive real and strictly positive real transfer functions.

Lemma 6.2 (Positive Real) *Let $G(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ transfer function matrix where (A, B) is controllable and (A, C) is observable. Then, $G(s)$ is positive real if and only if there exist matrices $P = P^T > 0$, L, and W such that*

$$\begin{aligned} PA + A^T P &= -L^T L && (6.11) \\ PB &= C^T - L^T W && (6.12) \\ W^T W &= D + D^T && (6.13) \end{aligned}$$

◇

Proof: See Appendix C.12.

Lemma 6.3 (Kalman–Yakubovich–Popov) *Let $G(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ transfer function matrix, where (A, B) is controllable and (A, C) is observable. Then, $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that*

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P && (6.14) \\ PB &= C^T - L^T W && (6.15) \\ W^T W &= D + D^T && (6.16) \end{aligned}$$

◇

Proof: Suppose there exist $P = P^T > 0$, L, W, and $\varepsilon > 0$ that satisfy (6.14) through (6.16). Set $\mu = \varepsilon/2$ and recall that $G(s-\mu) = C(sI - \mu I - A)^{-1}B + D$. From (6.14), we have

$$P(A + \mu I) + (A + \mu I)^T P = -L^T L \tag{6.17}$$

It follows from Lemma 6.2 that $G(s-\mu)$ is positive real. Hence, $G(s)$ is strictly positive real. On the other hand, suppose $G(s)$ is strictly positive real. There exists $\mu > 0$ such that $G(s-\mu)$ is positive real. It follows from Lemma 6.2 that there are matrices $P = P^T > 0$, L, and W, which satisfy (6.15) through (6.17). Setting $\varepsilon = 2\mu$ shows that P, L, W, and ε satisfy (6.14) through (6.16). □

Lemma 6.4 *The linear time-invariant minimal realization*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with $G(s) = C(sI - A)^{-1}B + D$ *is*

- *passive if* $G(s)$ *is positive real;*
- *strictly passive if* $G(s)$ *is strictly positive real.*

◇

Proof: Apply Lemmas 6.2 and 6.3, respectively, and use $V(x) = (1/2)x^T P x$ as the storage function.

$$\begin{aligned} u^T y - \frac{\partial V}{\partial x}(Ax + Bu) &= u^T(Cx + Du) - x^T P(Ax + Bu) \\ &= u^T Cx + \tfrac{1}{2}u^T(D + D^T)u - \tfrac{1}{2}x^T(PA + A^T P)x - x^T PBu \\ &= u^T(B^T P + W^T L)x + \tfrac{1}{2}u^T W^T W u \\ &\quad + \tfrac{1}{2}x^T L^T L x + \tfrac{1}{2}\varepsilon x^T P x - x^T PBu \\ &= \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) + \tfrac{1}{2}\varepsilon x^T P x \ \geq \ \tfrac{1}{2}\varepsilon x^T P x \end{aligned}$$

In the case of Lemma 6.2, $\varepsilon = 0$, and we conclude that the system is passive, while in the case of Lemma 6.3, $\varepsilon > 0$, and we conclude that the system is strictly passive. □

6.4 $\mathcal{L}_2$ and Lyapunov Stability

In this section, we study $\mathcal{L}_2$ and Lyapunov stability of passive systems of the form

$$\dot{x} = f(x, u) \tag{6.18}$$
$$y = h(x, u) \tag{6.19}$$

where $f : R^n \times R^p \to R^n$ is locally Lipschitz, $h : R^n \times R^p \to R^p$ is continuous, $f(0, 0) = 0$, and $h(0, 0) = 0$.

Lemma 6.5 *If the system* (6.18)–(6.19) *is output strictly passive with* $u^Ty \geq \dot{V} + \delta y^Ty$, *for some* $\delta > 0$, *then it is finite-gain* $\mathcal{L}_2$ *stable and its* $\mathcal{L}_2$ *gain is less than or equal to* $1/\delta$.

Proof: The derivative of the storage function $V(x)$ satisfies

$$\begin{aligned}\dot{V} &\leq u^Ty - \delta y^Ty = -\frac{1}{2\delta}(u-\delta y)^T(u-\delta y) + \frac{1}{2\delta}u^Tu - \frac{\delta}{2}y^Ty \\ &\leq \frac{1}{2\delta}u^Tu - \frac{\delta}{2}y^Ty\end{aligned}$$

integrating both sides over $[0, \tau]$ yields

$$\int_0^\tau y^T(t)y(t)\ dt \leq \frac{1}{\delta^2}\int_0^\tau u^T(t)u(t)\ dt - \frac{2}{\delta}[V(x(\tau)) - V(x(0))]$$

Thus,

$$\|y_\tau\|_{\mathcal{L}_2} \leq \frac{1}{\delta}\|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\delta}V(x(0))}$$

where we used the facts that $V(x) \geq 0$ and $\sqrt{a^2+b^2} \leq a+b$ for nonnegative numbers a and b. □

Lemma 6.6 *If the system* (6.18)–(6.19) *is passive with a positive definite storage function* $V(x)$, *then the origin of* $\dot{x} = f(x,0)$ *is stable.*

Proof: Take V as a Lyapunov function candidate for $\dot{x} = f(x,0)$. Then $\dot{V} \leq 0$. □

To show asymptotic stability of the origin of $\dot{x} = f(x,0)$, we need to either show that $\dot{V}$ is negative definite or apply the invariance principle. In the next lemma, we apply the invariance principle by considering a case where $\dot{V} = 0$ when $y = 0$ and then require the additional property that

$$y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \tag{6.20}$$

for all solutions of (6.18) when $u = 0$. Equivalently, no solutions of $\dot{x} = f(x,0)$ can stay identically in $S = \{x \in R^n \mid h(x,0) = 0\}$, other than the trivial solution $x(t) \equiv 0$. The property (6.20) can be interpreted as an observability condition. Recall that for the linear system

$$\dot{x} = Ax, \qquad y = Cx$$

observability is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \ \Leftrightarrow\ x(0) = 0 \ \Leftrightarrow\ x(t) \equiv 0$$

For easy reference, we define (6.20) as an observability property of the system.

Definition 6.5 *The system* (6.18)–(6.19) *is said to be zero-state observable if no solution of* $\dot{x} = f(x, 0)$ *can stay identically in* $S = \{x \in R^n \mid h(x, 0) = 0\}$, *other than the trivial solution* $x(t) \equiv 0$.

Lemma 6.7 *Consider the system* (6.18)–(6.19). *The origin of* $\dot{x} = f(x, 0)$ *is asymptotically stable if the system is*

- *strictly passive or*
- *output strictly passive and zero-state observable.*

Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable. ◇

Proof: Suppose the system is strictly passive and let $V(x)$ be its storage function. Then, with $u = 0$, $\dot{V}$ satisfies the inequality $\dot{V} \leq -\psi(x)$, where $\psi(x)$ is positive definite. We can use this inequality to show that $V(x)$ is positive definite. In particular, for any $x \in R^n$, the equation $\dot{x} = f(x, 0)$ has a solution $\phi(t; x)$, starting from x at $t = 0$ and defined on some interval $[0, \delta]$. Integrating the inequality $\dot{V} \leq -\psi(x)$ yields

$$V(\phi(\tau, x)) - V(x) \leq -\int_0^\tau \psi(\phi(t; x))\, dt, \quad \forall\ \tau \in [0, \delta]$$

Using $V(\phi(\tau, x)) \geq 0$, we obtain

$$V(x) \geq \int_0^\tau \psi(\phi(t; x))\, dt$$

Suppose now that there is $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. The foregoing inequality implies

$$\int_0^\tau \psi(\phi(t; \bar{x}))\, dt = 0, \ \forall\ \tau \in [0, \delta] \ \Rightarrow\ \psi(\phi(t; \bar{x})) \equiv 0 \ \Rightarrow\ \phi(t; \bar{x}) \equiv 0 \ \Rightarrow\ \bar{x} = 0$$

which contradicts the claim that $\bar{x} \neq 0$. Thus, $V(x) > 0$ for all $x \neq 0$. This qualifies $V(x)$ as a Lyapunov function candidate, and since $\dot{V}(x) \leq -\psi(x)$, we conclude that the origin is asymptotically stable.

Suppose now the system is output strictly passive and let $V(x)$ be its storage function. Then, with $u = 0$, $\dot{V}$ satisfies the inequality $\dot{V} \leq -y^T \rho(y)$, where $y^T \rho(y) > 0$ for all $y \neq 0$. By repeating the preceding argument, we can use the inequality to show that $V(x)$ is positive definite. In particular, for any $x \in R^n$, we have

$$V(x) \geq \int_0^\tau h^T(\phi(t; x), 0) \rho(h(\phi(t; x), 0))\, dt$$

Suppose now that there is $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. The foregoing inequality implies

$$\int_0^\tau h^T(\phi(t;\bar{x}),0)\rho(h(\phi(t;\bar{x}),0))\,dt = 0, \quad \forall\, \tau \in [0,\delta] \quad \Rightarrow \quad h(\phi(t;\bar{x}),0) \equiv 0$$

which, due to zero-state observability, implies

$$\phi(t;\bar{x}) \equiv 0 \quad \Rightarrow \quad \bar{x} = 0$$

Hence, $V(x) > 0$ for all $x \neq 0$. This qualifies $V(x)$ as a Lyapunov function candidate, and since $\dot{V}(x) \leq -y^T\rho(y)$ and $y(t) \equiv 0 \;\Rightarrow x(t) \equiv 0$, we conclude by the invariance principle that the origin is asymptotically stable. Finally, if $V(x)$ is radially unbounded, we can infer global asymptotic stability from Theorem 4.2 and Corollary 4.2, respectively. □

Example 6.5 Consider the p-input–p-output system[4]

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned}$$

where f is locally Lipschitz, G and h are continuous, $f(0) = 0$, and $h(0) = 0$. Suppose there is a continuously differentiable positive semidefinite function $V(x)$ such that

$$\frac{\partial V}{\partial x} f(x) \leq 0, \qquad \frac{\partial V}{\partial x} G(x) = h^T(x)$$

Then,

$$u^T y - \frac{\partial V}{\partial x}[f(x) + G(x)u] = u^T h(x) - \frac{\partial V}{\partial x} f(x) - h^T(x)u = -\frac{\partial V}{\partial x} f(x) \geq 0$$

which shows that the system is passive. If $V(x)$ is positive definite, we can conclude that the origin of $\dot{x} = f(x)$ is stable. If we have the stronger condition

$$\frac{\partial V}{\partial x} f(x) \leq -k h^T(x)h(x), \qquad \frac{\partial V}{\partial x} G(x) = h^T(x)$$

for some $k > 0$, then

$$u^T y - \frac{\partial V}{\partial x}[f(x) + G(x)u] \geq k y^T y$$

and the system is output strictly passive with $\rho(y) = ky$. It follows from Lemma 6.5 that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. If, in addition, the system is zero-state observable, then the origin of $\dot{x} = f(x)$ is asymptotically stable. Furthermore, if $V(x)$ is radially unbounded, the origin will be globally asymptotically stable. △

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[4] $\mathcal{L}_2$ stability of this system was studied in Examples 5.9 and 5.10.

Example 6.6 Consider the single-input–single-output system[5]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + u \\ y &= x_2 \end{aligned}$$

where a and k are positive constants. Consider also the positive definite, radially unbounded function $V(x) = (1/4)ax_1^4 + (1/2)x_2^2$ as a storage function candidate. The derivative $\dot{V}$ is given by

$$\dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

Therefore, the system is output strictly passive with $\rho(y) = ky$. It follows from Lemma 6.5 that the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to $1/k$. Moreover, when $u = 0$,

$$y(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Hence, the system is zero-state observable. It follows from Lemma 6.7 that the origin of the unforced system is globally asymptotically stable. △

6.5 Feedback Systems: Passivity Theorems

Consider the feedback connection of Figure 6.11 where each of the feedback components H_1 and H_2 is either a time-invariant dynamical system represented by the state model

$$\begin{aligned} \dot{x}_i &= f_i(x_i, e_i) && (6.21) \\ y_i &= h_i(x_i, e_i) && (6.22) \end{aligned}$$

or a (possibly time-varying) memoryless function represented by

$$y_i = h_i(t, e_i) \qquad (6.23)$$

In describing the feedback connection of Figure 6.11 it should be noted that u_1, y_1, u_2, and y_2 could be vectors of the same dimension. We are interested in using passivity properties of the feedback components H_1 and H_2 to analyze stability of the feedback connection. We will study both $\mathcal{L}_2$ and Lyapunov stability. We require the feedback connection to have a well-defined state model. When both components H_1 and H_2 are dynamical systems, the closed-loop state model takes the form

$$\begin{aligned} \dot{x} &= f(x, u) && (6.24) \\ y &= h(x, u) && (6.25) \end{aligned}$$

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[5] $\mathcal{L}_2$ and Lyapunov stability of this system were studied in Examples 5.8 and 4.9.

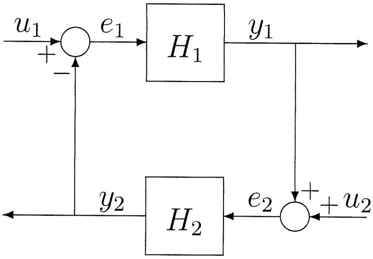

Figure 6.11: Feedback connection.

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We assume that f is locally Lipschitz, h is continuous, $f(0,0) = 0$, and $h(0,0) = 0$. It can be easily verified that the feedback connection will have a well-defined state model if the equations

$$e_1 = u_1 - h_2(x_2, e_2) \tag{6.26}$$

$$e_2 = u_2 + h_1(x_1, e_1) \tag{6.27}$$

have a unique solution (e_1, e_2) for every (x_1, x_2, u_1, u_2). The properties $f(0,0) = 0$ and $h(0,0) = 0$ follow from $f_i(0,0) = 0$ and $h_i(0,0) = 0$. It is also easy to see that (6.26) and (6.27) will always have a unique solution if h_1 is independent of e_1 or h_2 is independent of e_2. In this case, the functions f and h of the closed-loop state model inherit smoothness properties of the functions f_i and h_i of the feedback components. In particular, if f_i and h_i are locally Lipschitz, so are f and h. For linear systems, requiring h_i to be independent of e_i is equivalent to requiring the transfer function of H_i to be strictly proper.[6]

When one component, H_1 say, is a dynamical system, while the other one is a memoryless function, the closed-loop state model takes the form

$$\dot{x} = f(t, x, u) \tag{6.28}$$

$$y = h(t, x, u) \tag{6.29}$$

where

$$x = x_1, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We assume that f is piecewise continuous in t and locally Lipschitz in (x, u), h is piecewise continuous in t and continuous in (x, u), $f(t, 0, 0) = 0$, and $h(t, 0, 0) = 0$.

---

[6] The existence of solutions for (6.26) and (6.27) is pursued further in Exercise 6.12.

The feedback connection will have a well-defined state model if the equations

$$\begin{aligned} e_1 &= u_1 - h_2(t, e_2) && (6.30) \\ e_2 &= u_2 + h_1(x_1, e_1) && (6.31) \end{aligned}$$

have a unique solution (e_1, e_2) for every (x_1, t, u_1, u_2). This will be always the case when h_1 is independent of e_1. The case when both components are memoryless functions is less important and follows trivially as a special case when the state x does not exist. In this case, the feedback connection is represented by $y = h(t, u)$.

The starting point of our analysis is the following fundamental property:

Theorem 6.1 *The feedback connection of two passive systems is passive.*

Proof: Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2, respectively. If either component is a memoryless function, take $V_i = 0$. Then,

$$e_i^T y_i \geq \dot{V}_i$$

From the feedback connection of Figure 6.11, we see that

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

Hence,

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2$$

Taking $V(x) = V_1(x_1) + V_2(x_2)$ as the storage function for the feedback connection, we obtain

$$u^T y \geq \dot{V}$$

□

Using Theorem 6.1 and the results of the previous section on stability properties of passive systems, we can arrive at some straightforward conclusions on stability of the feedback connection. We start with $\mathcal{L}_2$ stability. The next lemma is an immediate consequence of Lemma 6.5.

Lemma 6.8 *The feedback connection of two output strictly passive systems with*

$$e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i, \quad \delta_i > 0$$

is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\min\{\delta_1, \delta_2\}$.

Proof: With $V = V_1 + V_2$ and $\delta = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} u^T y &= e_1^T y_1 + e_2^T y_2 \ \geq \ \dot{V}_1 + \delta_1 y_1^T y_1 + \dot{V}_2 + \delta_2 y_2^T y_2 \\ &\geq \dot{V} + \delta(y_1^T y_1 + y_2^T y_2) = \dot{V} + \delta y^T y \end{aligned}$$

□

Reading the proof of Lemma 6.5 shows that we use the inequality

$$u^T y \geq \dot{V} + \delta y^T y \tag{6.32}$$

to arrive at the inequality

$$\dot{V} \leq \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \tag{6.33}$$

which is then used to show finite-gain $\mathcal{L}_2$ stability. In Lemma 6.8, we establish (6.32) for the feedback connection, which then leads to (6.33). However, even if (6.32) does not hold for the feedback connection, we may still be able to show an inequality of the form (6.33). This idea is used in the next theorem to prove a more general result that includes Lemma 6.8 as a special case.

Theorem 6.2 *Consider the feedback connection of Figure 6.11 and suppose each feedback component satisfies the inequality*

$$e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i, \quad \text{for } i = 1, 2 \tag{6.34}$$

for some storage function $V_i(x_i)$. Then, the closed-loop map from u to y is finite gain $\mathcal{L}_2$ stable if

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0 \tag{6.35}$$

◇

Proof: Adding inequalities (6.34) for $i = 1, 2$ and using

$$\begin{aligned} e_1^T y_1 + e_2^T y_2 &= u_1^T y_1 + u_2^T y_2 \\ e_1^T e_1 &= u_1^T u_1 - 2u_1^T y_2 + y_2^T y_2 \\ e_2^T e_2 &= u_2^T u_2 + 2u_2^T y_1 + y_1^T y_1 \end{aligned}$$

we obtain

$$\dot{V} \leq -y^T L y - u^T M u + u^T N y$$

where

$$L = \begin{bmatrix} (\varepsilon_2 + \delta_1)I & 0 \\ 0 & (\varepsilon_1 + \delta_2)I \end{bmatrix}, \ M = \begin{bmatrix} \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \end{bmatrix}, \ N = \begin{bmatrix} I & 2\varepsilon_1 I \\ -2\varepsilon_2 I & I \end{bmatrix}$$

and $V(x) = V_1(x_1) + V_2(x_2)$. Let $a = \min\{\varepsilon_2 + \delta_1, \varepsilon_1 + \delta_2\} > 0$, $b = \|N\|_2 \geq 0$, and $c = \|M\|_2 \geq 0$. Then

$$\begin{aligned} \dot{V} &\leq -a\|y\|_2^2 + b\|u\|_2\|y\|_2 + c\|u\|_2^2 \\ &= -\frac{1}{2a}(b\|u\|_2 - a\|y\|_2)^2 + \frac{b^2}{2a}\|u\|_2^2 - \frac{a}{2}\|y\|_2^2 + c\|u\|_2^2 \\ &\leq \frac{k^2}{2a}\|u\|_2^2 - \frac{a}{2}\|y\|_2^2 \end{aligned}$$

where $k^2 = b^2 + 2ac$. Integrating over $[0, \tau]$, using $V(x) \geq 0$, and taking the square roots, we arrive at

$$\|y_\tau\|_{\mathcal{L}_2} \leq \frac{k}{a}\|u_\tau\|_{\mathcal{L}_2} + \sqrt{2V(x(0))/a}$$

which completes the proof of the theorem. $\square$

Theorem 6.2 reduces to Lemma 6.8 when (6.34) is satisfied with $\varepsilon_1 = \varepsilon_2 = 0$, $\delta_1 > 0$, and $\delta_2 > 0$. However, condition (6.35) is satisfied in several other cases. For example, it is satisfied when both H_1 and H_2 are input strictly passive with $e_i^T y_i \geq \dot{V}_i + \varepsilon_i u_i^T u_i$ for some $\varepsilon_i > 0$. It is also satisfied when one component (H_1 say) is passive, while the other component satisfies (6.34) with positive ε_2 and δ_2. What is more interesting is that (6.35) can be satisfied even when some of the constants ε_i and δ_i are negative. For example, a negative ε_1 can be compensated for by a positive δ_2. This is a case where shortage of passivity (at the input side) of H_1 is compensated for by excess of passivity (at the output side) of H_2. Similarly, a negative δ_2 can be compensated for by a positive ε_1. This is a case where shortage of passivity (at the output side) of H_2 is compensated for by excess of passivity (at the input side) of H_1.

Example 6.7 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + G(x)e_1 \\ y_1 & = & h(x) \end{array} \right. \qquad \text{and} \qquad H_2 : \quad y_2 = ke_2$$

where $k > 0$ and $e_i,\ y_i \in R^p$. Suppose there is a positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \qquad \forall\ x \in R^n$$

Both components are passive. Moreover, H_2 satisfies

$$e_2^T y_2 = k e_2^T e_2 = \gamma k e_2^T e_2 + \frac{(1-\gamma)}{k} y_2^T y_2, \qquad 0 < \gamma < 1$$

Thus, (6.34) is satisfied with $\varepsilon_1 = \delta_1 = 0$, $\varepsilon_2 = \gamma k$, and $\delta_2 = (1-\gamma)/k$. This shows that (6.35) is satisfied, and we conclude that the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable. $\triangle$

Example 6.8 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -ax_1^3 - \sigma(x_2) + e_1 \\ y_1 & = & x_2 \end{array} \right. \qquad \text{and} \qquad H_2 : \quad y_2 = ke_2$$

where $\sigma \in [-\alpha, \infty]$, $a > 0$, $\alpha > 0$, and $k > 0$. If σ was in the sector $[0, \infty]$, we could have shown that H_1 is passive with the storage function $V_1(x) = (a/4)x_1^4 + (1/2)x_2^2$. For $\sigma \in [-\alpha, \infty]$, we have

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - x_2\sigma(x_2) + x_2e_1 \leq \alpha x_2^2 + x_2e_1 = \alpha y_1^2 + y_1e_1$$

Hence, (6.34) is satisfied for H_1 with $\varepsilon_1 = 0$ and $\delta_1 = -\alpha$. Since

$$e_2 y_2 = k e_2^2 = \gamma k e_2^2 + \frac{(1-\gamma)}{k} y_2^2, \quad 0 < \gamma < 1$$

(6.34) is satisfied for H_2 with $\varepsilon_2 = \gamma k$ and $\delta_2 = (1-\gamma)/k$. If $k > \alpha$, we can choose γ such that $\gamma k > \alpha$. Then, $\varepsilon_1 + \delta_2 > 0$ and $\varepsilon_2 + \delta_1 > 0$. We conclude that the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable. △

Let us turn now to studying Lyapunov stability of the feedback connection. We are interested in studying stability and asymptotic stability of the origin of the closed-loop system when the input $u = 0$. Stability of the origin follows trivially from Theorem 6.1 and Lemma 6.6. Therefore, we focus our attention on studying asymptotic stability. The next theorem is an immediate consequence of Theorem 6.1 and Lemma 6.7.

Theorem 6.3 *Consider the feedback connection of two time-invariant dynamical systems of the form* (6.21)–(6.22). *The origin of the closed-loop system* (6.24) (*when* $u = 0$) *is asymptotically stable if*

- *both feedback components are strictly passive,*
- *both feedback components are output strictly passive and zero-state observable, or*
- *one component is strictly passive and the other one is output strictly passive and zero-state observable.*

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable. ◇

Proof: Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2, respectively. As in the proof of Lemma 6.7, we can show that $V_1(x_1)$ and $V_2(x_2)$ are positive definite functions. Take $V(x) = V_1(x_1) + V_2(x_2)$ as a Lyapunov function candidate for the closed-loop system. In the first case, the derivative $\dot{V}$ satisfies

$$\dot{V} \le u^T y - \psi_1(x_1) - \psi_2(x_2) = -\psi_1(x_1) - \psi_2(x_2)$$

since $u = 0$. Hence, the origin is asymptotically stable. In the second case,

$$\dot{V} \le -y_1^T \rho_1(y_1) - y_2^T \rho_2(y_2)$$

where $y_i^T \rho_i(y_i) > 0$ for all $y_i \neq 0$. Here $\dot{V}$ is only negative semidefinite and $\dot{V} = 0 \Rightarrow y = 0$. To apply the invariance principle, we need to show that $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$. Note that $y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0$. Then, zero-state observability of H_1 shows that $y_1(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$. Similarly, $y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0$ and zero-state observability of H_2 shows that $y_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$. Thus, the origin is asymptotically stable. In the third case (with H_1 as the strictly passive component),

$$\dot{V} \le -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

and $\dot{V} = 0$ implies $x_1 = 0$ and $y_2 = 0$. Note that $y_2(t) \equiv 0 \;\Rightarrow\; e_1(t) \equiv 0$, which together with $x_1(t) \equiv 0$ imply that $y_1(t) \equiv 0$. Hence, $e_2(t) \equiv 0$ and zero-state observability of H_2 shows that $y_2(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0$. Thus, the origin is asymptotically stable. Finally, if $V_1(x_1)$ and $V_2(x_2)$ are radially unbounded, so is $V(x)$, and we can conclude global asymptotic stability. $\square$

The proof uses a simple idea, namely, that the sum of the storage functions for the feedback components is used as a Lyapunov function candidate for the feedback connection. Beyond this simple idea, the rest of the proof is straightforward Lyapunov analysis. In fact, the analysis is restrictive because to show that $\dot{V} = \dot{V}_1 + \dot{V}_2 \le 0$, we insist that both $\dot{V}_1 \le 0$ and $\dot{V}_2 \le 0$. Clearly, this is not necessary. One term, $\dot{V}_1$ say, could be positive over some region as long as the sum $\dot{V} \le 0$ over the same region. This is again a manifestation of the idea that shortage of passivity of one component can be compensated for by excess of passivity of the other component. This idea is exploited in Examples 6.10 and 6.11, while Example 6.9 is a straightforward application of Theorem 6.3.

Example 6.9 Consider the feedback connection of

$$H_1: \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -ax_1^3 - kx_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \quad \text{and} \quad H_2: \left\{ \begin{array}{rcl} \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & -bx_3 - x_4^3 + e_2 \\ y_2 & = & x_4 \end{array} \right.$$

where a, b, and k are positive constants. Using $V_1 = (a/4)x_1^4 + (1/2)x_2^2$ as the storage function for H_1, we obtain

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

Hence, H_1 is output strictly passive. Besides, with $e_1 = 0$, we have

$$y_1(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

which shows that H_1 is zero-state observable. Using $V_2 = (b/2)x_3^2 + (1/2)x_4^2$ as the storage function for H_2, we obtain

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

Therefore, H_2 is output strictly passive. Moreover, with $e_2 = 0$, we have

$$y_2(t) \equiv 0 \;\Leftrightarrow\; x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$$

which shows that H_2 is zero-state observable. Thus, by the second case of Theorem 6.3 and the fact that V_1 and V_2 are radially unbounded, we conclude that the origin is globally asymptotically stable. $\triangle$

Example 6.10 Reconsider the feedback connection of the previous example, but change the output of H_1 to $y_1 = x_2 + e_1$. From the expression

$$\dot{V}_1 = -kx_2^2 + x_2 e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1 e_1$$

we can conclude that H_1 is passive, but we cannot conclude strict passivity or output strict passivity. Therefore, we cannot apply Theorem 6.3. Using

$$V = V_1 + V_2 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}bx_3^2 + \tfrac{1}{2}x_4^2$$

as a Lyapunov function candidate for the closed-loop system, we obtain

$$\begin{aligned} \dot{V} &= -kx_2^2 + x_2 e_1 - x_4^4 + x_4 e_2 \\ &= -kx_2^2 - x_2 x_4 - x_4^4 + x_4(x_2 - x_4) \\ &= -kx_2^2 - x_4^4 - x_4^2 \le 0 \end{aligned}$$

Moreover, $\dot{V} = 0$ implies that $x_2 = x_4 = 0$ and

$$x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) - x_4(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

$$x_4(t) \equiv 0 \;\Rightarrow\; -bx_3(t) + x_2(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$$

Thus, by the invariance principle and the fact that V is radially unbounded, we conclude that the origin is globally asymptotically stable. △

Example 6.11 Reconsider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2) \end{aligned}$$

from Examples 4.8 and 4.9, where h_1 and h_2 are locally Lipschitz and belong to the sector $(0, \infty)$. The system can be viewed as the state model of the feedback connection of Figure 6.12, where H_1 consists of a negative feedback loop around the integrator x_2 with h_2 in the feedback path, and H_2 consists of a cascade connection of the integrator x_1 with h_1. We saw in Example 6.2 that H_1 is output strictly passive with the storage function $V_1 = (1/2)x_2^2$ and, in Example 6.3, that H_2 is lossless with the storage function $V_2 = \int_0^{x_1} h_1(\sigma)\, d\sigma$. We cannot apply Theorem 6.3 because H_2 is neither strictly passive nor output strictly passive. However, using $V = V_1 + V_2 = \int_0^{x_1} h_1(\sigma)\, d\sigma + (1/2)x_2^2$ as a Lyapunov function candidate, we can proceed to investigate asymptotic stability of the origin. This is already done in Examples 4.8 and 4.9, where it is shown that the origin is asymptotically stable and will be globally asymptotically stable if $\int_0^y h_1(z)\, dz \to \infty$ as $|y| \to \infty$. We will not repeat the analysis of these two examples here, but let us note that if $h_1(y)$ and $h_2(y)$ belong to the sector $(0, \infty)$ only for $y \in (-a, a)$, then the Lyapunov analysis can be limited to some region around the origin, leading to a local asymptotic stability conclusion, as in Example 4.8. This shows that passivity remains useful as a tool for Lyapunov analysis even when it holds only on a finite region, rather than the whole space. △

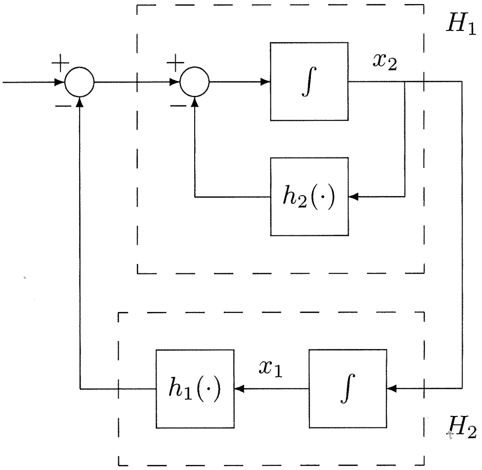

Figure 6.12: Example 6.11.

When the feedback connection has a dynamical system as one component and a memoryless function as the other component, we can perform Lyapunov analysis by using the storage function of the dynamical system as a Lyapunov function candidate. It is important, however, to distinguish between time-invariant and time-varying memoryless functions, for in the latter case the closed-loop system will be nonautonomous and we cannot apply the invariance principle as we did in the proof of Theorem 6.3. We treat these two cases separately in the next two theorems.

Theorem 6.4 *Consider the feedback connection of a strictly passive, time-invariant, dynamical system of the form* (6.21)–(6.22) *with a passive* (*possibly time-varying*) *memoryless function of the form* (6.23). *Then, the origin of the closed-loop system* (6.28) (*when* $u = 0$) *is uniformly asymptotically stable. Furthermore, if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable.* ◇

Proof: As in the proof of Lemma 6.7, it can be shown that $V_1(x_1)$ is positive definite. Its derivative is given by

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1) \leq -\psi_1(x_1)$$

The conclusion follows from Theorem 4.9. □

Theorem 6.5 *Consider the feedback connection of a time-invariant dynamical system* H_1 *of the form* (6.21)–(6.22) *with a time-invariant memoryless function* H_2 *of* *the form* (6.23). *Suppose that* H_1 *is zero-state observable and has a positive definite storage function, which satisfies*

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1) \tag{6.36}$$

and that H_2 *satisfies*

$$e_2^T y_2 \geq e_2^T \varphi_2(e_2) \tag{6.37}$$

Then, the origin of the closed-loop system (6.28) (*when* $u = 0$) *is asymptotically stable if*

$$v^T[\rho_1(v) + \varphi_2(v)] > 0, \quad \forall\ v \neq 0 \tag{6.38}$$

Furthermore, if V_1 *is radially unbounded, the origin will be globally asymptotically stable.* ◇

Proof: Use $V_1(x_1)$ as a Lyapunov function candidate, to obtain

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \ \leq \ e_1^T y_1 - y_1^T \rho_1(y_1) \\ &= -e_2^T y_2 - y_1^T \rho_1(y_1) \ \leq \ -[y_1^T \varphi_2(y_1) + y_1^T \rho_1(y_1)] \end{aligned}$$

Inequality (6.38) shows that $\dot{V}_1 \leq 0$ and $\dot{V}_1 = 0 \ \Rightarrow y_1 = 0$. Noting that $y_1(t) \equiv 0 \ \Rightarrow \ e_2(t) \equiv 0 \ \Rightarrow e_1(t) \equiv 0$, we see that zero-state observability of H_1 implies that $x_1(t) \equiv 0$. The conclusion follows from the invariance principle. □

Example 6.12 Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function. From Lemma 6.4, we know that the dynamical system is strictly passive with a positive definite storage function of the form $V(x) = (1/2)x^T Px$. From Theorem 6.4, we conclude that the origin of the closed-loop system is globally uniformly asymptotically stable. This is a version of the circle criterion of Section 7.1. △

Example 6.13 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + G(x)e_1 \\ y_1 &=& h(x) \end{array} \right. \qquad \text{and} \qquad H_2 : \ y_2 = \sigma(e_2)$$

where $\sigma \in (0, \infty)$ and $e_i, \ y_i \in R^p$. Suppose there is a radially unbounded positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \qquad \forall\ x \in R^n$$

and H_1 is zero-state observable. Both components are passive. Moreover, H_2 satisfies

$$e_2^T y_2 = e_2^T \sigma(e_2)$$

Thus, (6.36) is satisfied with $\rho_1 = 0$, and (6.37) is satisfied with $\varphi_2 = \sigma$. Since $\sigma \in (0, \infty)$, (6.38) is satisfied. It follows from Theorem 6.5 that the origin of the closed-loop system is globally asymptotically stable. △

We conclude this section by presenting *loop transformations*, which extend the utility of passivity theorems. Starting with a feedback connection in which one of the two feedback components is not passive or does not satisfy a condition that is needed in one of the theorems, we may be able to reconfigure the feedback connection into an equivalent connection that has the desired properties. We illustrate the process first for loop transformations that use constant gains. Suppose H_1 is a time-invariant dynamical system, while H_2 is a (possibly time-varying) memoryless function that belongs to the sector $[K_1, K_2]$, where $K = K_2 - K_1$ is a positive definite symmetric matrix. We saw in Section 6.1 that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 6.7. Input feedforward on H_2 can be nullified by output feedback on H_1, as shown in Figure 6.13(b), resulting in an equivalent feedback connection, as far as asymptotic stability of the origin is concerned. Similarly, premultiplying the modified H_2 by K^{-1} can be nullified by postmultiplying the modified H_1 by K, as shown in Figure 6.13(c). Finally, output feedback on the component in the feedback path can be nullified by input feedforward on the component in the forward path, as shown in Figure 6.13(d). The reconfigured feedback connection has two components $\tilde{H}_1$ and $\tilde{H}_2$, where $\tilde{H}_2$ is a memoryless function that belongs to the sector $[0, \infty]$. We can now apply Theorem 6.4 or 6.5 if $\tilde{H}_1$ satisfies the conditions of the respective theorem.

Example 6.14 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) + bx_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \qquad \text{and} \qquad H_2 : \;\; y_2 = \sigma(e_2)$$

where $\sigma \in [\alpha, \beta]$, $h \in [\alpha_1, \infty]$, $b > 0$, $\alpha_1 > 0$, and $k = \beta - \alpha > 0$. Applying the loop transformation of Figure 6.13(d) (with $K_1 = \alpha$ and $K_2 = \beta$) results in the feedback connection of

$$\tilde{H}_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) - ax_2 + \tilde{e}_1 \\ \tilde{y}_1 & = & kx_2 + \tilde{e}_1 \end{array} \right. \qquad \text{and} \qquad \tilde{H}_2 : \;\; \tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)$$

where $\tilde{\sigma} \in [0, \infty]$ and $a = \alpha - b$. If $\alpha > b$, it can be shown (Exercise 6.4) that $\tilde{H}_1$ is strictly passive with a storage function of the form $V_1 = k \int_0^{x_1} h(s)\, ds + x^T P x$, where $P = P^T > 0$. Thus, we conclude from Theorem 6.4 that the origin of the feedback connection is globally asymptotically stable. △

Next, we consider loop transformations with dynamic multipliers, as shown in Figure 6.14. Premultiplying H_2 by a transfer function $W(s)$ can by nullified by postmultiplying H_1 by $W^{-1}(s)$, provided the inverse exists. For example, when H_2 is a passive, time-invariant, memoryless function h, we saw in Example 6.3 that premultiplying h by the transfer function $1/(as + 1)$ results in a strictly passive

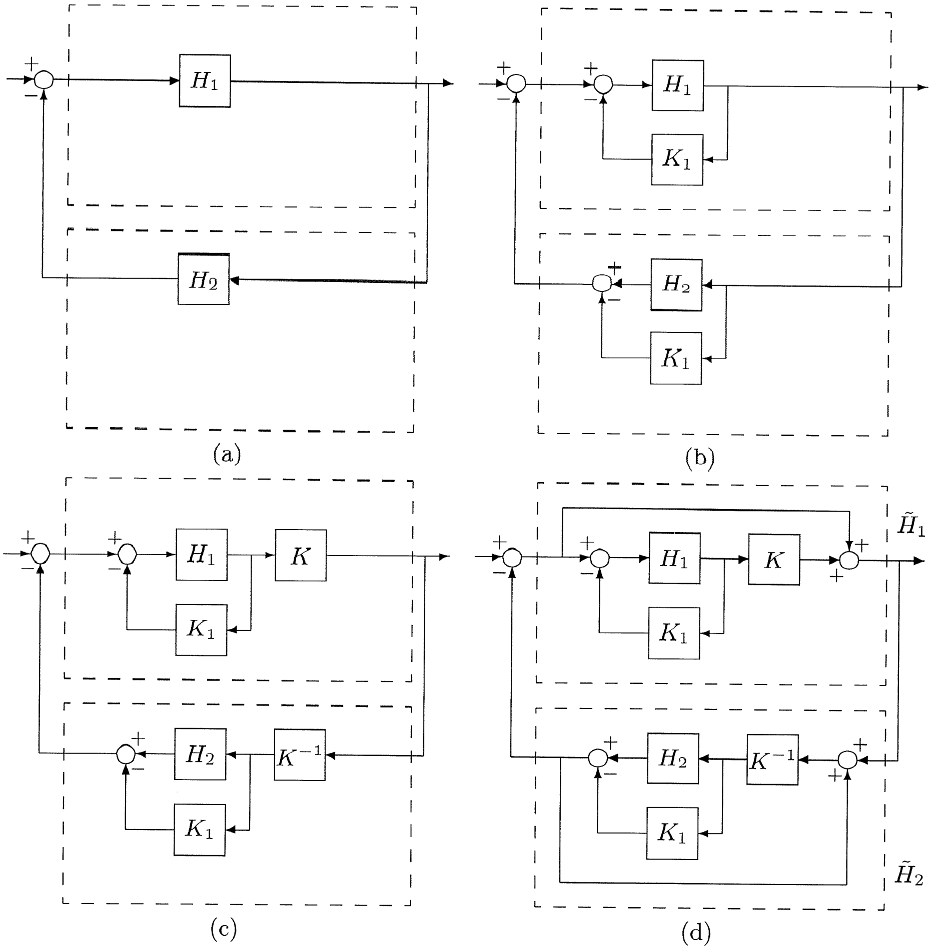
(a)
(b)
(c)
(d)
H_1
H_2
K_1
K
K^{-1}
$\tilde{H}_1$
$\tilde{H}_2$
+
−

Figure 6.13: Loop transformation with constant gains. A memoryless function H_2 in the sector $[K_1, K_2]$ is transformed into a memoryless function $\tilde{H}_2$ in the sector $[0, \infty]$.

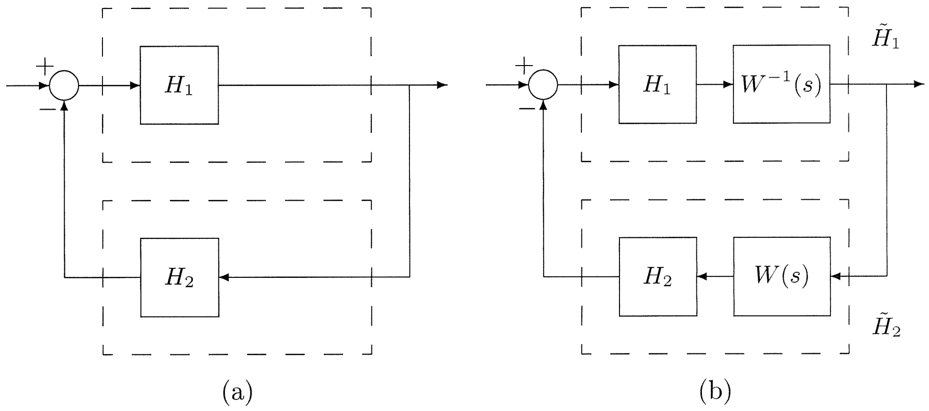

Figure 6.14: Loop transformation with dynamic multipliers.

dynamical system. If postmultiplying H_1 by $(as + 1)$ results in a strictly passive system or an output strictly passive system that is zero-state observable, we can employ Theorem 6.3 to conclude asymptotic stability of the origin. This idea is illustrated in the next two examples for cases where H_1 is linear and nonlinear, respectively.

Example 6.15 Let H_1 be a linear time-invariant system represented by the state model

$$\dot{x} = Ax + Be_1, \qquad y_1 = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Its transfer function $1/(s^2 + s + 1)$ has relative degree two; hence, it is not positive real. Postmultiplying H_1 by $(as+1)$ results in $\tilde{H}_1$, which can be represented by the state model

$$\dot{x} = Ax + Be_1, \qquad \tilde{y}_1 = \tilde{C}x$$

where $\tilde{C} = C + aCA = \begin{bmatrix} 1 & a \end{bmatrix}$. Its transfer function $(as+1)/(s^2+s+1)$ satisfies the condition

$$Re\left[\frac{1 + j\omega a}{1 - \omega^2 + j\omega}\right] = \frac{1 + (a-1)\omega^2}{(1-\omega^2)^2 + \omega^2} > 0, \quad \forall\ \omega \in R$$

if $a \geq 1$. Thus, choosing $a \geq 1$, we can apply Lemmas 6.3 and 6.4 to conclude that $\tilde{H}_1$ is strictly passive with the storage function $(1/2)x^TPx$ where P satisfies the equations

$$PA + A^TP = -L^TL - \varepsilon P, \qquad PB = \tilde{C}^T$$

for some L and $\varepsilon > 0$. On the other hand, let H_2 be given by $y_2 = h(e_2)$, where $h \in [0, \infty]$. We saw in Example 6.3 that premultiplying h by the transfer function $1/(as+1)$ results in a strictly passive system with the storage function $a \int_0^{e_2} h(s)\, ds$. Application of Theorem 6.3 shows that the origin of the transformed feedback connection of Figure 6.14(b) (with zero input) is asymptotically stable with the Lyapunov function $V = (1/2)x^TPx + a \int_0^{e_2} h(s)\, ds$. Notice, however, that the transformed feedback connection of Figure 6.14(b) has a state model of dimension three, while the original feedback connection has a state model of dimension two; so more work is needed to establish asymptotic stability of the origin of the original feedback connection. The extra work can be alleviated if we use the transformed feedback connection only to come up with the Lyapunov function V and then proceed to calculate the derivative of V with respect to the original feedback connection. Such derivative is given by

$$\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^TP\dot{x} + \tfrac{1}{2}\dot{x}^TPx + ah(e_2)\dot{e}_2 \\
&= \tfrac{1}{2}x^TP[Ax - Bh(e_2)] + \tfrac{1}{2}[Ax - Bh(e_2)]^TPx + ah(e_2)C[Ax - Bh(e_2)] \\
&= -\tfrac{1}{2}x^TL^TLx - (\varepsilon/2)x^TPx - x^T\tilde{C}^Th(e_2) + ah(e_2)CAx \\
&= -\tfrac{1}{2}x^TL^TLx - (\varepsilon/2)x^TPx - x^T[C + aCA]^Th(e_2) + ah(e_2)CAx \\
&= -\tfrac{1}{2}x^TL^TLx - (\varepsilon/2)x^TPx - e_2^Th(e_2) \ \le\ -(\varepsilon/2)x^TPx
\end{aligned}$$

which shows that the origin is asymptotically stable. In fact, since V is radially unbounded, we conclude that the origin is globally asymptotically stable. △

Example 6.16 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -bx_1^3 - kx_2 + e_1 \\ y_1 &=& x_1 \end{array} \right. \qquad \text{and} \qquad H_2 : \quad y_2 = h(e_2)$$

where $b > 0$, $k > 0$, and $h \in [0, \infty]$. Postmultiplying H_1 by $(as + 1)$ results in a system $\tilde{H}_1$ represented by the same state equation but with a new output $\tilde{y}_1 = x_1 + ax_2$. Using $V_1 = (1/4)bx_1^4 + (1/2)x^TPx$ as a storage function candidate for $\tilde{H}_1$, we obtain

$$\begin{aligned}
\dot{V}_1 &= b(1 - p_{22})x_1^3x_2 - p_{12}bx_1^4 + (p_{11}x_1 + p_{12}x_2)x_2 \\
&\quad - (p_{12}x_1 + p_{22}x_2)kx_2 + (p_{12}x_1 + p_{22}x_2)e_1
\end{aligned}$$

Taking $p_{11} = k$, $p_{12} = p_{22} = 1$, $a = 1$, and assuming that $k > 1$, we obtain

$$\dot{V}_1 = -bx_1^4 - (k - 1)x_2^2 + \tilde{y}_1e_1$$

which shows that $\tilde{H}_1$ is strictly passive. On the other hand, premultiplying h by the transfer function $1/(s + 1)$ results in a strictly passive system with the storage function $\int_0^{e_2} h(s)\, ds$. Using the storage function (of the transformed feedback connection)

$$V = (1/4)bx_1^4 + (1/2)x^T Px + \int_0^{e_2} h(s)\ ds$$

as a Lyapunov function candidate for the original feedback connection (when $u = 0$) yields

$$\begin{aligned} \dot{V} &= bx_1^3 x_2 + (kx_1 + x_2)x_2 + (x_1 + x_2)[-bx_1^3 - kx_2 - h(e_2)] + h(e_2)x_2 \\ &= -(k-1)x_2^2 - bx_1^4 - x_1 h(x_1) \end{aligned}$$

which is negative definite. Since V is positive definite and radially unbounded, we conclude that the origin is globally asymptotically stable. △

6.6 Exercises

6.1 Verify that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 6.7.

6.2 Consider the system

$$a\dot{x} = -x + \frac{1}{k}h(x) + u, \qquad y = h(x)$$

where a and k are positive constants and $h \in [0, k]$. Show that the system is passive with $V(x) = a\int_0^x h(\sigma)\ d\sigma$ as the storage function.

6.3 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = \alpha x_1 + x_2$$

where $0 < \alpha < a$ and $h \in (0, \infty]$. Show that the system is strictly passive. Hint: Use $V(x)$ of Example 4.5 as the storage function.

6.4 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = kx_2 + u$$

where $a > 0$, $k > 0$, $h \in [\alpha_1, \infty]$, and $\alpha_1 > 0$. Let $V(x) = k\int_0^{x_1} h(s)\ ds + x^T Px$, where $p_{11} = ap_{12}$, $p_{22} = k/2$, and $0 < p_{12} < \min\{2\alpha_1,\ ak/2\}$. Using $V(x)$ as a storage function, show that the system is strictly passive.

6.5 Consider the system represented by the block diagram of Figure 6.15, where $u, y \in R^p$, M and K are positive definite symmetric matrices, $h \in [0, K]$, and $\int_0^x h^T(\sigma)M\ d\sigma \geq 0$ for all x. Show that the system is output strictly passive.

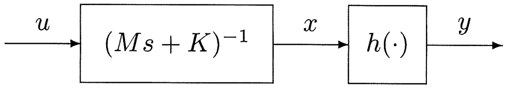

Figure 6.15: Exercise 6.5

6.6 Show that the parallel connection of two passive (respectively, input strictly passive, output strictly passive, strictly passive) dynamical systems is passive (respectively, input strictly passive, output strictly passive, strictly passive).

6.7 Show that the transfer function $(b_0 s + b_1)/(s^2 + a_1 s + a_2)$ is strictly positive real if and only if all coefficients are positive and $b_1 < a_1 b_0$.

6.8 Consider equations (6.14) through (6.16) and suppose that $(D + D^T)$ is nonsingular. Show that P satisfies the Riccati equation

$$PA_0 + A_0^T P - PB_0 P + C_0 = 0$$

where $A_0 = -(\epsilon/2)I - A + B(D + D^T)^{-1}C$, $B_0 = B(D + D^T)^{-1}B^T$, and $C_0 = -C^T(D + D^T)^{-1}C$.

6.9 Show that if a system is input strictly passive, with $\varphi(u) = \epsilon u$, and finite-gain $\mathcal{L}_2$ stable, then there is a storage function V and positive constants ϵ_1 and δ_1 such that

$$u^T y \geq \dot{V} + \epsilon_1 u^T u + \delta_1 y^T y$$

6.10 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) Using the total energy $V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a storage function, show that the map from u to $\dot{q}$ is passive.

(b) With $u = -K_d\dot{q} + v$, where K_d is a positive diagonal constant matrix, show that the map from v to $\dot{q}$ is output strictly passive.

(c) Show that $u = -K_d\dot{q}$, where K_d is a positive diagonal constant matrix, makes the origin ($q = 0$, $\dot{q} = 0$) asymptotically stable. Under what additional conditions will it be globally asymptotically stable?

6.11 (**[151]**) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that the map from $u = [u_1, u_2, u_3]^T$ to $\omega = [\omega_1, \omega_2, \omega_3]^T$ is lossless.

(b) Let $u = -K\omega + v$, where K is a positive definite symmetric matrix. Show that the map from v to ω is finite-gain $\mathcal{L}_2$ stable.

(c) Show that, when $v = 0$, the origin $\omega = 0$ is globally asymptotically stable.

6.12 Consider the feedback system of Figure 6.11 where H_1 and H_2 have the state models

$$\dot{x}_i = f_i(x_i) + G_i(x_i)e_i, \qquad y_i = h_i(x_i) + J_i(x_i)e_i$$

for $i = 1, 2$. Show that the feedback system has a well-defined state model if the matrix $I + J_2(x_2)J_1(x_1)$ is nonsingular for all x_1 and x_2.

6.13 Consider equations (6.26)–(6.27) and (6.30)–(6.31), and suppose $h_1 = h_1(x_1)$, independent of e_1. Show, in each case, that the equations have a unique solution (e_1, e_2).

6.14 Consider the feedback connection of Figure 6.11 with

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 - h_1(x_2) + e_1 \\ y_1 & = & x_2 \end{array} \right. \quad \text{and} \quad H_2 : \left\{ \begin{array}{rcl} \dot{x}_3 & = & -x_3 + e_2 \\ y_2 & = & h_2(x_3) \end{array} \right.$$

where h_1 and h_2 are locally Lipschitz functions, which satisfy $h_1 \in (0, \infty]$, $h_2 \in (0, \infty]$, and $|h_2(z)| \geq |z|/(1 + z^2)$ for all z.

(a) Show that the feedback connection is passive.

(b) Show that the origin of the unforced system is globally asymptotically stable.

6.15 Repeat the previous exercise for

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_2 \\ \dot{x}_2 & = & -x_1^3 - x_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \quad \text{and} \quad H_2 : \left\{ \begin{array}{rcl} \dot{x}_3 & = & -x_3 + e_2 \\ y_2 & = & x_3^3 \end{array} \right.$$

6.16 (**[78]**) Consider the feedback system of Figure 6.11, where H_1 and H_2 are passive dynamical systems of the form (6.21)–(6.22). Suppose the feedback connection has a well-defined state model and the series connection $H_1(-H_2)$, with input e_2 and output y_1, is zero-state observable. Show that the origin is asymptotically stable if H_2 is input strictly passive or H_1 is output strictly passive.

6.17 (**[78]**) Consider the feedback system of Figure 6.11, where H_1 and H_2 are passive dynamical systems of the form (6.21)–(6.22). Suppose the feedback connection has a well-defined state model and the series connection H_2H_1, with input e_1 and output y_2, is zero-state observable. Show that the origin is asymptotically stable if H_1 is input strictly passive or H_2 is output strictly passive.

6.18 ([78]) As a generalization of the concept of passivity, a dynamical system of the form (6.6)–(6.7) is said to be dissipative with respect to a supply rate $w(u, y)$ if there is a positive definite storage function $V(x)$ such that $\dot{V} \leq w$. Consider the feedback system of Figure 6.11 where H_1 and H_2 are zero-state observable, dynamical systems of the form (6.21)–(6.22). Suppose each of H_1 and H_2 is dissipative with storage function $V_i(x_i)$ and supply rate $w_i(u_i, y_i) = y_i^T Q_i y_i + 2y_i^T S_i u_i + u_i^T R_i u_i$, where Q_i and R_i are real symmetric matrices and S_i is a real matrix. Show that the origin is stable (respectively, asymptotically stable) if the matrix

$$\hat{Q} = \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^T \\ -S_1^T + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix}$$

is negative semidefinite (respectively, negative definite) for some $\alpha > 0$.

6.19 Consider the feedback connection of two time-invariant dynamical systems of the form (6.21)–(6.22). Suppose both feedback components are zero-state observable and there exist positive definite storage functions which satisfy

$$e_i^T y_i \geq \dot{V}_i + e_i^T \varphi_i(e_i) + y_i^T \rho_i(y_i), \quad \text{for } i = 1, 2$$

Show that the origin of the closed-loop system (6.24) when $u = 0$ is asymptotically stable if

$$v^T[\rho_1(v) + \varphi_2(v)] > 0 \quad \text{and} \quad v^T[\rho_2(v) - \varphi_1(-v)] > 0, \quad \forall\, v \neq 0$$

Under what additional conditions will the origin be globally asymptotically stable?

Chapter 7

Frequency Domain Analysis of Feedback Systems

Many nonlinear physical systems can be represented as a feedback connection of a linear dynamical system and a nonlinear element, as shown in Figure 7.1. The process of representing a system in this form depends on the particular system involved. For instance, in the case in which a control system's only nonlinearity is in the form of a relay or actuator/sensor nonlinearity, there is no difficulty in representing the system in the feedback form of Figure 7.1. In other cases, the representation may be less obvious. We assume that the external input $r = 0$ and study the behavior of the unforced system. What is unique about this chapter is the use of the frequency response of the linear system, which builds on classical control tools like the Nyquist plot and the Nyquist criterion. In Section 7.1, we study absolute stability. The system is said to absolutely stable if it has a globally uniformly asymptotically stable equilibrium point at the origin for all nonlinearities in a given sector. The circle and Popov criteria give frequency-domain sufficient conditions for absolute stability in

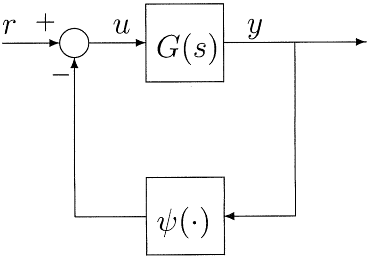

Figure 7.1: Feedback connection.

the form of strict positive realness of certain transfer functions. In the single-input–single-output case, both criteria can be applied graphically. In Section 7.2, we use the describing function method to study the existence of periodic solutions for a single-input–single-output system. We derive frequency-domain conditions, which can be applied graphically, to predict the existence or absence of oscillations and estimate the frequency and amplitude of oscillation when there is one.

7.1 Absolute Stability

Consider the feedback connection of Figure 7.1. We assume that the external input $r = 0$ and study the behavior of the unforced system, represented by

$$\dot{x} = Ax + Bu \tag{7.1}$$

$$y = Cx + Du \tag{7.2}$$

$$u = -\psi(t, y) \tag{7.3}$$

where $x \in R^n$, $u, y \in R^p$, (A, B) is controllable, (A, C) is observable, and $\psi : [0, \infty) \times R^p \to R^p$ is a memoryless, possibly time-varying, nonlinearity, which is piecewise continuous in t and locally Lipschitz in y. We assume that the feedback connection has a well-defined state model, which is the case when

$$u = -\psi(t, Cx + Du) \tag{7.4}$$

has a unique solution u for every (t, x) in the domain of interest. This is always the case when $D = 0$. The transfer function matrix of the linear system

$$G(s) = C(sI - A)^{-1}B + D \tag{7.5}$$

is square and proper. The controllability and observability assumptions ensure that $\{A, B, C, D\}$ is a minimal realization of $G(s)$. From linear system theory, we know that for any rational proper $G(s)$, a minimal realization always exists. The nonlinearity ψ is required to satisfy a sector condition per Definition 6.2. The sector condition may be satisfied globally, that is, for all $y \in R^p$, or satisfied only for $y \in Y$, a subset of R^p, whose interior is connected and contains the origin.

For all nonlinearities satisfying the sector condition, the origin $x = 0$ is an equilibrium point of the system (7.1)–(7.3). The problem of interest here is to study the stability of the origin, not for a given nonlinearity, but rather for a class of nonlinearities that satisfy a given sector condition. If we succeed in showing that the origin is uniformly asymptotically stable for all nonlinearities in the sector, the system is said to be absolutely stable. The problem was originally formulated by Lure and is sometimes called *Lure's problem*. Traditionally, absolute stability has been defined for the case when the origin is globally uniformly asymptotically stable. To keep up this tradition, we will use the phrase "absolute stability" when the sector condition is satisfied globally and the origin is globally uniformly asymptotically stable. Otherwise, we will use the phrase "absolute stability with a finite domain."

Definition 7.1 *Consider the system* (7.1)–(7.3), *where* ψ *satisfies a sector condition per Definition* 6.2. *The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector. It is absolutely stable with a finite domain if the origin is uniformly asymptotically stable.*

We will investigate asymptotic stability of the origin by using Lyapunov analysis. A Lyapunov function candidate can be chosen by using the passivity tools of the previous chapter. In particular, if the closed-loop system can be represented as a feedback connection of two passive systems, then the sum of the two storage functions can be used as a Lyapunov function candidate for the closed-loop system. The use of loop transformations allows us to cover various sectors and Lyapunov function candidates, leading to the circle and Popov criteria.

7.1.1 Circle Criterion

Theorem 7.1 *The system* (7.1)–(7.3) *is absolutely stable if*

- $\psi \in [K_1, \infty]$ *and* $G(s)[I + K_1 G(s)]^{-1}$ *is strictly positive real, or*
- $\psi \in [K_1, K_2]$, *with* $K = K_2 - K_1 = K^T > 0$, *and* $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ *is strictly positive real.*

If the sector condition is satisfied only on a set $Y \subset R^p$, *then the foregoing conditions ensure that the system is absolutely stable with a finite domain.* ◇

We refer to this theorem as the *multivariable circle criterion*, although the reason for using this name will not be clear until we specialize it to the scalar case. A necessary condition for equation (7.4) to have a unique solution u for every $\psi \in [K_1, \infty]$ or $\psi \in [K_1, K_2]$ is the nonsingularity of the matrix $(I + K_1 D)$. This can be seen by taking $\psi = K_1 y$ in (7.4). Therefore, the transfer function $[I + K_1 G(s)]^{-1}$ is proper.

Proof of Theorem 7.1: We prove the theorem first for the sector $[0, \infty]$ and recover the other cases by loop transformations. If $\psi \in [0, \infty]$ and $G(s)$ is strictly positive real, we have a feedback connection of two passive systems. From Lemma 6.4, we know that the storage function for the linear dynamical system is $V(x) = (1/2)x^T P x$, where $P = P^T > 0$ satisfies the Kalman–Yakubovich–Popov equations

$$PA + A^T P = -L^T L - \varepsilon P \tag{7.6}$$

$$PB = C^T - L^T W \tag{7.7}$$

$$W^T W = D + D^T \tag{7.8}$$

and $\varepsilon > 0$. Using $V(x)$ as a Lyapunov function candidate, we obtain

$$\dot{V} = \tfrac{1}{2} x^T P \dot{x} + \tfrac{1}{2} \dot{x}^T P x = \tfrac{1}{2} x^T (PA + A^T P) x + x^T P B u$$

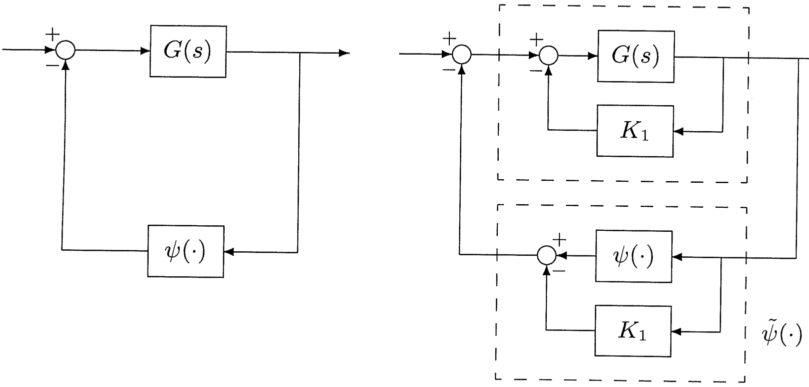

Figure 7.2: $\psi \in [K_1, \infty]$ is transformed to $\tilde{\psi} \in [0, \infty]$ via a loop transformation.

Using (7.6) and (7.7) yields

$$\begin{aligned} \dot{V} &= -\tfrac{1}{2}x^T L^T L x - \tfrac{1}{2}\varepsilon x^T P x + x^T(C^T - L^T W)u \\ &= -\tfrac{1}{2}x^T L^T L x - \tfrac{1}{2}\varepsilon x^T P x + (Cx + Du)^T u - u^T D u - x^T L^T W u \end{aligned}$$

Using (7.8) and the fact that $u^T D u = \frac{1}{2}u^T(D + D^T)u$, we obtain

$$\dot{V} = -\tfrac{1}{2}\varepsilon x^T P x - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - y^T \psi(t, y)$$

Since $y^T \psi(t, y) \geq 0$, we have

$$\dot{V} \leq -\tfrac{1}{2}\varepsilon x^T P x$$

which shows that the origin is globally exponentially stable. If ψ satisfies the sector condition only for $y \in Y$, the foregoing analysis will be valid in some neighborhood of the origin, showing that the origin is exponentially stable. The case $\psi \in [K_1, \infty]$ can be transformed to a case where the nonlinearity belongs to $[0, \infty]$ via the loop transformation of Figure 7.2. Hence, the system is absolutely stable if $G(s)[I + K_1 G(s)]^{-1}$ is strictly positive real. The case $\psi \in [K_1, K_2]$ can be transformed to a case where the nonlinearity belongs to $[0, \infty]$ via the loop transformation of Figure 7.3. Hence, the system is absolutely stable if

$$I + KG(s)[I + K_1 G(s)]^{-1} = [I + K_2 G(s)][I + K_1 G(s)]^{-1}$$

is strictly positive real. □

Example 7.1 Consider the system (7.1)–(7.3) and suppose $G(s)$ is Hurwitz and strictly proper. Let

$$\gamma_1 = \sup_{\omega \in R} \sigma_{\max}[G(j\omega)] = \sup_{\omega \in R} \|G(j\omega)\|_2$$

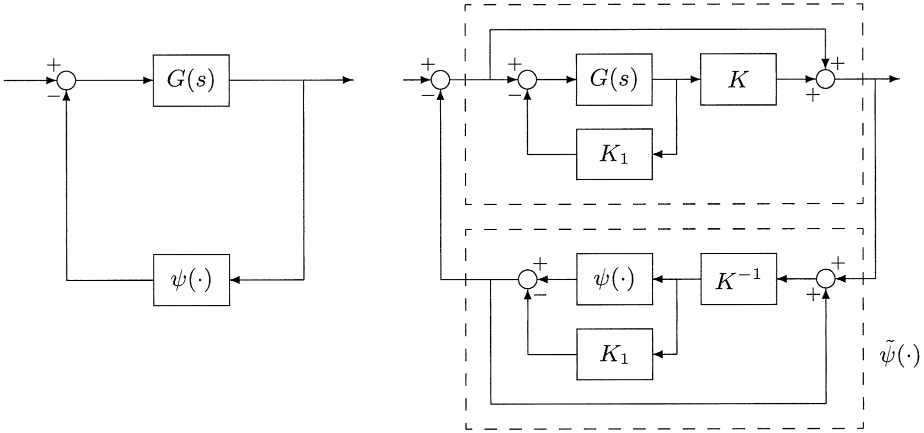

Figure 7.3: $\psi \in [K_1, K_2]$ is transformed to $\tilde{\psi} \in [0, \infty]$ via a loop transformation.

where $\sigma_{\max}[\cdot]$ denotes the maximum singular value of a complex matrix. The constant γ_1 is finite, since $G(s)$ is Hurwitz. Suppose ψ satisfies the inequality

$$\|\psi(t,y)\|_2 \le \gamma_2 \|y\|_2, \quad \forall\, t \ge 0, \ \forall\, y \in R^p \tag{7.9}$$

then it belongs to the sector $[K_1, K_2]$ with $K_1 = -\gamma_2 I$ and $K_2 = \gamma_2 I$. To apply Theorem 7.1, we need to show that

$$Z(s) = [I + \gamma_2 G(s)][I - \gamma_2 G(s)]^{-1}$$

is strictly positive real. We note that $\det[Z(s) + Z^T(-s)]$ is not identically zero because $Z(\infty) = I$. We apply Lemma 6.1. Since $G(s)$ is Hurwitz, $Z(s)$ will be Hurwitz if $[I - \gamma_2 G(s)]^{-1}$ is Hurwitz. Noting that[1]

$$\sigma_{\min}[I - \gamma_2 G(j\omega)] \ge 1 - \gamma_1\gamma_2$$

we see that if $\gamma_1\gamma_2 < 1$, the plot of $\det[I - \gamma_2 G(j\omega)]$ will not go through nor encircle the origin. Hence, by the multivariable Nyquist criterion,[2] $[I - \gamma_2 G(s)]^{-1}$ is Hurwitz; consequently, $Z(s)$ is Hurwitz. Next, we show that

$$Z(j\omega) + Z^T(-j\omega) > 0, \quad \forall\, \omega \in R$$

---

[1] The following properties of singular values of a complex matrix are used:

$$\det G \ne 0 \Leftrightarrow \sigma_{\min}[G] > 0$$

$$\sigma_{\max}[G^{-1}] = 1/\sigma_{\min}[G], \ \text{if } \sigma_{\min}[G] > 0$$

$$\sigma_{\min}[I + G] \ge 1 - \sigma_{\max}[G]$$

$$\sigma_{\max}[G_1 G_2] \le \sigma_{\max}[G_1]\sigma_{\max}[G_2]$$

[2] See [33, pp. 160–161] for a statement of the multivariable Nyquist criterion.

The left-hand side of this inequality is given by

$$\begin{aligned} Z(j\omega) + Z^T(-j\omega) &= [I + \gamma_2 G(j\omega)][I - \gamma_2 G(j\omega)]^{-1} \\ &\quad + [I - \gamma_2 G^T(-j\omega)]^{-1}[I + \gamma_2 G^T(-j\omega)] \\ &= [I - \gamma_2 G^T(-j\omega)]^{-1}\left[2I - 2\gamma_2^2 G^T(-j\omega)G(j\omega)\right] \\ &\quad \times [I - \gamma_2 G(j\omega)]^{-1} \end{aligned}$$

Hence, $Z(j\omega) + Z^T(-j\omega)$ is positive definite for all ω if and only if

$$\sigma_{\min}\left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right] > 0, \quad \forall\, \omega \in R$$

Now, for $\gamma_1\gamma_2 < 1$, we have

$$\begin{aligned} \sigma_{\min}[I - \gamma_2^2 G^T(-j\omega)G(j\omega)] &\geq 1 - \gamma_2^2 \sigma_{\max}[G^T(-j\omega)]\sigma_{\max}[G(j\omega)] \\ &\geq 1 - \gamma_1^2\gamma_2^2 > 0 \end{aligned}$$

Finally, $Z(\infty) + Z^T(\infty) = 2I$. Thus, all the conditions of Lemma 6.1 are satisfied and we conclude that $Z(s)$ is strictly positive real and the system is absolutely stable if $\gamma_1\gamma_2 < 1$. This is a robustness result, which shows that closing the loop around a Hurwitz transfer function with a nonlinearity satisfying (7.9), with a sufficiently small γ_2, does not destroy the stability of the system.[3] △

In the scalar case $p = 1$, the conditions of Theorem 7.1 can be verified graphically by examining the Nyquist plot of $G(j\omega)$. For $\psi \in [\alpha, \beta]$, with $\beta > \alpha$, the system is absolutely stable if the scalar transfer function

$$Z(s) = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is strictly positive real. To verify that $Z(s)$ is strictly positive real, we can use Lemma 6.1 which states that $Z(s)$ is strictly positive real if it is Hurwitz and

$$\text{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \quad \forall\, \omega \in [-\infty, \infty] \tag{7.10}$$

To relate condition (7.10) to the Nyquist plot of $G(j\omega)$, we have to distinguish between three different cases, depending on the sign of α. Consider first the case when $\beta > \alpha > 0$. In this case, condition (7.10) can be rewritten as

$$\text{Re}\left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right] > 0, \quad \forall\, \omega \in [-\infty, \infty] \tag{7.11}$$

For a point q on the Nyquist plot of $G(j\omega)$, the two complex numbers $(1/\beta) + G(j\omega)$ and $(1/\alpha) + G(j\omega)$ can be represented by the lines connecting q to $-(1/\beta) + j0$ and

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[3] The inequality $\gamma_1\gamma_2 < 1$ can be derived also from the small-gain theorem. (See Example 5.13.)

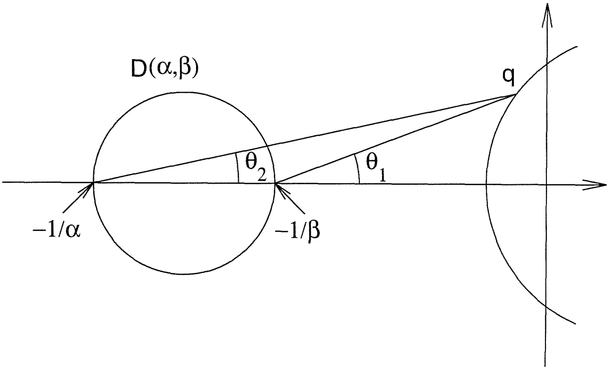

Figure 7.4: Graphical representation of the circle criterion.

$-(1/\alpha)+j0$, respectively, as shown in Figure 7.4. The real part of the ratio of two complex numbers is positive when the angle difference between the two numbers is less than $\pi/2$; that is, the angle $(\theta_1-\theta_2)$ in Figure 7.4 is less than $\pi/2$. If we define $D(\alpha,\beta)$ to be the closed disk in the complex plane whose diameter is the line segment connecting the points $-(1/\alpha)+j0$ and $-(1/\beta)+j0$, then it is simple to see that the angle $(\theta_1-\theta_2)$ is less than $\pi/2$ when q is outside the disk $D(\alpha,\beta)$. Since (7.11) is required to hold for all ω, all points on the Nyquist plot of $G(j\omega)$ must be strictly outside the disk $D(\alpha,\beta)$. On the other hand, $Z(s)$ is Hurwitz if $G(s)/[1+\alpha G(s)]$ is Hurwitz. The Nyquist criterion states that $G(s)/[1+\alpha G(s)]$ is Hurwitz if and only if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha)+j0$ and encircles it exactly m times in the counterclockwise direction, where m is the number of poles of $G(s)$ in the open right-half complex plane.[4] Therefore, the conditions of Theorem 7.1 are satisfied if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha,\beta)$ and encircles it m times in the counterclockwise direction. Consider, next, the case when $\beta>0$ and $\alpha=0$. For this case, Theorem 7.1 requires $1+\beta G(s)$ to be strictly positive real. This is the case if $G(s)$ is Hurwitz and

$$\text{Re}[1+\beta G(j\omega)]>0, \quad \forall\, \omega\in[-\infty,\infty]$$

The latter condition can be rewritten as

$$\text{Re}[G(j\omega)]>-\frac{1}{\beta}, \quad \forall\, \omega\in[-\infty,\infty]$$

which is equivalent to the graphical condition that the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\text{Re}[s]=-1/\beta$. Finally, consider the case when $\alpha < 0 < \beta$. In this case, condition (7.10) is equivalent to

---

[4] When $G(s)$ has poles on the imaginary axis, the Nyquist path is indented in the right-half plane, as usual.

$$\text{Re}\left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right] < 0, \quad \forall\, \omega \in [-\infty, \infty] \tag{7.12}$$

where the inequality sign is reversed because, as we go from (7.10) to (7.12), we multiply by α/β, which is now negative. Repeating previous arguments, it can be easily seen that for (7.12) to hold, the Nyquist plot of $G(j\omega)$ must lie inside the disk $D(\alpha, \beta)$. Consequently, the Nyquist plot cannot encircle the point $-(1/\alpha) + j0$. Therefore, from the Nyquist criterion, we see that $G(s)$ must be Hurwitz for $G(s)/[1 + \alpha G(s)]$ to be so. The stability criteria for the three cases are summarized in the following theorem, which is known as the *circle criterion.*

Theorem 7.2 *Consider a scalar system of the form* (7.1)–(7.3), *where* $\{A, B, C, D\}$ *is a minimal realization of* $G(s)$ *and* $\psi \in [\alpha, \beta]$. *Then, the system is absolutely stable if one of the following conditions is satisfied, as appropriate:*

1. *If* $0 < \alpha < \beta$, *the Nyquist plot of* $G(j\omega)$ *does not enter the disk* $D(\alpha, \beta)$ *and encircles it* m *times in the counterclockwise direction, where* m *is the number of poles of* $G(s)$ *with positive real parts.*

2. *If* $0 = \alpha < \beta$, $G(s)$ *is Hurwitz and the Nyquist plot of* $G(j\omega)$ *lies to the right of the vertical line defined by* $\text{Re}[s] = -1/\beta$.

3. *If* $\alpha < 0 < \beta$, $G(s)$ *is Hurwitz and the Nyquist plot of* $G(j\omega)$ *lies in the interior of the disk* $D(\alpha, \beta)$.

If the sector condition is satisfied only on an interval $[a, b]$, *then the foregoing conditions ensure that the system is absolutely stable with a finite domain.* ◇

The circle criterion allows us to investigate absolute stability by using only the Nyquist plot of $G(j\omega)$. This is important because the Nyquist plot can be determined directly from experimental data. Given the Nyquist plot of $G(j\omega)$, we can determine permissible sectors for which the system is absolutely stable. The next two examples illustrate the use of the circle criterion.

Example 7.2 Let

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

The Nyquist plot of $G(j\omega)$ is shown in Figure 7.5. Since $G(s)$ is Hurwitz, we can allow α to be negative and apply the third case of the circle criterion. So, we need to determine a disk $D(\alpha, \beta)$ that encloses the Nyquist plot. Clearly, the choice of the disk is not unique. Suppose we decide to locate the center of the disk at the origin of the complex plane. This means that we will work with a disk $D(-\gamma_2, \gamma_2)$, where the radius $(1/\gamma_2) > 0$ is to be chosen. The Nyquist plot will be inside this

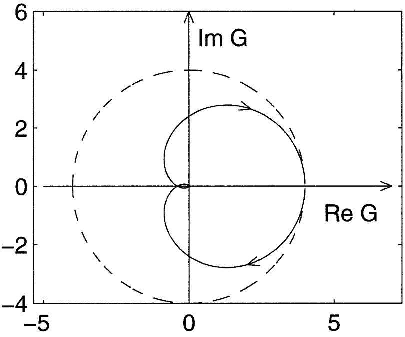

Figure 7.5: Nyquist plot for Example 7.2.

disk if $|G(j\omega)| < 1/\gamma_2$. In particular, if we set $\gamma_1 = \sup_{\omega \in R} |G(j\omega)|$, then γ_2 must be chosen to satisfy $\gamma_1\gamma_2 < 1$. This is the same condition we found in Example 7.1. It is not hard to see that $|G(j\omega)|$ is maximum at $\omega = 0$ and $\gamma_1 = 4$. Thus, γ_2 must be less than 0.25. Hence, we can conclude that the system is absolutely stable for all nonlinearities in the sector $[-0.25+\varepsilon, 0.25-\varepsilon]$, where $\varepsilon > 0$ can be arbitrarily small. Inspection of the Nyquist plot and the disk $D(-0.25, 0.25)$ in Figure 7.5 suggests that the choice to locate the center at the origin may not be the best one. By locating the center at another point, we might be able to obtain a disk that encloses the Nyquist plot more tightly. For example, let us locate the center at the point $1.5 + j0$. The maximum distance from this point to the Nyquist plot is 2.834. Hence, choosing the radius of the disk to be 2.9 ensures that the Nyquist plot is inside the disk $D(-1/4.4, 1/1.4)$, and we can conclude that the system is absolutely stable for all nonlinearities in the sector $[-0.227, 0.714]$. Comparing this sector with the previous one (see Figure 7.6) shows that by giving in a little bit on the lower bound of the sector, we achieve a significant improvement in the upper bound. Clearly, there is still room for optimizing the choice of the center of the disk, but we will not pursue it. The point we wanted to show is that the graphical representation used in the circle criterion gives us a closer look at the problem, compared with the use of norm inequalities as in Example 7.1, which allows us to obtain less conservative estimates of the sector. Another direction we can pursue in applying the circle criterion is to restrict α to zero and apply the second case of the circle criterion. The Nyquist plot lies to the right of the vertical line $\text{Re}[s] = -0.857$. Hence, we can conclude that the system is absolutely stable for all nonlinearities in the sector $[0, 1.166]$. This sector is sketched in Figure 7.6, together with the previous two sectors. It gives the best estimate of β, which is achieved at the expense of limiting the nonlinearity to be a first-quadrant–third-quadrant nonlinearity. To appreciate how this flexibility in using the circle criterion could be useful in applications, let us suppose that we are interested in studying the stability of the system of Figure 7.7, which includes a limiter or saturation nonlinearity (a typical nonlinearity in feedback control systems

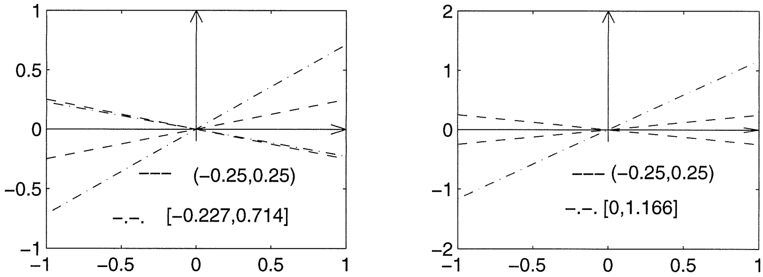

Figure 7.6: Sectors for Example 7.2.

due to constraints on physical variables). The saturation nonlinearity belongs to a sector $[0, 1]$. Therefore, it is included in the sector $[0, 1.166]$, but not in the sector $(-0.25, 0.25)$ or $[-0.227, 0.714]$. Thus, based on the application of the second case of the circle criterion, we can conclude that the feedback system of Figure 7.7 has a globally asymptotically stable equilibrium point at the origin. △

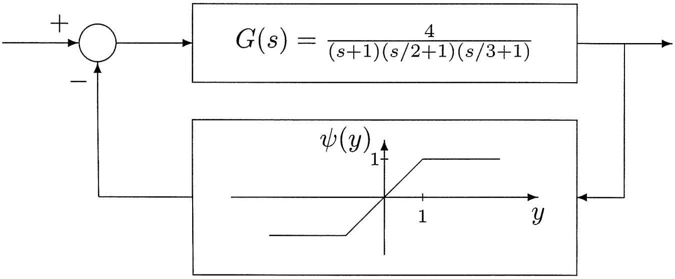

Figure 7.7: Feedback connection with saturation nonlinearity.

Example 7.3 Let

$$G(s) = \frac{4}{(s-1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

This transfer function is not Hurwitz, since it has a pole in the open right-half plane. So, we must restrict α to be positive and apply the first case of the circle criterion. The Nyquist plot of $G(j\omega)$ is shown in Figure 7.8. From the circle criterion, we know that the Nyquist plot must encircle the disk $D(\alpha, \beta)$ once in the counterclockwise direction. Inspection of the Nyquist plot shows that a disk can be encircled by the Nyquist plot only if it is totally inside one of the two lobes formed by the Nyquist plot in the left-half plane. A disk inside the right lobe is encircled once in the clockwise direction. Hence, it does not satisfy the circle criterion. A disk inside the left lobe is encircled once in the counterclockwise direction. Thus, we need to

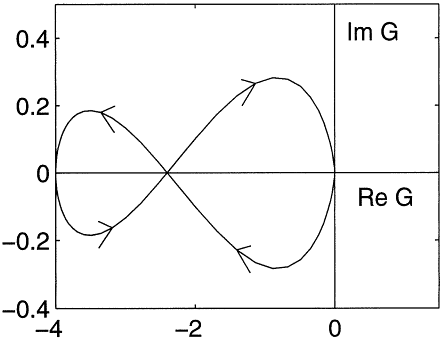

Figure 7.8: Nyquist plot for Example 7.3.

choose α and β to locate the disk $D(\alpha, \beta)$ inside the left lobe. Let us locate the center of the disk at the point $-3.2 + j0$, about halfway between the two ends of the lobe on the real axis. The minimum distance from this center to the Nyquist plot is 0.1688. Hence, choosing the radius to be 0.168, we conclude that the system is absolutely stable for all nonlinearities in the sector $[0.2969, 0.3298]$. △

In Examples 7.1 through 7.3, we have considered cases where the sector condition is satisfied globally. In the next example, the sector condition is satisfied only on a finite interval.

Example 7.4 Consider the feedback connection of Figure 7.1, where the linear system is represented by the transfer function

$$G(s) = \frac{s+2}{(s+1)(s-1)}$$

and the nonlinear element is $\psi(y) = \text{sat}(y)$. The nonlinearity belongs globally to the sector $[0, 1]$. However, since $G(s)$ is not Hurwitz, we must apply the first case of the circle criterion, which requires the sector condition to hold with a positive α. Thus, we cannot conclude absolute stability by using the circle criterion.[5] The best we can hope for is to show absolute stability with a finite domain. Figure 7.9 shows that on the interval $[-a, a]$, the nonlinearity ψ belongs to the sector $[\alpha, \beta]$ with $\alpha = 1/a$ and $\beta = 1$. Since $G(s)$ has a pole with positive real part, the Nyquist plot of $G(j\omega)$, shown in Figure 7.10, must encircle the disk $D(\alpha, 1)$ once in the counterclockwise direction. It can be verified, analytically, that condition (7.10) is satisfied for $\alpha > 0.5359$. Thus, choosing $\alpha = 0.55$, the sector condition is satisfied on the interval $[-1.818, 1.818]$ and the disk $D(0.55, 1)$ is encircled once by the Nyquist plot in the counterclockwise direction. From the first case of the circle criterion, we

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[5] In fact, the origin is not globally asymptotically stable because the system has three equilibrium points.

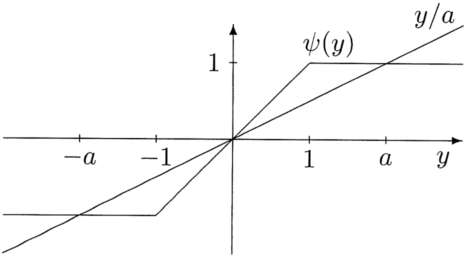

Figure 7.9: Sector for Example 7.4.

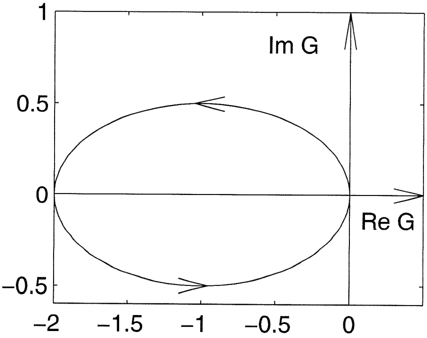

Figure 7.10: Nyquist plot for Example 7.4.

conclude that the system is absolutely stable with a finite domain. We can also use a quadratic Lyapunov function $V(x) = x^T P x$ to estimate the region of attraction. Consider the state model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u \\ y &= 2x_1 + x_2 \\ u &= -\psi(y) \end{aligned}$$

The loop transformation of Figure 7.3 is given by

$$\begin{aligned} u &= -\alpha y + \tilde{u} = -0.55y + \tilde{u} \\ \tilde{y} &= (\beta - \alpha)y + \tilde{u} = 0.45y + \tilde{u} \end{aligned}$$

Thus, the transformed linear system is given by

$$\dot{x} = Ax + B\tilde{u}, \qquad \tilde{y} = Cx + D\tilde{u}$$

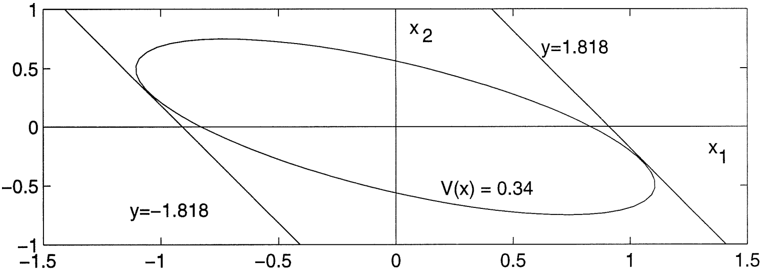

Figure 7.11: Region of attraction for Example 7.4.

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.9 & 0.45 \end{bmatrix}, \quad \text{and } D = 1$$

The matrix P is the solution of equations (7.6) through (7.8). It can be verified that[6]

$$\varepsilon = 0.02, \quad P = \begin{bmatrix} 0.4946 & 0.4834 \\ 0.4834 & 1.0774 \end{bmatrix}, \quad L = \begin{bmatrix} 0.2946 & -0.4436 \end{bmatrix}, \quad \text{and } W = \sqrt{2}$$

satisfy (7.6) through (7.8). Thus, $V(x) = x^TPx$ is a Lyapunov function for the system. We estimate the region of attraction by

$$\Omega_c = \{x \in R^2 \mid V(x) \le c\}$$

where $c \le \min_{\{|y|=1.818\}} V(x) = 0.3445$ to ensure that Ω_c is contained in the set $\{|y| \le 1.818\}$. Taking $c = 0.34$ gives the estimate shown in Figure 7.11. △

7.1.2 Popov Criterion

Consider a special case of the system (7.1)–(7.3), given by

$$\begin{aligned} \dot{x} &= Ax + Bu && (7.13) \\ y &= Cx && (7.14) \\ u_i &= -\psi_i(y_i), \quad 1 \le i \le p && (7.15) \end{aligned}$$

where $x \in R^n$, $u, y \in R^p$, (A, B) is controllable, (A, C) is observable, and $\psi_i : R \to R$ is a locally Lipschitz memoryless nonlinearity that belongs to the sector $[0, k_i]$. In this special case, the transfer function $G(s) = C(sI - A)^{-1}B$ is strictly proper

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[6] The value of ε is chosen such that $G(s - \varepsilon/2)$ is positive real and $[(\varepsilon/2)I + A]$ is Hurwitz, where $G(s) = C(sI - A)^{-1}B + D$. Then, P is calculated by solving a Riccati equation, as described in Exercise 6.8.

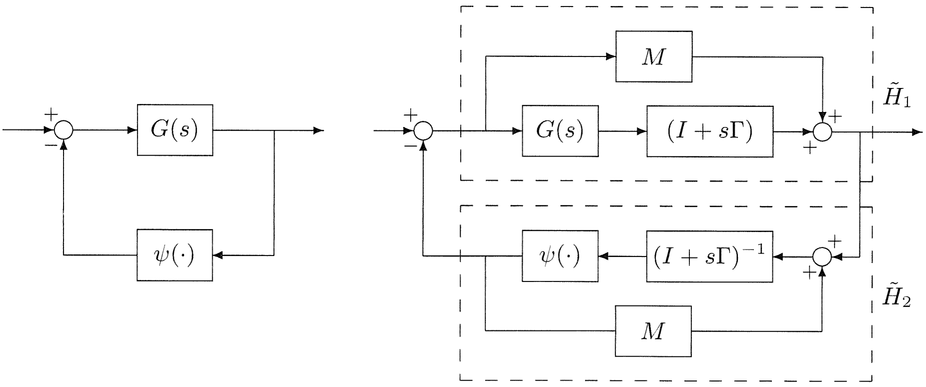

Figure 7.12: Loop transformation.

and ψ is time invariant and decoupled; that is, $\psi_i(y) = \psi_i(y_i)$. Since $D = 0$, the feedback connection has a well-defined state model. The following theorem, known as the multivariable Popov criterion, is proved by using a (Lure-type) Lyapunov function of the form $V = (1/2)x^TPx + \sum \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$, which is motivated by the application of a loop transformation that transforms the system (7.13)–(7.15) into the feedback connection of two passive dynamical systems.

Theorem 7.3 *The system* (7.13)–(7.15) *is absolutely stable if, for* $1 \le i \le p$, $\psi_i \in [0, k_i]$, $0 < k_i \le \infty$, *and there exists a constant* $\gamma_i \ge 0$, *with* $(1 + \lambda_k \gamma_i) \ne 0$ *for every eigenvalue* λ_k *of* A, *such that* $M + (I + s\Gamma)G(s)$ *is strictly positive real, where* $\Gamma = \text{diag}(\gamma_1, \cdots, \gamma_p)$ *and* $M = \text{diag}(1/k_1, \cdots, 1/k_p)$. *If the sector condition* $\psi_i \in [0, k_i]$ *is satisfied only on a set* $Y \subset R^p$, *then the foregoing conditions ensure that the system is absolutely stable with a finite domain.* ◇

Proof: The loop transformation of Figure 7.12 results in a feedback connection of $\tilde{H}_1$ and $\tilde{H}_2$, where $\tilde{H}_1$ is a linear system whose transfer function is

$$\begin{aligned} M + (I + s\Gamma)G(s) &= M + (I + s\Gamma)C(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\ &= M + (C + \Gamma CA)(sI - A)^{-1}B + \Gamma CB \end{aligned}$$

Thus, $M + (I + s\Gamma)G(s)$ can be realized by the state model $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, where $\mathcal{A} = A$, $\mathcal{B} = B$, $\mathcal{C} = C + \Gamma CA$, and $\mathcal{D} = M + \Gamma CB$. Let λ_k be an eigenvalue of A and v_k be the associated eigenvector. Then

$$(C + \Gamma CA)v_k = (C + \Gamma C\lambda_k)v_k = (I + \lambda_k\Gamma)Cv_k$$

The condition $(1+\lambda_k\gamma_i) \neq 0$ implies that $(\mathcal{A},\mathcal{C})$ is observable; hence, the realization $\{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\}$ is minimal. If $M + (I + s\Gamma)G(s)$ is strictly positive real, we can apply the Kalman–Yakubovich–Popov lemma to conclude that there are matrices $P = P^T > 0$, L, and W, and a positive constant ε that satisfy

$$\begin{aligned} PA + A^TP &= -L^TL - \varepsilon P && (7.16)\\ PB &= (C + \Gamma CA)^T - L^TW && (7.17)\\ W^TW &= 2M + \Gamma CB + B^TC^T\Gamma && (7.18) \end{aligned}$$

and $V = (1/2)x^TPx$ is a storage function for $\tilde{H}_1$. One the other hand, it can be verified (Exercise 6.2) that $\tilde{H}_2$ is passive with the storage function $\sum_{i=1}^{p}\gamma_i\int_0^{y_i}\psi_i(\sigma)\;d\sigma$. Thus, the storage function for the transformed feedback connection of Figure 7.12 is

$$V = \tfrac{1}{2}x^TPx + \sum_{i=1}^{p}\gamma_i\int_0^{y_i}\psi_i(\sigma)\;d\sigma$$

We use V as a Lyapunov function candidate for the original feedback connection (7.13)–(7.15). The derivative $\dot{V}$ is given by

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}x^TP\dot{x} + \tfrac{1}{2}\dot{x}^TPx + \psi^T(y)\Gamma\dot{y}\\ &= \tfrac{1}{2}x^T(PA + A^TP)x + x^TPBu + \psi^T(y)\Gamma C(Ax + Bu) \end{aligned}$$

Using (7.16) and (7.17) yields

$$\begin{aligned} \dot{V} = &-\tfrac{1}{2}x^TL^TLx - \tfrac{1}{2}\varepsilon x^TPx + x^T(C^T + A^TC^T\Gamma - L^TW)u\\ &+ \psi^T(y)\Gamma CAx + \psi^T(y)\Gamma CBu \end{aligned}$$

Substituting $u = -\psi(y)$ and using (7.18), we obtain

$$\dot{V} = -\tfrac{1}{2}\varepsilon x^TPx - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - \psi(y)^T[y - M\psi(y)] \leq -\tfrac{1}{2}\varepsilon x^TPx$$

which shows that the origin is globally asymptotically stable. If ψ satisfies the sector condition only for $y \in Y$, the foregoing analysis will be valid in some neighborhood of the origin, showing that the origin is asymptotically stable. □

For $M + (I + s\Gamma)G(s)$ to be strictly positive real, $G(s)$ must be Hurwitz. As we have done in the circle criterion, this restriction on $G(s)$ may be removed by performing a loop transformation that replaces $G(s)$ by $G(s)[I + K_1G(s)]^{-1}$. We will not repeat this idea in general, but will illustrate it by an example. In the scalar case $p = 1$, we can test the strict positive realness of $Z(s) = (1/k) + (1 + s\gamma)G(s)$ graphically. By Lemma 6.1, $Z(s)$ is strictly positive real if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] > 0, \quad \forall\; \omega \in [-\infty,\infty] \qquad (7.19)$$

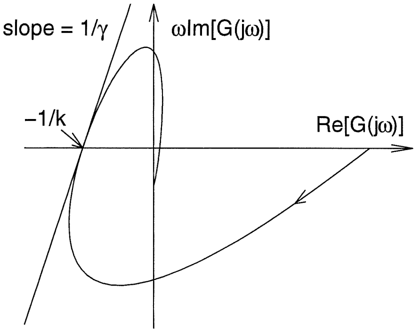

Figure 7.13: Popov plot.

where $G(j\omega) = \text{Re}[G(j\omega)] + j\text{Im}[G(j\omega)]$. If we plot $\text{Re}[G(j\omega)]$ versus $\omega\text{Im}[G(j\omega)]$ with ω as a parameter, then condition (7.19) is satisfied if the plot lies to the right of the line that intercepts the point $-(1/k) + j0$ with a slope $1/\gamma$. (See Figure 7.13.) Such a plot is known as a Popov plot, in contrast to a Nyquist plot, which is a plot of $\text{Re}[G(j\omega)]$ versus $\text{Im}[G(j\omega)]$. If condition (7.19) is satisfied only for $\omega \in (-\infty, \infty)$, while the left-hand side approaches zero as ω tends to ∞, then we need to analytically verify that

$$\lim_{\omega\to\infty} \omega^2 \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] \right\} > 0$$

This case arises when $k = \infty$ and $G(s)$ has relative degree two.

With $\gamma = 0$, condition (7.19) reduces to the circle criterion condition $\text{Re}[G(j\omega)] > -1/k$, which shows that, for the system (7.13)–(7.15), the conditions of the Popov criterion are weaker than those of the circle criterion. In other words, with $\gamma > 0$, absolute stability can be established under less stringent conditions.

Example 7.5 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - h(y) \\ y &= x_1 \end{aligned}$$

This system would fit the form (7.13)–(7.15) if we took $\psi = h$, but the matrix A would not be Hurwitz. Adding and subtracting the term αy to the right-hand side of the second state equation, where $\alpha > 0$, and defining $\psi(y) = h(y) - \alpha y$, the system takes the form (7.13)–(7.15), with

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

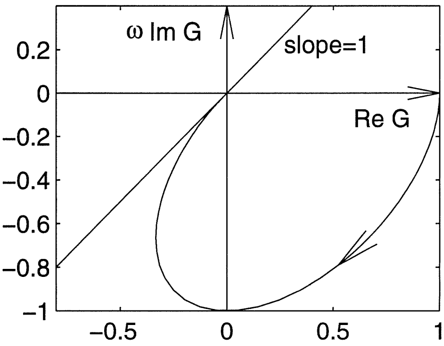

Figure 7.14: Popov plot for Example 7.5.

Assume that h belongs to a sector $[\alpha, \beta]$, where $\beta > \alpha$. Then, ψ belongs to the sector $[0, k]$, where $k = \beta - \alpha$. Condition (7.19) takes the form

$$\frac{1}{k} + \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [-\infty, \infty]$$

For all finite positive values of α and k, this inequality is satisfied by choosing $\gamma > 1$. Even at $k = \infty$, the foregoing inequality is satisfied for all $\omega \in (-\infty, \infty)$ and

$$\lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

Hence, the system is absolutely stable for all nonlinearities h in the sector $[\alpha, \infty]$, where α can be arbitrarily small. Figure 7.14 shows the Popov plot of $G(j\omega)$ for $\alpha = 1$. The plot is drawn only for $\omega \geq 0$, since $\text{Re}[G(j\omega)]$ and $\omega\text{Im}[G(j\omega)]$ are even functions of ω. The Popov plot asymptotically approaches the line through the origin of unity slope from the right side. Therefore, it lies to the right of any line of slope less than one that intersects the real axis at the origin and approaches it asymptotically as ω tends to ∞. To see the advantage of having $\gamma > 0$, let us take $\gamma = 0$ and apply the circle criterion. From the second case of Theorem 7.2, the system is absolutely stable if the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\text{Re}[s] = -1/k$. Since a portion of the Nyquist plot lies in the left-half plane, k cannot be arbitrarily large. The maximum permissible value of k can be determined analytically from the condition

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [-\infty, \infty]$$

which yields $k < 1 + 2\sqrt{\alpha}$. Thus, using the circle criterion, we can only conclude that the system is absolutely stable for all nonlinearities h in the sector $[\alpha, 1 + \alpha + 2\sqrt{\alpha} - \varepsilon]$, where $\alpha > 0$ and $\varepsilon > 0$ can be arbitrarily small. △

7.2 The Describing Function Method

Consider a single-input–single-output nonlinear system represented by the feedback connection of Figure 7.1, where $G(s)$ is a strictly proper, rational transfer function and ψ is a time-invariant, memoryless nonlinearity. We assume that the external input $r = 0$ and study the existence of periodic solutions. A periodic solution satisfies $y(t + 2\pi/\omega) = y(t)$ for all t, where ω is the frequency of oscillation. We will use a general method for finding periodic solutions, known as *the method of harmonic balance.* The idea of the method is to represent a periodic solution by a Fourier series and seek a frequency ω and a set of Fourier coefficients that satisfy the system's equation. Suppose $y(t)$ is periodic and let

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega t)$$

be its Fourier series, where a_k are complex coefficients,[7] $a_k = \bar{a}_{-k}$ and $j = \sqrt{-1}$. Since $\psi(\cdot)$ is a time-invariant nonlinearity, $\psi(y(t))$ is periodic with the same frequency ω and can be written as

$$\psi(y(t)) = \sum_{k=-\infty}^{\infty} c_k \exp(jk\omega t)$$

where each complex coefficient c_k is a function of all a_i's. For $y(t)$ to be a solution of the feedback system, it must satisfy the differential equation

$$d(p)y(t) + n(p)\psi(y(t)) = 0$$

where p is the differential operator $p(\cdot) = d(\cdot)/dt$ and $n(s)$ and $d(s)$ are the numerator and denominator polynomials of $G(s)$. Because

$$p\exp(jk\omega t) = \frac{d}{dt}\exp(jk\omega t) = jk\omega \exp(jk\omega t)$$

we have

$$d(p)\sum_{k=-\infty}^{\infty} a_k \exp(jk\omega t) = \sum_{k=-\infty}^{\infty} d(jk\omega)a_k \exp(jk\omega t)$$

and

$$n(p)\sum_{k=-\infty}^{\infty} c_k \exp(jk\omega t) = \sum_{k=-\infty}^{\infty} n(jk\omega)c_k \exp(jk\omega t)$$

Substituting these expressions back into the differential equation yields

$$\sum_{k=-\infty}^{\infty} [d(jk\omega)a_k + n(jk\omega)c_k] \exp(jk\omega t) = 0$$

---

[7] A bar over a complex variable denotes its complex conjugate.

Using the orthogonality of the functions $\exp(jk\omega t)$ for different values of k, we find that the Fourier coefficients must satisfy

$$G(jk\omega)c_k + a_k = 0 \tag{7.20}$$

for all integers k. Because $G(jk\omega) = \bar{G}(-jk\omega)$, $a_k = \bar{a}_{-k}$, and $c_k = \bar{c}_{-k}$, we need only look at (7.20) for $k \geq 0$. Equation (7.20) is an infinite-dimensional equation, which we can hardly solve. We need to find a finite-dimensional approximation of (7.20). Noting that the transfer function $G(s)$ is strictly proper, that is, $G(j\omega) \rightarrow 0$ as $\omega \rightarrow \infty$, it is reasonable to assume that there is an integer $q > 0$ such that for all $k > q$, $|G(jk\omega)|$ is small enough to replace $G(jk\omega)$ (and consequently a_k) by 0. This approximation reduces (7.20) to a finite-dimensional problem

$$G(jk\omega)\hat{c}_k + \hat{a}_k = 0, \quad k = 0, 1, 2, \ldots, q \tag{7.21}$$

where the Fourier coefficients are written with a hat accent to emphasize that a solution of (7.21) is only an approximation to the solution of (7.20). In essence, we can proceed to solve (7.21). However, the complexity of the problem will grow with q and, for a large q, the finite-dimensional problem (7.21) might still be difficult to solve. The simplest problem results if we can choose $q = 1$. This, of course, requires the transfer function $G(s)$ to have sharp "low-pass filtering" characteristics to allow us to approximate $G(jk\omega)$ by 0 for all $k > 1$. Even though we know $G(s)$, we cannot judge whether this is a good approximation, since we do not know the frequency of oscillation ω. Nevertheless, the classical describing function method makes this approximation and sets $\hat{a}_k = 0$ for $k > 1$ to reduce the problem to one of solving the two equations

$$G(0)\hat{c}_0(\hat{a}_0, \hat{a}_1) + \hat{a}_0 = 0 \tag{7.22}$$

$$G(j\omega)\hat{c}_1(\hat{a}_0, \hat{a}_1) + \hat{a}_1 = 0 \tag{7.23}$$

Notice that (7.22) and (7.23) define one real equation (7.22) and one complex equation (7.23) in two real unknowns, ω and $\hat{a}_0$, and a complex unknown $\hat{a}_1$. When expressed as real quantities, they define three equations in four unknowns. This is expected because the time origin is arbitrary for an autonomous system, so if $(\hat{a}_0, \hat{a}_1)$ satisfies the equation, then $(\hat{a}_0, \hat{a}_1 e^{j\theta})$ will give another solution for arbitrary real θ. To take care of this nonuniqueness, we take the first harmonic of $y(t)$ to be $a \sin \omega t$, with $a \geq 0$; that is, we choose the time origin such that the phase of the first harmonic is zero. Using

$$a \sin \omega t = \frac{a}{2j}[\exp(j\omega t) - \exp(-j\omega t)] \;\Rightarrow\; \hat{a}_1 = \frac{a}{2j}$$

we rewrite (7.22) and (7.23) as

$$G(0)\hat{c}_0\left(\hat{a}_0, \frac{a}{2j}\right) + \hat{a}_0 = 0 \tag{7.24}$$

$$G(j\omega)\hat{c}_1\left(\hat{a}_0, \frac{a}{2j}\right) + \frac{a}{2j} = 0 \tag{7.25}$$

Since (7.24) does not depend on ω, it may be solved for $\hat{a}_0$ as a function of a. Note that if $\psi(\cdot)$ is an odd function, that is,

$$\psi(-y) = -\psi(y)$$

then $\hat{a}_0 = \hat{c}_0 = 0$ is a solution of (7.24) because

$$\hat{c}_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \psi(\hat{a}_0 + a \sin \omega t)\ dt$$

For convenience, let us restrict our attention to nonlinearities with odd symmetry and take $\hat{a}_0 = \hat{c}_0 = 0$. Then, we can rewrite (7.25) as

$$G(j\omega)\hat{c}_1\left(0, \frac{a}{2j}\right) + \frac{a}{2j} = 0 \tag{7.26}$$

The coefficient $\hat{c}_1(0, a/2j)$ is the complex Fourier coefficient of the first harmonic at the output of the nonlinearity when its input is the sinusoidal signal $a \sin \omega t$. It is given by

$$\begin{aligned}
\hat{c}_1(0, a/2j) &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \psi(a \sin \omega t) \exp(-j\omega t)\ dt \\
&= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} [\psi(a \sin \omega t) \cos \omega t - j\psi(a \sin \omega t) \sin \omega t]\ dt
\end{aligned}$$

The first term under the integral sign is an odd function, while the second term is an even function. Therefore, the integration of the first term over one complete cycle is zero, and the integral simplifies to

$$\hat{c}_1(0, a/2j) = -j\frac{\omega}{\pi} \int_0^{\pi/\omega} \psi(a \sin \omega t) \sin \omega t\ dt$$

Define a function $\Psi(a)$ by

$$\Psi(a) = \frac{\hat{c}_1(0, a/2j)}{a/2j} = \frac{2\omega}{\pi a} \int_0^{\pi/\omega} \psi(a \sin \omega t) \sin \omega t\ dt \tag{7.27}$$

so that (7.26) can be rewritten as

$$[G(j\omega)\Psi(a) + 1]a = 0 \tag{7.28}$$

Since we are not interested in a solution with $a = 0$, we can solve (7.28) completely by finding all solutions of

$$G(j\omega)\Psi(a) + 1 = 0 \tag{7.29}$$

Equation (7.29) is known as the *first-order harmonic balance equation*, or simply the *harmonic balance equation*. The function $\Psi(a)$ defined by (7.27) is called *the* *describing function* of the nonlinearity ψ. It is obtained by applying a sinusoidal signal $a \sin \omega t$ at the input of the nonlinearity and by calculating the ratio of the Fourier coefficient of the first harmonic at the output to a. It can be thought of as an "equivalent gain" of a linear time-invariant element whose response to $a \sin \omega t$ is $\Psi(a) a \sin \omega t$. This equivalent gain concept (sometimes called *equivalent linearization*) can be applied to more general time-varying nonlinearities or nonlinearities with memory, like hysteresis and backlash.[8] In that general context, the describing function might be complex and dependent on both a and ω. We will only deal with describing functions of odd, time-invariant, memoryless nonlinearities for which $\Psi(a)$ is real, dependent only on a, and given by the expression

$$\Psi(a) = \frac{2}{\pi a} \int_0^{\pi} \psi(a \sin \theta) \sin \theta \; d\theta \tag{7.30}$$

which is obtained from (7.27) by changing the integration variable from t to $\theta = \omega t$.

The describing function method states that if (7.29) has a solution (a_s, ω_s), then there is "probably" a periodic solution of the system with frequency and amplitude (at the input of the nonlinearity) close to ω_s and a_s. Conversely, if (7.29) has no solutions, then the system "probably" does not have a periodic solution. More analysis is needed to replace the word "probably" with "certainly" and to quantify the phrase "close to ω_s and a_s" when there is a periodic solution. We will postpone these investigations until a later point in the section. For now, we would like to look more closely at the calculation of the describing function and the question of solving the harmonic balance equation (7.29). The next three examples illustrate the calculation of the describing function for odd nonlinearities.

Example 7.6 Consider the signum nonlinearity $\psi(y) = \text{sgn}(y)$. The describing function is given by

$$\Psi(a) = \frac{2}{\pi a} \int_0^{\pi} \psi(a \sin \theta) \sin \theta \; d\theta = \frac{2}{\pi a} \int_0^{\pi} \sin \theta \; d\theta = \frac{4}{\pi a}$$

△

Example 7.7 Consider the *piecewise-linear* function of Figure 7.15. If a sinusoidal input to this nonlinearity has amplitude $a \leq \delta$, the nonlinearity will act as a linear gain. The output will be a sinusoid with amplitude $s_1 a$. Hence, the describing function is $\Psi(a) = s_1$, independent of a. When $a > \delta$, we divide the integral on the right-hand side of (7.30) into pieces, with each piece corresponding to a linear portion of $\psi(\cdot)$. Furthermore, using the odd symmetry of the output waveform, we simplify the integration to

$$\Psi(a) \;=\; \frac{2}{\pi a} \int_0^{\pi} \psi(a \sin \theta) \sin \theta \; d\theta \;=\; \frac{4}{\pi a} \int_0^{\pi/2} \psi(a \sin \theta) \sin \theta \; d\theta$$

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[8] See [18] or [85].

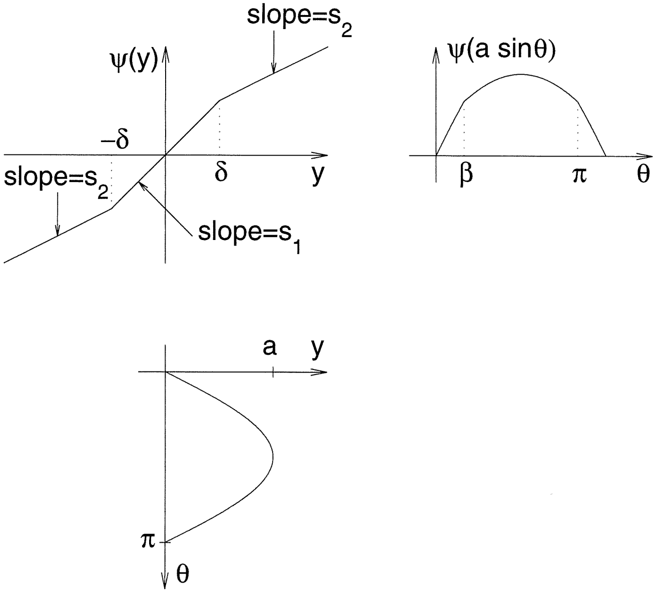

Figure 7.15: Piecewise-linear function.

$$
\begin{aligned}
&= \frac{4}{\pi a}\int_0^{\beta} a s_1 \sin^2\theta \; d\theta \\
&\quad + \frac{4}{\pi a}\int_{\beta}^{\pi/2} [\delta s_1 + s_2(a\sin\theta - \delta)]\sin\theta \; d\theta, \quad \beta = \sin^{-1}\left(\frac{\delta}{a}\right) \\
&= \frac{2s_1}{\pi}\left(\beta - \frac{1}{2}\sin 2\beta\right) + \frac{4\delta(s_1 - s_2)}{\pi a}\left(\cos\beta - \cos\frac{\pi}{2}\right) \\
&\quad + \frac{2s_2}{\pi}\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi - \beta + \frac{1}{2}\sin 2\beta\right) \\
&= \frac{2(s_1 - s_2)}{\pi}\left(\beta + \frac{\delta}{a}\cos\beta\right) + s_2
\end{aligned}
$$

Thus,

$$
\Psi(a) = \frac{2(s_1 - s_2)}{\pi}\left[\sin^{-1}\left(\frac{\delta}{a}\right) + \frac{\delta}{a}\sqrt{1 - \left(\frac{\delta}{a}\right)^2}\,\right] + s_2
$$

A sketch of the describing function is shown in Figure 7.16. By selecting specific values for δ and the slopes s_1 and s_2, we can obtain the describing function of several common nonlinearities. For example, the saturation nonlinearity is a special case of the piecewise-linear function of Figure 7.15 with $\delta = 1$, $s_1 = 1$, and $s_2 = 0$. Hence,

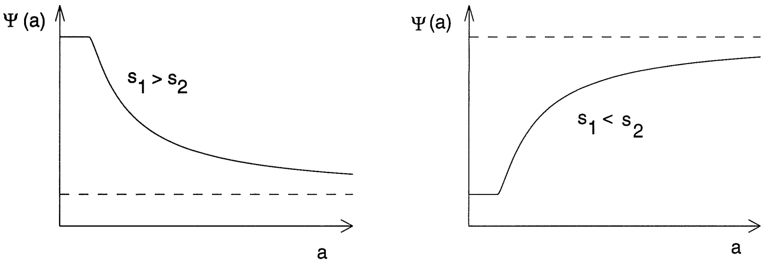

Figure 7.16: Describing function for the piecewise-linear function of Figure 7.15.

its describing function is given by

$$\Psi(a) = \begin{cases} 1, & \text{if } 0 \le a \le 1 \\ \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{1-\left(\frac{1}{a}\right)^2}\right], & \text{if } a > 1 \end{cases}$$

△

Example 7.8 Consider an odd nonlinearity that satisfies the sector condition

$$\alpha y^2 \le y\psi(y) \le \beta y^2$$

for all $y \in R$. The describing function $\Psi(a)$ satisfies the lower bound

$$\Psi(a) = \frac{2}{\pi a}\int_0^\pi \psi(a\sin\theta)\sin\theta\ d\theta \ge \frac{2\alpha}{\pi}\int_0^\pi \sin^2\theta\ d\theta = \alpha$$

and the upper bound

$$\Psi(a) = \frac{2}{\pi a}\int_0^\pi \psi(a\sin\theta)\sin\theta\ d\theta \le \frac{2\beta}{\pi}\int_0^\pi \sin^2\theta\ d\theta = \beta$$

Therefore,

$$\alpha \le \Psi(a) \le \beta, \ \ \forall\ a \ge 0$$

△

Since the describing function $\Psi(a)$ is real, (7.29) can be rewritten as

$$\{\text{Re}[G(j\omega)] + j\text{Im}[G(j\omega)]\}\,\Psi(a) + 1 = 0$$

This equation is equivalent to the two real equations

$$1 + \Psi(a)\text{Re}[G(j\omega)] = 0 \qquad (7.31)$$

$$\text{Im}[G(j\omega)] = 0 \qquad (7.32)$$

Because (7.32) is independent of a, we can solve it first for ω to determine the possible frequencies of oscillation. For each solution ω, we solve (7.31) for a. Note that the possible frequencies of oscillation are determined solely by the transfer function $G(s)$; they are independent of the nonlinearity $\psi(\cdot)$. The nonlinearity determines the corresponding value of a, that is, the possible amplitude of oscillation. This procedure can be carried out analytically for low-order transfer functions, as illustrated by the next examples.

Example 7.9 Let

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

and consider two nonlinearities: the signum nonlinearity and the saturation nonlinearity. By simple manipulation, we can write $G(j\omega)$ as

$$G(j\omega) = \frac{-3\omega - j\left(2-\omega^2\right)}{9\omega^3 + \omega\left(2-\omega^2\right)^2}$$

Equation (7.32) takes the form

$$\frac{\left(2-\omega^2\right)}{9\omega^3 + \omega\left(2-\omega^2\right)^2} = 0$$

which has one positive root $\omega = \sqrt{2}$. Note that for each positive root of (7.32), there is a negative root of equal magnitude. We only consider the positive roots. Note also that a root at $\omega = 0$ would be of no interest because it would not give rise to a nontrivial periodic solution. Evaluating $\text{Re}[G(j\omega)]$ at $\omega = \sqrt{2}$ and substituting it in (7.31), we obtain $\Psi(a) = 6$. All this information has been gathered without specifying the nonlinearity $\psi(\cdot)$. Consider now the signum nonlinearity. We found in Example 7.6 that $\Psi(a) = 4/\pi a$. Therefore, $\Psi(a) = 6$ has a unique solution $a = 2/3\pi$. Now we can say that the nonlinear system formed of $G(s)$ and the signum nonlinearity will "probably" oscillate with frequency close to $\sqrt{2}$ and amplitude (at the input of the nonlinearity) close to $2/3\pi$. Consider next the saturation nonlinearity. We found in Example 7.7 that $\Psi(a) \leq 1$ for all a. Therefore, $\Psi(a) = 6$ has no solutions. Therefore, we expect that the nonlinear system formed of $G(s)$ and the saturation nonlinearity will not have a sustained oscillation. △

Example 7.10 Let

$$G(s) = \frac{-s}{s^2 + 0.8s + 8}$$

and consider two nonlinearities: the saturation nonlinearity and a dead-zone nonlinearity that is a special case of the piecewise-linear function of Example 7.7 with $s_1 = 0$, $s_2 = 0.5$, and $\delta = 1$. We can write $G(j\omega)$ as

$$G(j\omega) = \frac{-0.8\omega^2 - j\omega\left(8-\omega^2\right)}{0.64\omega^2 + \left(8-\omega^2\right)^2}$$

Equation (7.32) has a unique positive root $\omega = 2\sqrt{2}$. Evaluating $\mathrm{Re}[G(j\omega)]$ at $\omega = 2\sqrt{2}$ and substituting it in (7.31), we obtain $\Psi(a) = 0.8$. For the saturation nonlinearity, the describing function is given in Example 7.7, and $\Psi(a) = 0.8$ has the unique solution $a = 1.455$. Therefore, we expect that the nonlinear system formed of $G(s)$ and the saturation nonlinearity will oscillate with frequency close to $2\sqrt{2}$ and amplitude (at the input of the nonlinearity) close to 1.455. For the dead-zone nonlinearity, the describing function $\Psi(a)$ is less than 0.8 for all a. Thus, $\Psi(a) = 0.8$ has no solutions, and we expect that the nonlinear system formed of $G(s)$ and the dead-zone nonlinearity will not have sustained oscillations. In this particular example, we can confirm the no oscillation conjecture by showing that the system is absolutely stable for a class of sector nonlinearities, which includes the given dead-zone nonlinearity. It can be easily checked that

$$\mathrm{Re}[G(j\omega)] \geq -1.25, \quad \forall\ \omega \in R$$

Therefore, from the circle criterion (Theorem 7.2), we know that the system is absolutely stable for a sector $[0, \beta]$ with $\beta < 0.8$. The given dead-zone nonlinearity belongs to this sector. Consequently, the origin of the state space is globally asymptotically stable and the system cannot have a sustained oscillation. △

Example 7.11 Consider Raleigh's equation

$$\ddot{z} + z = \varepsilon\left(\dot{z} - \tfrac{1}{3}\dot{z}^3\right)$$

where ε is a positive constant. To study existence of periodic solutions, we represent the equation in the feedback form of Figure 7.1. Let $u = -\dot{z}^3/3$ and rewrite the system's equation as

$$\begin{aligned} \ddot{z} - \varepsilon\dot{z} + z &= \varepsilon u \\ u &= -\tfrac{1}{3}\dot{z}^3 \end{aligned}$$

The first equation defines a linear system. Taking $y = \dot{z}$ to be its output, its transfer function is

$$G(s) = \frac{\varepsilon s}{s^2 - \varepsilon s + 1}$$

The second equation defines a nonlinearity $\psi(y) = y^3/3$. The two equations together represent the system in the feedback form of Figure 7.1. The describing function of $\psi(y) = y^3/3$ is given by

$$\Psi(a) = \frac{2}{3\pi a}\int_0^{\pi} (a\sin\theta)^3 \sin\theta\ d\theta = \tfrac{1}{4}a^2$$

The function $G(j\omega)$ can be written as

$$G(j\omega) = \frac{j\varepsilon\omega[(1-\omega^2) + j\varepsilon\omega]}{(1-\omega^2)^2 + \varepsilon^2\omega^2}$$

The equation $\mathrm{Im}[G(j\omega)] = 0$ yields $\omega(1 - \omega^2) = 0$; hence, there is a unique positive solution $\omega = 1$. Then,

$$1 + \Psi(a)\mathrm{Re}[G(j)] = 0 \;\Rightarrow\; a = 2$$

Therefore, we expect that Raleigh's equation has a periodic solution of frequency near 1 rad/sec and that the amplitude of oscillation in $\dot{z}$ is near 2. △

For higher-order transfer functions, solving the harmonic balance equation (7.29) analytically might be very complicated. Of course, we can always resort to numerical methods for solving (7.29). However, the power of the describing function method is not in solving (7.29) analytically or numerically; rather, it is the graphical solution of (7.29) that made the method popular. Equation (7.29) can be rewritten as

$$G(j\omega) = -\frac{1}{\Psi(a)} \tag{7.33}$$

or

$$\frac{1}{G(j\omega)} = -\Psi(a) \tag{7.34}$$

Equation (7.33) suggests that we can solve (7.29) by plotting the Nyquist plot of $G(j\omega)$ for $\omega > 0$ and the locus of $-1/\Psi(a)$ for $a \geq 0$. Intersections of these loci give the solutions of (7.29). Since $\Psi(a)$ is real for odd nonlinearities, the locus of $-1/\Psi(a)$ in the complex plane will be confined to the real axis. Equation (7.34) suggests a similar procedure by plotting the inverse Nyquist plot of $G(j\omega)$ (that is, the locus in the complex plane of $1/G(j\omega)$ as ω varies) and the locus of $-\Psi(a)$. The important role of Nyquist plots in classical control theory made this graphical implementation of the describing function method a popular tool with control engineers as they faced nonlinearities.

Example 7.12 Consider again the transfer function $G(s)$ of Example 7.9. The Nyquist plot of $G(j\omega)$ is shown in Figure 7.17. It intersects the real axis at $(-1/6, 0)$. For odd nonlinearities, (7.29) will have a solution if the locus of $-1/\Psi(a)$ on the real axis includes this point of intersection. △

Let us turn now to the question of justifying the describing function method. Being an approximate method for solving the infinite-dimensional equation (7.20), the describing function method can be justified by providing estimates of the error caused by the approximation. In the interest of simplicity, we will pursue this analysis only for nonlinearities with the following two features:[9]

- Odd nonlinearity, that is, $\psi(y) = -\psi(-y)$, $\forall\ y \neq 0$.
- Single-valued nonlinearity with a slope between α and β; that is,

  $$\alpha(y_2 - y_1) \leq [\psi(y_2) - \psi(y_1)] \leq \beta(y_2 - y_1)$$

  for all real numbers y_1 and $y_2 > y_1$.

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[9] See [24], [129], and [189] for describing function theory for more general nonlinearities.

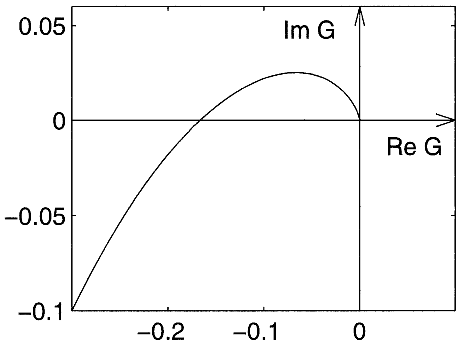

Figure 7.17: Nyquist plot for Example 7.12.

A nonlinearity $\psi(\cdot)$ with these features belongs to a sector $[\alpha, \beta]$. Hence, from Example 7.8, its describing function satisfies $\alpha \leq \Psi(a) \leq \beta$ for all $a \geq 0$. It should be noted, however, that the slope restriction is not the same as the sector condition. A nonlinearity may satisfy the foregoing slope restriction with bounds α and β, and could belong to a sector $[\bar{\alpha}, \bar{\beta}]$ with different bounds $\bar{\alpha}$ and $\bar{\beta}$.[10] We emphasize that in the forthcoming analysis, we should use the slope bounds α and β, not the sector boundaries $\bar{\alpha}$ and $\bar{\beta}$.

Example 7.13 Consider the piecewise-linear odd nonlinearity

$$\psi(y) = \begin{cases} y, & \text{for } 0 \leq y \leq 2 \\ 4 - y, & \text{for } 2 \leq y \leq 3 \\ y - 2, & \text{for } y \geq 3 \end{cases}$$

shown in Figure 7.18. The nonlinearity satisfies the slope restriction

$$-1 \leq \frac{\psi(y_2) - \psi(y_1)}{y_2 - y_1} \leq 1$$

as well as the sector condition

$$\frac{1}{3} \leq \frac{\psi(y)}{y} \leq 1$$

In the forthcoming analysis, we should take $\alpha = -1$ and $\beta = 1$. △

We will restrict our attention to the question of the existence of half-wave symmetric periodic solutions;[11] that is, periodic solutions that only have odd harmonics. This is a reasonable restriction in view of the odd symmetry of ψ. The Fourier coefficients

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[10] Verify that $[\bar{\alpha}, \bar{\beta}] \subset [\alpha, \beta]$.

[11] This restriction is made only for convenience. See [128] for a more general analysis that does not make this assumption.

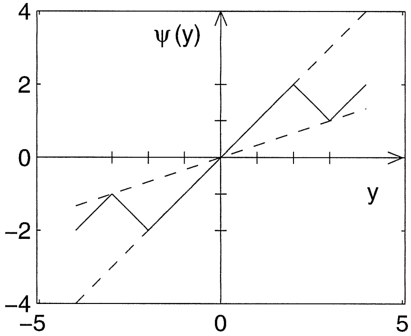

Figure 7.18: Nonlinearity of Example 7.13.

of the odd harmonics of a periodic solution $y(t)$ satisfy (7.20) for $k = 1, 3, 5, \ldots$. The basic idea of the error analysis is to split the periodic solution $y(t)$ into a first harmonic $y_1(t)$ and higher harmonics $y_h(t)$. We choose the time origin such that the phase of the first harmonic is zero; that is, $y_1(t) = a \sin \omega t$. Thus,

$$y(t) = a \sin \omega t + y_h(t)$$

Using this representation, the Fourier coefficients of the first harmonic of $y(t)$ and $\psi(y(t))$ are

$$\begin{aligned} a_1 &= \frac{a}{2j} \\ c_1 &= \frac{\omega}{\pi} \int_0^{\pi/\omega} \psi(a \sin \omega t + y_h(t)) \exp(-j\omega t)\ dt \end{aligned}$$

From (7.20), with $k = 1$, we have

$$G(j\omega)c_1 + a_1 = 0$$

Introducing the function

$$\Psi^*(a, y_h) = \frac{c_1}{a_1} = j\frac{2\omega}{\pi a} \int_0^{\pi/\omega} \psi(a \sin \omega t + y_h(t)) \exp(-j\omega t)\ dt$$

we can rewrite the equation as

$$\frac{1}{G(j\omega)} + \Psi^*(a, y_h) = 0 \tag{7.35}$$

Adding $\Psi(a)$ to both sides of (7.35), we can rewrite it as

$$\frac{1}{G(j\omega)} + \Psi(a) = \delta\Psi \tag{7.36}$$

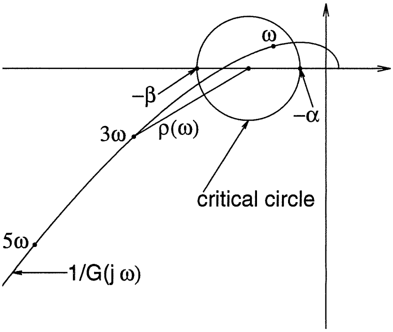

Figure 7.19: Finding $\rho(\omega)$.

where

$$\delta\Psi = \Psi(a) - \Psi^*(a, y_h)$$

When $y_h = 0$, $\Psi^*(a, 0) = \Psi(a)$. Thus, $\delta\Psi = 0$ and (7.36) reduces to the harmonic balance equation

$$\frac{1}{G(j\omega)} + \Psi(a) = 0 \tag{7.37}$$

Therefore, the harmonic balance equation (7.37) is an approximate version of the exact equation (7.36). The error term $\delta\Psi$ cannot be found exactly, but its size can often be estimated. Our next step is to find an upper bound on $\delta\Psi$. To that end, let us define two functions $\rho(\omega)$ and $\sigma(\omega)$. Start by drawing the locus of $1/G(j\omega)$ in the complex plane. On the same graph paper, draw a (critical) circle with the interval $[-\beta, -\alpha]$ on the real axis as a diameter. Notice that the locus of $-\Psi(a)$ lies inside this circle on the real axis, since $\alpha \leq \Psi(a) \leq \beta$. Now consider an ω such that the points on the locus $1/G$ corresponding to $k\omega$ ($k > 1$ and odd) lie outside the critical circle, as shown in Figure 7.19. The distance from any one of these points to the center of the critical circle is

$$\left| \frac{\alpha + \beta}{2} + \frac{1}{G(jk\omega)} \right|$$

Define

$$\rho(\omega) = \inf_{k>1;k \text{ odd}} \left| \frac{\alpha + \beta}{2} + \frac{1}{G(jk\omega)} \right| \tag{7.38}$$

Note that we have defined $\rho(\omega)$ only for ω at which $1/G(jk\omega)$ lies outside the critical circle for all $k = 3, 5, \ldots$; that is, for ω in the set

$$\Omega = \{\omega \mid \rho(\omega) > \tfrac{1}{2}(\beta - \alpha)\}$$

On any connected subset Ω' of Ω, define

$$\sigma(\omega) = \frac{\left(\frac{\beta-\alpha}{2}\right)^2}{\rho(\omega) - \frac{\beta-\alpha}{2}} \tag{7.39}$$

The positive quantity $\sigma(\omega)$ is an upper bound on the error term $\delta\Psi$, as stated in the next lemma.

Lemma 7.1 *Under the stated assumptions,*

$$\frac{\omega}{\pi}\int_0^{2\pi/\omega} y_h^2(t)\ dt \le \left[\frac{2\sigma(\omega)a}{\beta-\alpha}\right]^2, \quad \forall\ \omega \in \Omega' \tag{7.40}$$

$$|\delta\Psi| \le \sigma(\omega), \quad \forall\ \omega \in \Omega' \tag{7.41}$$

◇

Proof: See Appendix C.13.

The proof of Lemma 7.1 is based on writing an equation for $y_h(t)$ in the form $y_h = T(y_h)$ and showing that $T(\cdot)$ is a contraction mapping. This allows us to calculate the upper bound of (7.40), which is then used to calculate the upper bound of (7.41) on the error term. The slope restrictions on the nonlinearity ψ are used in showing that $T(\cdot)$ is a contraction mapping.

Using the bound of (7.41) in (7.36), we see that a necessary condition for the existence of a half-wave symmetric periodic solution with $\omega \in \Omega'$ is

$$\left|\frac{1}{G(j\omega)} + \Psi(a)\right| \le \sigma(\omega)$$

Geometrically, this condition states that the point $-\Psi(a)$ must be contained in a circle with a center at $1/G(j\omega)$ and radius $\sigma(\omega)$. For each $\omega \in \Omega' \subset \Omega$, we can draw such an error circle. The envelope of all error circles over the connected set Ω' forms an uncertainty band. The reason for choosing a subset of Ω is that, as ω approaches the boundary of Ω, the error circles become arbitrarily large and cease to give any useful information. The subset Ω' should be chosen with the objective of drawing a narrow band. If $G(j\omega)$ has sharp low-pass filtering characteristics, the uncertainty band can be quite narrow over Ω'. Note that $\rho(\omega)$ is a measure of the low-pass filtering characteristics of $G(j\omega)$; for the smaller $|G(jk\omega)|$ for $k > 1$, the larger $\rho(\omega)$, as seen from (7.38). A large $\rho(\omega)$ results in a small radius $\sigma(\omega)$ for the error circle, as seen from (7.39).

We are going to look at intersections of the uncertainty band with the locus of $-\Psi(a)$. If no part of the band intersects the $-\Psi(a)$ locus, then clearly (7.36) has no solution with $\omega \in \Omega'$. If the band intersects the locus completely, as in Figure 7.20, then we expect that there is a solution. This is indeed true, provided

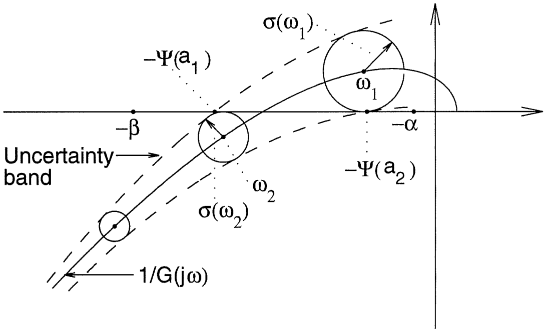

Figure 7.20: A complete intersection.

we exclude some degenerate cases. Actually, we can find error bounds by examining the intersection. Let a_1 and a_2 be the amplitudes corresponding to the intersections of the boundary of the uncertainty band with the $-\Psi(a)$ locus. Let ω_1 and ω_2 be the frequencies corresponding to the error circles of radii $\sigma(\omega_1)$ and $\sigma(\omega_2)$, which are tangent to the $-\Psi(a)$ locus on either side of it. Define a rectangle Γ in the (ω, a) plane by

$$\Gamma = \{(\omega, a) \mid \omega_1 < \omega < \omega_2,\ a_1 < a < a_2\}$$

The rectangle Γ contains the point (ω_s, a_s) for which the loci of $1/G$ and $-\Psi$ intersect, that is, the solution of the harmonic balance equation (7.37). It turns out that if certain regularity conditions hold, then it is possible to show that (7.36) has a solution in the closure of Γ. These regularity conditions are

$$\left.\frac{d}{da}\Psi(a)\right|_{a=a_s} \neq 0; \quad \left.\frac{d}{d\omega}\mathrm{Im}[G(j\omega)]\right|_{\omega=\omega_s} \neq 0$$

A complete intersection between the uncertainty band and the $-\Psi(a)$ locus can now be precisely defined as taking place when the $1/G(j\omega)$ locus intersects the $-\Psi(a)$ locus and a finite set Γ can be defined, as shown, such that (ω_s, a_s) is the unique intersection point in Γ and the regularity conditions hold.

Finally, notice that at high frequencies for which all harmonics (including the first) have the corresponding $1/G(j\omega)$ points outside the critical circle, we do not need to draw the uncertainty band. Therefore, we define a set

$$\tilde{\Omega} = \left\{\omega \;\middle|\; \left|\frac{\alpha+\beta}{2} + \frac{1}{G(jk\omega)}\right| > \frac{\beta-\alpha}{2},\ k = 1, 3, 5, \ldots\right\}$$

and take the smallest frequency in $\tilde{\Omega}$ as the largest frequency in Ω', then decrease ω until the error circles become uncomfortably large.

The next theorem on the justification of the describing function method is the main result of this section.

Theorem 7.4 *Consider the feedback connection of Figure* 7.1, *where the nonlinearity* $\psi(\cdot)$ *is memoryless, time invariant, odd, and single valued with slopes between* α *and* β. *Draw the loci of* $1/G(j\omega)$ *and* $-\Psi(a)$ *in the complex plane and construct the critical circle and the band of uncertainty as described earlier. Then,*

- *the system has no half-wave symmetric periodic solutions with fundamental frequency* $\omega \in \tilde{\Omega}$.
- *the system has no half-wave symmetric periodic solutions with fundamental frequency* $\omega \in \Omega'$ *if the corresponding error circle does not intersect the* $-\Psi(a)$ *locus.*
- *for each complete intersection defining a set* Γ *in the* (ω, a) *plane, there is at least one half-wave symmetric periodic solution*

$$y(t) = a \sin \omega t + y_h(t)$$

with (ω, a) *in* $\bar{\Gamma}$ *and* $y_h(t)$ *satisfies the bound of* (7.40). ◇

Proof: See Appendix C.14.

Note that the theorem gives a sufficient condition for oscillation and a sufficient condition for nonoscillation. Between the two conditions, there is an area of ambiguity where we cannot reach conclusions of oscillation or nonoscillation.

Example 7.14 Consider again

$$G(s) = \frac{-s}{s^2 + 0.8s + 8}$$

together with the saturation nonlinearity. We have seen in Example 7.10 that the harmonic balance equation has a unique solution $\omega_s = 2\sqrt{2} \approx 2.83$ and $a_s = 1.455$. The saturation nonlinearity satisfies the slope restrictions with $\alpha = 0$ and $\beta = 1$. Therefore, the critical circle is centered at -0.5 and its radius is 0.5. The function $1/G(j\omega)$ is given by

$$\frac{1}{G(j\omega)} = -0.8 + j\frac{8 - \omega^2}{\omega}$$

Hence, the locus of $1/G(j\omega)$ lies on the line $\text{Re}[s] = -0.8$, as shown in Figure 7.21. The radius of the error circle $\sigma(\omega)$ has been calculated for eight frequencies starting with $\omega = 2.65$ and ending with $\omega = 3.0$, with uniform increments of 0.05. The centers of the error circles are spread on the line $\text{Re}[s] = -0.8$ inside the critical circle. The value of $\sigma(\omega)$ at $\omega = 2.65$ is 0.0388 and at $\omega = 3.0$ is 0.0321, with monotonic change between the two extremes. In all cases, the closest harmonic to the critical circle is the third harmonic, so that the infimum in (7.38) is achieved at $k = 3$. The boundaries of the uncertainty band are almost vertical. The intersection of the uncertainty band with the $-\Psi(a)$ locus correspond to the points $a_1 = 1.377$

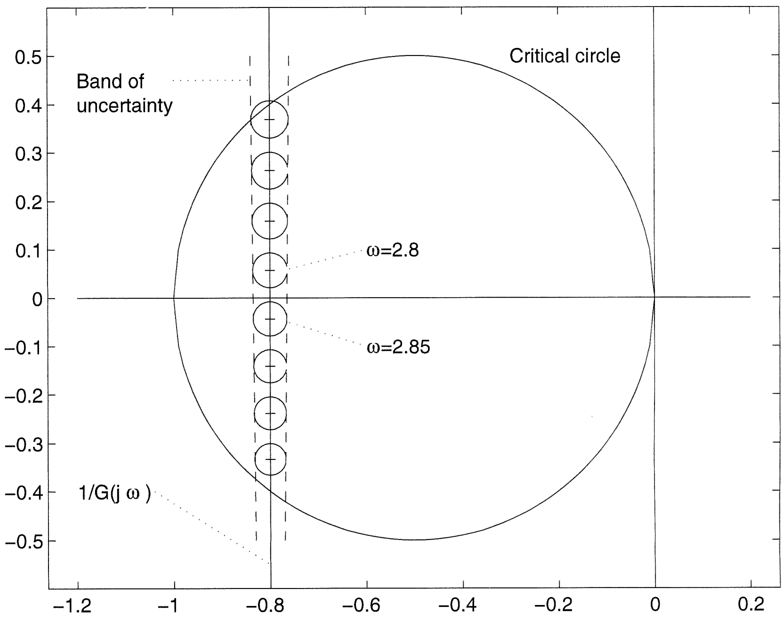

Figure 7.21: Uncertainty band for Example 7.14.

and $a_2 = 1.539$. The error circle corresponding to $\omega = 2.85$ is almost tangent to the real axis from the lower side, so we take $\omega_2 = 2.85$. The error circle corresponding to $\omega = 2.8$ is the closest circle to be tangent to the real axis from the upper side. This means that $\omega_1 > 2.8$. Trying $\omega = 2.81$, we have obtained a circle that is almost tangent to the real axis. Therefore, we define the set Γ as

$$\Gamma = \{(\omega, a) \mid 2.81 < \omega < 2.85,\ 1.377 < a < 1.539\}$$

There is only one intersection point in Γ. We need to check the regularity conditions. The derivative

$$\frac{d}{da}\Psi(a) = \frac{2}{\pi}\frac{d}{da}\left[\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{1-\left(\frac{1}{a}\right)^2}\,\right] = -\frac{4}{\pi a^3}\sqrt{a^2-1}$$

is different from zero at $a = 1.455$, and

$$\left.\frac{d}{d\omega}\mathrm{Im}[G(j\omega)]\right|_{\omega=\sqrt{8}} = \frac{2}{(0.8)^2} \neq 0$$

Thus, by Theorem 7.4, we conclude that the system indeed has a periodic solution. Moreover, we conclude that the frequency of oscillation ω belongs to the interval

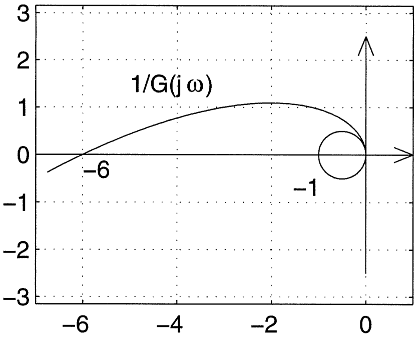

Figure 7.22: Inverse Nyquist plot and critical circle for Example 7.15.

[2.81, 2.85], and the amplitude of the first harmonic at the input of the nonlinearity belongs to the interval [1.377, 1.539]. From the bound of (7.40), we know also that the higher harmonic component $y_h(t)$ satisfies

$$\frac{\omega}{\pi}\int_0^{2\pi/\omega} y_h^2(t)\ dt \le 0.0123, \quad \forall\ (\omega, a) \in \Gamma$$

which shows that the waveform of the oscillating signal at the nonlinearity input is fairly close to its first harmonic $a \sin \omega t$. △

Example 7.15 Reconsider Example 7.9 with

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

and the saturation nonlinearity. The nonlinearity satisfies the slope restriction with $\alpha = 0$ and $\beta = 1$. The inverse Nyquist plot of $G(j\omega)$, shown in Figure 7.22, lies outside the critical circle for all $\omega > 0$. Hence, $\tilde{\Omega} = (0, \infty)$, and we conclude that there is no oscillation. △

7.3 Exercises

7.1 Using the circle criterion, study absolute stability for each of the scalar transfer functions given next. In each case, find a sector $[\alpha, \beta]$ for which the system is absolutely stable.

$$(1) \qquad G(s) = \frac{s}{s^2 - s + 1}$$

$$(2) \qquad G(s) = \frac{1}{(s+2)(s+3)}$$

$$(3) \qquad G(s) = \frac{1}{s^2 + s + 1}$$

$$(4) \qquad G(s) = \frac{s^2 - 1}{(s+1)(s^2+1)}$$

$$(5) \qquad G(s) = \frac{1 - s}{(s+1)^2}$$

$$(6) \qquad G(s) = \frac{s+1}{(s+2)^2(s-1)}$$

$$(7) \qquad G(s) = \frac{1}{(s+1)^4}$$

$$(8) \qquad G(s) = \frac{1}{(s+1)^2(s+2)^2}$$

7.2 Consider the feedback connection of Figure 7.1 with $G(s) = 2s/(s^2 + s + 1)$.

(a) Show that the system is absolutely stable for nonlinearities in the sector $[0, 1]$.

(b) Show that the system has no limit cycles when $\psi(y) = \text{sat}(y)$.

7.3 Consider the system

$$\dot{x}_1 = -x_1 - h(x_1 + x_2), \qquad \dot{x}_2 = x_1 - x_2 - 2h(x_1 + x_2)$$

where h is a smooth function satisfying

$$yh(y) \geq 0, \ \forall \ y \in R, \qquad h(y) = \begin{cases} c, & y \geq a_2 \\ 0, & |y| \leq a_1 \\ -c & y \leq -a_2 \end{cases}$$

$$|h(y)| \leq c, \quad \text{for } a_1 < y < a_2 \text{ and } -a_2 < y < -a_1$$

(a) Show that the origin is the unique equilibrium point.

(b) Show, using the circle criterion, that the origin is globally asymptotically stable.

7.4 (**[201]**) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(\mu^2 + a^2 - q\cos\omega t)x_1 - 2\mu x_2$$

where μ, a, q, and ω are positive constants. Represent the system in the form of Figure 7.1 with $\psi(t, y) = qy\cos\omega t$ and use the circle criterion to derive conditions on μ, a, q, and ω, which ensure that the origin is exponentially stable.

7.5 Consider the linear time-varying system $\dot{x} = [A + BE(t)C]x$, where A is Hurwitz, $\|E(t)\|_2 \leq 1, \ \forall \ t \geq 0$, and $\sup_{\omega \in R} \sigma_{\max}[C(j\omega I - A)^{-1}B] < 1$. Show that the origin is uniformly asymptotically stable.

7.6 Consider the system $\dot{x} = Ax + Bu$ and let $u = -Fx$ be a stabilizing state feedback control; that is, the matrix $(A - BF)$ is Hurwitz. Suppose that, due to physical limitations, we have to use a limiter to limit the value of u_i to $|u_i(t)| \leq L$. The closed-loop system can be represented by $\dot{x} = Ax - BL \operatorname{sat}(Fx/L)$, where $\operatorname{sat}(v)$ is a vector whose ith component is the saturation function.

(a) Show that the system can be represented in the form of Figure 7.1 with $G(s) = F(sI - A + BF)^{-1}B$ and $\psi(y) = L \operatorname{sat}(y/L) - y$.

(b) Derive a condition for asymptotic stability of the origin using the multivariable circle criterion.

(c) Apply the result to the case

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \text{and} \quad L = 1$$

and estimate the region of attraction.

7.7 Repeat Exercise 7.1 using the Popov criterion.

7.8 In this exercise, we derive a version of the Popov criterion for a scalar transfer function $G(s)$ with all poles in the open left-half plane, except for a simple pole on the imaginary axis having a positive residue. The system is represented by

$$\dot{z} = Az - B\psi(y), \qquad \dot{v} = -\psi(y), \qquad \text{and} \qquad y = Cz + dv$$

where $d > 0$, A is Hurwitz, (A, B) is controllable, (A, C) is observable, and ψ belongs to a sector $(0, k]$. Let $V(z, v) = z^T P z + a(y - Cz)^2 + b \int_0^y \psi(\sigma)\, d\sigma$, where $P = P^T > 0$, $a > 0$, and $b \geq 0$.

(a) Show that V is positive definite and radially unbounded.

(b) Show that $\dot{V}$ satisfies the inequality

$$\dot{V} \leq z^T(PA + A^T P)z - 2z^T(PB - w)\psi(y) - \gamma\psi^2(y)$$

where $w = adC^T + (1/2)bA^T C^T$ and $\gamma = (2ad/k) + b(d + CB)$. Assume b is chosen such that $\gamma \geq 0$.

(c) Show that the system is absolutely stable if

$$\frac{1}{k} + \operatorname{Re}[(1 + j\omega\eta)G(j\omega)] > 0, \quad \forall\ \omega \in R, \text{ where } \eta = \frac{b}{2ad}$$

7.9 (**[85]**) The feedback system of Figure 7.23 represents a control system where $H(s)$ is the (scalar) transfer function of the plant and the inner loop models the actuator. Let $H(s) = (s + 6)/(s + 2)(s + 3)$ and suppose $k \geq 0$ and ψ belongs to a sector $(0, \beta]$, where β could be arbitrarily large, but finite.

(a) Show that the system can be represented as the feedback connection of Figure 7.1 with $G(s) = [H(s) + k]/s$.

(b) Using the version of the Popov criterion in Exercise 7.8, find a lower bound k_c such that the system is absolutely stable for all $k > k_c$.

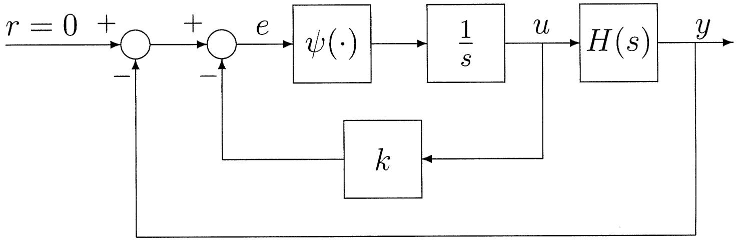

Figure 7.23: Exercise 7.9.

7.10 For each odd nonlinearity $\psi(y)$ on the following list, verify the given expression of the describing function $\Psi(a)$:

(1) $\psi(y) = y^5; \quad \Psi(a) = 5a^4/8$

(2) $\psi(y) = y^3|y|; \quad \Psi(a) = 32a^3/15\pi$

(3) $\psi(y)$: Figure 7.24(a); $\quad \Psi(a) = k + \dfrac{4A}{\pi a}$

(4) $\psi(y)$: Figure 7.24(b)

$$\Psi(a) = \begin{cases} 0, & \text{for } a \le A \\ (4B/\pi a)[1 - (A/a)^2]^{1/2}, & \text{for } a \ge A \end{cases}$$

(5) $\psi(y)$: Figure 7.24(c)

$$\Psi(a) = \begin{cases} 0, & \text{for } a \le A \\ k[1 - N(a/A)], & \text{for } A \le a \le B \\ k[N(a/B) - N(a/A)], & \text{for } a \ge B \end{cases}$$

where

$$N(x) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{x}\sqrt{1 - \left(\frac{1}{x}\right)^2}\right]$$

7.11 Using the describing function method, investigate the existence of periodic solutions and the possible frequency and amplitude of oscillation in the feedback connection of Figure 7.1 for each of the following cases:

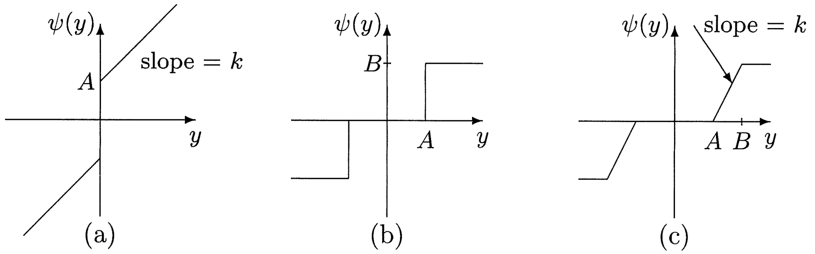

Figure 7.24: Exercise 7.10.

(1) $G(s) = (1 - s)/s(s+1)$ and $\psi(y) = y^5$.

(2) $G(s) = (1 - s)/s(s+1)$ and ψ is the nonlinearity of Exercise 7.10, part (5), with $A = 1$, $B = 3/2$, and $k = 2$.

(3) $G(s) = 1/(s+1)^6$ and $\psi(y) = \text{sgn}(y)$.

(4) $G(s) = (s+6)/s(s+2)(s+3)$ and $\psi(y) = \text{sgn}(y)$.

(5) $G(s) = s/(s^2 - s + 1)$ and $\psi(y) = y^5$.

(6) $G(s) = 5(s+0.25)/s^2(s+2)^2$ and ψ is the nonlinearity of Exercise 7.10, part (3), with $A = 1$ and $k = 2$.

(7) $G(s) = 5(s+0.25)/s^2(s+2)^2$ and ψ is the nonlinearity of Exercise 7.10, part (4), with $A = 1$ and $B = 1$.

(8) $G(s) = 5(s+0.25)/s^2(s+2)^2$ and ψ is the nonlinearity of Exercise 7.10, part (5), with $A = 1$, $B = 3/2$, and $k = 2$.

(9) $G(s) = 1/(s+1)^3$ and $\psi(y) = \text{sgn}(y)$.

(10) $G(s) = 1/(s+1)^3$ and $\psi(y) = \text{sat}(y)$.

7.12 Apply the describing function method to study the existence of periodic solutions in the negative resistance oscillator of Section 1.2.4 with $h(v) = -v + v^3 - v^5/5$ and $\varepsilon = 1$. For each possible periodic solution, estimate the frequency and amplitude of oscillation. Using computer simulation, determine how accurate the describing function results are.

7.13 Consider the feedback connection of Figure 7.1 with $G(s) = 2bs/(s^2 - bs + 1)$ and $\psi(y) = \text{sat}(y)$. Using the describing function method, show that for sufficiently small $b > 0$ the system has a periodic solution. Confirm your conclusion by applying Theorem 7.4 and estimate the frequency and amplitude of oscillation.

7.14 Consider the feedback connection of Figure 7.1 with

$$G(s) = \frac{1}{(s+1)^2(s+2)^2}, \qquad \psi(y) = \begin{cases} by & -1 \leq by \leq 1 \\ \operatorname{sgn}(y) & b|y| > 1 \end{cases}$$

where $b > 0$.

(a) Using the circle criterion, find the largest b for which we can confirm that the origin of the closed-loop system is globally asymptotically stable.

(b) Using the Popov criterion, find the largest b for which we can confirm that the origin of the closed-loop system is globally asymptotically stable.

(c) Using the describing function method, find the smallest b for which the system will oscillate and estimate the frequency of oscillation.

(d) For $b = 10$, study the existence of periodic solutions by using Theorem 7.4. For each oscillation, if any,

i. find the frequency interval $[\omega_1, \omega_2]$ and the amplitude interval $[a_1, a_2]$;

ii. use Lemma 7.1 to find an upper bound on the energy content of the higher-order harmonics and express it as a percentage of the energy content of the first harmonic; and

iii. simulate the system and compare the simulation results with the foregoing analytical results.

(e) Repeat part (d) for $b = 30$.

7.15 Repeat parts (a) to (c) of the previous exercise for $G(s) = 10/(s+1)^2(s+2)$.

7.16 Consider the feedback connection of Figure 7.1, where $G(s) = 1/(s+1)^3$ and $\psi(y)$ is the piecewise-linear function of Figure 7.15 with $\delta = 1/k$, $s_1 = k$, and $s_2 = 0$.

(a) Using the describing function method, investigate the existence of periodic solutions and the possible frequency and amplitude of oscillation when $k = 10$.

(b) Continuing with $k = 10$, apply Theorem 7.4. For each oscillation, if any, find the frequency interval $[\omega_1, \omega_2]$ and the amplitude interval $[a_1, a_2]$.

(c) What is the largest slope $k > 0$ for which Theorem 7.4 ensures that there is no oscillation?

7.17 For each of the following cases, apply Theorem 7.4 to study the existence of periodic solutions in the feedback connection of Figure 7.1. For each oscillation, if any, find the frequency interval $[\omega_1, \omega_2]$ and the amplitude interval $[a_1, a_2]$.

(1) $G(s) = 2(s-1)/s^3(s+1)$ and $\psi(y) = \text{sat}(y)$.

(2) $G(s) = -s/(s^2 + 0.8s + 8)$ and $\psi(y) = (1/2)\sin y$.

(3) $G(s) = -s/(s^2 + 0.8s + 8)$ and $\psi(y)$ is the nonlinearity of Example 7.13.

(4) $G(s) = -24/s^2(s+1)^3$ and $\psi(y)$ is an odd nonlinearity defined by

$$\psi(y) = \begin{cases} y^3 + y/2, & \text{for } 0 \le y \le 1 \\ 2y - 1/2, & \text{for } y \ge 1 \end{cases}$$

Chapter 8

Advanced Stability Analysis

In Chapter 4, we gave the basic concepts and tools of Lyapunov stability. In this chapter, we examine some of these concepts more closely and present a number of extensions and refinements.

We saw in Chapter 4 how to use linearization to study stability of equilibrium points of autonomous systems. We saw also that linearization fails when the Jacobian matrix, evaluated at the equilibrium point, has some eigenvalues with zero real parts and no eigenvalues with positive real parts. In Section 8.1, we introduce the center manifold theorem and use it to study stability of equilibrium points of autonomous systems in the critical case when linearization fails.

The concept of the region of attraction of an asymptotically stable equilibrium point was introduced in Section 4.1. In Section 8.2, we elaborate further on that concept and present some ideas for providing estimates of this region.

LaSalle's invariance principle for autonomous systems is very useful in applications. For a general nonautonomous system, there is no invariance principle in the same form that was presented in Theorem 4.4. There are, however, theorems which capture some features of the invariance principle. Two such theorems are given in Section 8.3. The first theorem shows convergence of the trajectory to a set, while the second one shows uniform asymptotic stability of the origin.

Finally, in Section 8.4, we introduce notions of stability of periodic solutions and invariant sets.

8.1 The Center Manifold Theorem

Consider the autonomous system

$$\dot{x} = f(x) \tag{8.1}$$

where $f : D \to R^n$ is continuously differentiable and $D \subset R^n$ is a domain that contains the origin $x = 0$. Suppose that the origin is an equilibrium point of (8.1).

Theorem 4.7 states that if the linearization of f at the origin, that is, the matrix

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

has all eigenvalues with negative real parts, then the origin is asymptotically stable; if it has some eigenvalues with positive real parts, then the origin is unstable. If A has some eigenvalues with zero real parts with the rest of the eigenvalues having negative real parts, then linearization fails to determine the stability properties of the origin. In this section, we take a closer look into the case when linearization fails. Failure of linearization leaves us with the task of analyzing the nth-order nonlinear system (8.1) in order to determine stability of the origin. The interesting finding that we are going to present in the next few pages is that stability properties of the origin can be determined by analyzing a lower order nonlinear system — a system whose order is exactly equal to the number of eigenvalues of A with zero real parts. This will follow as an application of the *center manifold theory.*[1]

A k-dimensional manifold in R^n $(1 \leq k < n)$ has a rigorous mathematical definition.[2] For our purpose here, it is sufficient to think of a k-dimensional manifold as the solution of the equation

$$\eta(x) = 0$$

where $\eta : R^n \to R^{n-k}$ is sufficiently smooth (that is, sufficiently many times continuously differentiable). For example, the unit circle

$$\{x \in R^2 \mid x_1^2 + x_2^2 = 1\}$$

is a one-dimensional manifold in R^2. Similarly, the unit sphere

$$\{x \in R^n \mid \sum_{i=1}^{n} x_i^2 = 1\}$$

is an $(n-1)$-dimensional manifold in R^n. A manifold $\{\eta(x) = 0\}$ is said to be an invariant manifold for (8.1) if

$$\eta(x(0)) = 0 \Rightarrow \eta(x(t)) \equiv 0, \quad \forall\ t \in [0, t_1) \subset R$$

where $[0, t_1)$ is any time interval over which the solution $x(t)$ is defined.

Suppose now that $f(x)$ is twice continuously differentiable. Equation (8.1) can be represented as

$$\dot{x} = Ax + \left[f(x) - \frac{\partial f}{\partial x}(0)\ x \right] = Ax + \tilde{f}(x)$$

---

[1] The center manifold theory has several applications to dynamical systems. It is presented here only insofar as it relates to determining the stability of the origin. For a broader viewpoint of the theory, the reader may consult [34].

[2] See, for example, [71].

where

$$\tilde{f}(x) = f(x) - \frac{\partial f}{\partial x}(0)\; x$$

is twice continuously differentiable and

$$\tilde{f}(0) = 0; \;\; \frac{\partial \tilde{f}}{\partial x}(0) = 0$$

Since our interest is in the case when linearization fails, assume that A has k eigenvalues with zero real parts and $m = n - k$ eigenvalues with negative real parts. We can always find a similarity transformation T that transforms A into a block diagonal matrix, that is,

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where all eigenvalues of A_1 have zero real parts and all eigenvalues of A_2 have negative real parts. Clearly, A_1 is $k \times k$ and A_2 is $m \times m$. The change of variables

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx; \;\; y \in R^k; \;\; z \in R^m$$

transforms (8.1) into the form

$$\dot{y} = A_1 y + g_1(y, z) \tag{8.2}$$

$$\dot{z} = A_2 z + g_2(y, z) \tag{8.3}$$

where g_1 and g_2 inherit properties of $\tilde{f}$. In particular, they are twice continuously differentiable and

$$g_i(0,0) = 0; \;\; \frac{\partial g_i}{\partial y}(0,0) = 0; \;\; \frac{\partial g_i}{\partial z}(0,0) = 0 \tag{8.4}$$

for $i = 1, 2$. If $z = h(y)$ is an invariant manifold for (8.2)–(8.3) and h is smooth, then it is called a *center manifold* if

$$h(0) = 0; \;\; \frac{\partial h}{\partial y}(0) = 0$$

Theorem 8.1 *If g_1 and g_2 are twice continuously differentiable and satisfy* (8.4), *all eigenvalues of A_1 have zero real parts, and all eigenvalues of A_2 have negative real parts, then there exist a constant $\delta > 0$ and a continuously differentiable function $h(y)$, defined for all $\|y\| < \delta$, such that $z = h(y)$ is a center manifold for* (8.2)–(8.3).
◇

Proof: See Appendix C.15.

If the initial state of the system (8.2)–(8.3) lies in the center manifold; that is, $z(0) = h(y(0))$, then the solution $(y(t), z(t))$ will lie in the manifold for all $t \geq 0$;

that is, $z(t) \equiv h(y(t))$. In this case, the motion of the system in the center manifold is described by the kth-order differential equation

$$\dot{y} = A_1 y + g_1(y, h(y)) \tag{8.5}$$

which we refer to as the *reduced system*. If $z(0) \neq h(y(0))$, then the difference $z(t) - h(y(t))$ represents the deviation of the trajectory from the center manifold at any time t. The change of variables

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} y \\ z - h(y) \end{bmatrix}$$

transforms (8.2)–(8.3) into

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, w + h(y)) && (8.6) \\ \dot{w} &= A_2[w + h(y)] + g_2(y, w + h(y)) - \frac{\partial h}{\partial y}(y)\ [A_1 y + g_1(y, w + h(y))] && (8.7) \end{aligned}$$

In the new coordinates, the center manifold is $w = 0$. The motion in the manifold is characterized by

$$w(t) \equiv 0 \ \Rightarrow \ \dot{w}(t) \equiv 0$$

Substituting these identities into (8.7) results in

$$0 = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}(y)\ [A_1 y + g_1(y, h(y))] \tag{8.8}$$

Since the equation must be satisfied by any solution that lies in the center manifold, we conclude that the function $h(y)$ must satisfy the partial differential equation (8.8). Adding and subtracting $g_1(y, h(y))$ to the right-hand side of (8.6), and subtracting (8.8) from (8.7), we can rewrite the equation in the transformed coordinates as

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, h(y)) + N_1(y, w) && (8.9) \\ \dot{w} &= A_2 w + N_2(y, w) && (8.10) \end{aligned}$$

where

$$N_1(y, w) = g_1(y, w + h(y)) - g_1(y, h(y))$$

and

$$N_2(y, w) = g_2(y, w + h(y)) - g_2(y, h(y)) - \frac{\partial h}{\partial y}(y)\ N_1(y, w)$$

It is not difficult to verify that N_1 and N_2 are twice continuously differentiable, and

$$N_i(y, 0) = 0; \quad \frac{\partial N_i}{\partial w}(0, 0) = 0$$

for $i = 1, 2$. Consequently, in the domain

$$\left\| \begin{array}{c} y \\ w \end{array} \right\|_2 < \rho$$

N_1 and N_2 satisfy

$$\|N_i(y, w)\|_2 \le k_i \|w\|, \quad i = 1, 2$$

where the positive constants k_1 and k_2 can be made arbitrarily small by choosing ρ small enough. These inequalities, together with the fact that A_2 is Hurwitz, suggest that the stability properties of the origin are determined by the reduced system (8.5). The next theorem, known as the *reduction principle*, confirms this conjecture.

Theorem 8.2 *Under the assumptions of Theorem* 8.1, *if the origin* $y = 0$ *of the reduced system* (8.5) *is asymptotically stable (respectively, unstable) then the origin of the full system* (8.2) and (8.3) *is also asymptotically stable (respectively, unstable).* ◇

Proof: The change of coordinates from (y, z) to (y, w) does not change the stability properties of the origin (Exercise 4.26); therefore, we can work with the system (8.9) and (8.10). If the origin of the reduced system (8.5) is unstable, then, by invariance, the origin of (8.9) and (8.10) is unstable. This is so because for any solution $y(t)$ of (8.5), there is a corresponding solution $(y(t), 0)$ of (8.9) and (8.10). Suppose now that the origin of the reduced system (8.5) is asymptotically stable. By (the converse Lyapunov) Theorem 4.16, there is a continuously differentiable function $V(y)$ that is positive definite and satisfies the inequalities

$$\frac{\partial V}{\partial y}[A_1 y + g_1(y, h(y))] \le -\alpha_3(\|y\|_2)$$

$$\left\| \frac{\partial V}{\partial y} \right\|_2 \le \alpha_4(\|y\|_2) \le k$$

in a neighborhood of the origin, where α_3 and α_4 are class $\mathcal{K}$ functions. On the other hand, since A_2 is Hurwitz, the Lyapunov equation

$$PA_2 + A_2^T P = -I$$

has a unique positive definite solution P. Consider

$$\nu(y, w) = V(y) + \sqrt{w^T P w}$$

as a Lyapunov function candidate[3] for the full system (8.9) and (8.10). The derivative of ν along the trajectories of the system is given by

$$\dot{\nu}(y, w) \;=\; \frac{\partial V}{\partial y}\;[A_1 y + g_1(y, h(y)) + N_1(y, w)]$$

---

[3] The function $\nu(y, w)$ is continuously differentiable everywhere around the origin, except on the manifold $w = 0$. Both $\nu(y, w)$ and $\dot{\nu}(y, w)$ are defined and continuous around the origin. It can be easily seen that the statement of Theorem 4.1 is still valid.

$$\begin{aligned}
&+ \frac{1}{2\sqrt{w^TPw}}\,[w^T(PA_2 + A_2^TP)w + 2w^TPN_2(y,w)] \\
\le\;& -\alpha_3(\|y\|_2) + kk_1\|w\|_2 - \frac{\|w\|_2}{2\sqrt{\lambda_{\max}(P)}} + \frac{k_2\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\|w\|_2 \\
=\;& -\alpha_3(\|y\|_2) - \frac{1}{4\sqrt{\lambda_{\max}(P)}}\|w\|_2 \\
&- \left[\frac{1}{4\sqrt{\lambda_{\max}(P)}} - kk_1 - k_2\frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\right]\|w\|_2
\end{aligned}$$

Since k_1 and k_2 can be made arbitrarily small by restricting the domain around the origin to be sufficiently small, we can choose them small enough to ensure that

$$\frac{1}{4\sqrt{\lambda_{\max}(P)}} - kk_1 - k_2\frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} > 0$$

Hence,

$$\dot{\nu}(y,w) \le -\alpha_3(\|y\|_2) - \frac{1}{4\sqrt{\lambda_{\max}(P)}}\|w\|_2$$

which shows that $\dot{\nu}(y,w)$ is negative definite. Consequently, the origin of the full system (8.9)–(8.10) is asymptotically stable. □

We leave it to the reader (Exercises 8.1 and 8.2) to extend the proof of Theorem 8.2 to prove the next two corollaries.

Corollary 8.1 *Under the assumptions of Theorem* 8.1, *if the origin* $y = 0$ *of the reduced system* (8.5) *is stable and there is a continuously differentiable Lyapunov function* $V(y)$ *such that*[4]

$$\frac{\partial V}{\partial y}\,[A_1y + g_1(y, h(y))] \le 0$$

in some neighborhood of $y = 0$, *then the origin of the full system* (8.2)–(8.3) *is stable.* ◇

Corollary 8.2 *Under the assumptions of Theorem* 8.1, *the origin of the reduced system* (8.5) *is asymptotically stable if and only if the origin of the full system* (8.2)–(8.3) *is asymptotically stable.* ◇

---

[4] The existence of the Lyapunov function $V(y)$ cannot be inferred from a converse Lyapunov theorem. The converse Lyapunov theorem for stability [72, 107] guarantees the existence of a Lyapunov function $V(t,y)$ whose derivative satisfies $\dot{V}(t,y) \le 0$. In general, this function cannot be made independent of t. (See [72, page 228].) Even though we can choose $V(t,y)$ to be continuously differentiable in its arguments, it cannot be guaranteed that the partial derivatives $\partial V/\partial y_i$, $\partial V/\partial t$ will be uniformly bounded in a neighborhood of the origin for all $t \ge 0$. (See [107, page 53].)

To use Theorem 8.2, we need to find the center manifold $z = h(y)$. The function h is a solution of the partial differential equation

$$\mathcal{N}(h(y)) \stackrel{\text{def}}{=} \frac{\partial h}{\partial y}(y)\ [A_1 y + g_1(y, h(y))] - A_2 h(y) - g_2(y, h(y)) = 0 \tag{8.11}$$

with boundary conditions

$$h(0) = 0; \quad \frac{\partial h}{\partial y}(0) = 0 \tag{8.12}$$

This equation for h cannot be solved exactly in most cases (to do so would imply that a solution of the full system (8.2)–(8.3) has been found), but its solution can be approximated arbitrarily closely as a Taylor series in y.

Theorem 8.3 *If a continuously differentiable function* $\phi(y)$ *with* $\phi(0) = 0$ *and* $[\partial\phi/\partial y](0) = 0$ *can be found such that* $\mathcal{N}(\phi(y)) = O(\|y\|^p)$ *for some* $p > 1$*, then for sufficiently small* $\|y\|$

$$h(y) - \phi(y) = O(\|y\|^p)$$

and the reduced system can be represented as

$$\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^{p+1})$$

◇

Proof: See Appendix C.15.

The order of magnitude notation $O(\cdot)$ will be formally introduced in Chapter 10 (Definition 10.1). For our purpose here, it is enough to think of $f(y) = O(\|y\|^p)$ as a shorthand notation for $\|f(y)\| \leq k\|y\|^p$ for sufficiently small $\|y\|$. Let us now illustrate the application of the center manifold theorem by examples. In the first two examples, we will make use of the observation that for a scalar state equation of the form

$$\dot{y} = a y^p + O\left(|y|^{p+1}\right)$$

where p is a positive integer, the origin is asymptotically stable if p is odd and $a < 0$. It is unstable if p is odd and $a > 0$, or p is even and $a \neq 0$.[5]

Example 8.1 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + a x_1^2 + b x_1 x_2 \end{aligned}$$

where $a \neq 0$. The system has a unique equilibrium point at the origin. The linearization at the origin results in the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

---

[5] See Exercise 4.2.

which has eigenvalues at 0 and -1. Let M be a matrix whose columns are the eigenvectors of A; that is,

$$M = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

and take $T = M^{-1}$. Then,

$$TAT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

The change of variables

$$\begin{bmatrix} y \\ z \end{bmatrix} = T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_2 \end{bmatrix}$$

puts the system into the form

$$\begin{aligned} \dot{y} &= a(y+z)^2 - b(yz + z^2) \\ \dot{z} &= -z - a(y+z)^2 + b(yz+z^2) \end{aligned}$$

The center manifold equation (8.11) with the boundary condition (8.12) becomes

$$\begin{aligned} \mathcal{N}(h(y)) &= h'(y)[a(y+h(y))^2 - b(yh(y) + h^2(y))] + h(y) \\ &\quad + a(y+h(y))^2 - b(yh(y)+h^2(y)) = 0, \quad h(0) = h'(0) = 0 \end{aligned}$$

We set $h(y) = h_2y^2 + h_3y^3 + \cdots$ and substitute this series in the center manifold equation to find the unknown coefficients $h_2, h_3, \ldots$ by matching coefficients of like powers in y (since the equation holds as an identity in y). We do not know in advance how many terms of the series we need. We start with the simplest approximation $h(y) \approx 0$. We substitute $h(y) = O(|y|^2)$ into the reduced system and study stability of its origin. If the stability properties of the origin can be determined, we are done. Otherwise, we calculate the coefficient h_2, substitute $h(y) = h_2y^2 + O(|y|^3)$, and study stability of the origin. If it cannot be resolved, we proceed to the approximation $h(y) \approx h_2y^2 + h_3y^3$, and so on. Let us start with the approximation $h(y) \approx 0$. The reduced system is

$$\dot{y} = ay^2 + O(|y|^3)$$

Notice that an $O(|y|^2)$ error in $h(y)$ results in an $O(|y|^3)$ error in the right-hand side of the reduced system. This is a consequence of the fact that the function $g_1(y,z)$, which appears on the right-hand side of the reduced system (8.5) as $g_1(y, h(y))$, has a partial derivative with respect to z that vanishes at the origin. Clearly, this observation is also valid for higher order approximations; that is, an error of order $O(|y|^k)$ in $h(y)$ results in an error of order $O(|y|^{k+1})$ in $g_1(y, h(y))$, for $k \geq 2$. The term ay^2 is the dominant term on the right-hand side of the reduced system. For $a \neq 0$, the origin of the reduced system is unstable. Consequently, by Theorem 8.2, the origin of the full system is unstable. △

Example 8.2 Consider the system

$$\begin{aligned}\dot{y} &= yz\\ \dot{z} &= -z + ay^2\end{aligned}$$

which is already represented in the (y, z) coordinates. The center manifold equation (8.11) with the boundary condition (8.12) is

$$h'(y)[yh(y)] + h(y) - ay^2 = 0, \quad h(0) = h'(0) = 0$$

We start by trying $\phi(y) = 0$. The reduced system is

$$\dot{y} = O(|y|^3)$$

Clearly, we cannot reach any conclusion about the stability of the origin. Therefore, we substitute $h(y) = h_2 y^2 + O(|y|^3)$ into the center manifold equation and calculate h_2, by matching coefficients of y^2, to obtain $h_2 = a$. The reduced system is[6]

$$\dot{y} = ay^3 + O(|y|^4)$$

Therefore, the origin is asymptotically stable if $a < 0$ and unstable if $a > 0$. Consequently, by Theorem 8.2, we conclude that the origin of the full system is asymptotically stable if $a < 0$ and unstable if $a > 0$. If $a = 0$, the center manifold equation (8.11) with the boundary condition (8.12) reduces to

$$h'(y)[yh(y)] + h(y) = 0, \quad h(0) = h'(0) = 0$$

which has the exact solution $h(y) = 0$. The reduced system $\dot{y} = 0$ has a stable origin with $V(y) = y^2$ as a Lyapunov function. Therefore, by Corollary 8.1, we conclude that the origin of the full system is stable if $a = 0$ △

Example 8.3 Consider the system (8.2)–(8.3) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; g_1 = \begin{bmatrix} -y_1^3 \\ -y_2^3 + z^2 \end{bmatrix}, \; A_2 = -1, \text{ and } g_2 = y_1^3 - 3y_1^5 + 3y_1^2 y_2$$

It can be verified that $\phi(y) = 0$ results in $\mathcal{N}(\phi(y)) = O\left(\|y\|_2^3\right)$ and

$$\dot{y} = \begin{bmatrix} -y_1^3 + y_2 \\ -y_1 - y_2^3 \end{bmatrix} + O\left(\|y\|_2^4\right)$$

Using $V(y) = (y_1^2 + y_2^2)/2$ as a Lyapunov function candidate, we obtain

$$\dot{V} = -y_1^4 - y_2^4 + y^T O\left(\|y\|_2^4\right) \le -\|y\|_2^4 + k\|y\|_2^5$$

---

[6] The error on the right-hand side of the reduced system is actually $O(|y|^5)$ since, if we write $h(y) = h_2 y^2 + h_3 y^3 + \cdots$, we will find that $h_3 = 0$.

in some neighborhood of the origin where $k > 0$. Hence,

$$\dot{V} \leq -\tfrac{1}{2}\|y\|_2^4, \quad \text{for } \|y\|_2 < \frac{1}{2k}$$

which shows that the origin of the reduced system is asymptotically stable. Consequently, the origin of the full system is asymptotically stable. △

Notice that in the preceding example it is not enough to study the system

$$\dot{y} = \begin{bmatrix} -y_1^3 + y_2 \\ -y_1 - y_2^3 \end{bmatrix}$$

We have to find a Lyapunov function that confirms asymptotic stability of the origin for all perturbations of the order $O\left(\|y\|_2^4\right)$. The importance of this observation is illustrated by the next example.

Example 8.4 Consider the previous example, but change A_1 to

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

With $\phi(y) = 0$, the reduced system can be represented as

$$\dot{y} = \begin{bmatrix} -y_1^3 + y_2 \\ -y_2^3 \end{bmatrix} + O\left(\|y\|_2^4\right)$$

Without the perturbation term $O\left(\|y\|_2^4\right)$, the origin of this system is asymptotically stable.[7] If you try to find a Lyapunov function $V(y)$ to show asymptotic stability in the presence of the perturbation term, you will not succeed. In fact, it can be verified that the center manifold equation (8.11) with the boundary condition (8.12) has the exact solution $h(y) = y_1^3$, so that the reduced system is given by the equation

$$\dot{y} = \begin{bmatrix} -y_1^3 + y_2 \\ y_1^6 - y_2^3 \end{bmatrix}$$

whose origin is unstable.[8] △

8.2 Region of Attraction

Quite often, it is not sufficient to determine that a given system has an asymptotically stable equilibrium point. Rather, it is important to find the region of attraction of that point, or at least an estimate of it. To appreciate the importance of determining the region of attraction, let us run a scenario of events that could

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[7] See Exercise 4.56.

[8] See Exercise 4.13.

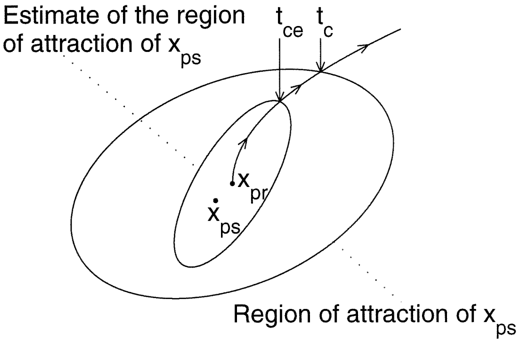
Estimate of the region of attraction of x_{ps}
t_{ce}
t_c
x_{pr}
x_{ps}
Region of attraction of x_{ps}

Figure 8.1: Critical clearance time.

happen in the operation of a nonlinear system. Suppose that a nonlinear system has an asymptotically stable equilibrium point, which is denoted by x_{pr} in Suppose the system is operating at steady state at x_{pr}. Then, at time t_0 a fault that changes the structure of the system takes place, for example, a short circuit in an electrical network. Suppose the faulted system does not have equilibrium points at x_{pr} or in its neighborhood. The trajectory of the system will be driven away from x_{pr}. Suppose further that the fault is cleared at time t_1 and the postfault system has an asymptotically stable equilibrium point at x_{ps}, where either $x_{ps} = x_{pr}$ or x_{ps} is sufficiently close to x_{pr} so that steady-state operation at x_{ps} is still acceptable. At time t_1 the state of the system, say, $x(t_1)$, could be far from the postfault equilibrium x_{ps}. Whether or not the system will return to steady-state operation at x_{ps} depends on whether $x(t_1)$ belongs to the region of attraction of x_{ps}, as determined by the postfault system equation. A crucial factor in determining how far $x(t_1)$ could be from x_{ps} is the time it takes the operators of the system to remove the fault, that is, the time difference $(t_1 - t_0)$. If $(t_1 - t_0)$ is very short, then, by continuity of the solution with respect to t, it is very likely that $x(t_1)$ will be in the region of attraction of x_{ps}. However, operators need time to detect the fault and fix it. How much time they have is a critical question. In planning such a system, it is valuable to give operators a "critical clearance time," say t_c, such that they have to clear the fault within this time; that is, $(t_1 - t_0)$ must be less than t_c. If we know the region of attraction of x_{ps}, we can find t_c by Figure 8.1. integrating the faulted system equation starting from the prefault equilibrium x_{pr} until the trajectory hits the boundary of the region of attraction. The time it takes the trajectory to reach the boundary can be taken as the critical clearance time because if the fault is cleared before that time the state $x(t_1)$ will be within the region of attraction. Of course, we are assuming that x_{pr} belongs to the region of attraction of x_{ps}, which is reasonable. If the actual region of attraction is not known, and an estimate t_{ce} of t_c is obtained by using an estimate of the region of attraction, then $t_{ce} < t_c$, since the boundary of the estimate of the region of attraction will be inside the actual boundary of the region. (See Figure 8.1.) This scenario shows an example where finding the region of attraction is needed in planning the operation of a nonlinear system. It also shows the importance of finding estimates of the region of attraction that are not too conservative. A very conservative estimate of the region of attraction would result in t_{ce} that is too small to be useful. Let us conclude this motivating discussion by saying that the scenario of events described here is not hypothetical. It is the essence of the transient stability problem in power systems.[9]

Let the origin $x = 0$ be an asymptotically stable equilibrium point for the nonlinear system

$$\dot{x} = f(x) \tag{8.13}$$

where $f : D \to R^n$ is locally Lipschitz and $D \subset R^n$ is a domain containing the origin. Let $\phi(t; x)$ be the solution of (8.13) that starts at initial state x at time $t = 0$. The region of attraction of the origin, denoted by R_A, is defined by

$$R_A = \{x \in D \mid \phi(t; x) \text{ is defined } \forall\ t \geq 0 \text{ and } \phi(t; x) \to 0 \text{ as } t \to \infty\}$$

Some properties of the region of attraction are stated in the next lemma, whose proof is given in Appendix C.16.

Lemma 8.1 *If* $x = 0$ *is an asymptotically stable equilibrium point for* (8.13)*, then its region of attraction* R_A *is an open, connected, invariant set. Moreover, the boundary of* R_A *is formed by trajectories.* ◇

Lemma 8.1 suggests that one way to determine the region of attraction is to characterize those trajectories that lie on the boundary of R_A. There are some methods that approach the problem from this viewpoint, but they use geometric notions from the theory of dynamical systems that are not introduced in this book. Therefore, we will not describe this class of methods.[10] We may, however, get a flavor of these geometric methods in the case of second-order systems ($n = 2$) by employing phase portraits. Examples 8.5 and 8.6 show typical cases in the state plane. In the first example, the boundary of the region of attraction is a limit cycle, while in the second one the boundary is formed of stable trajectories of saddle points. Example 8.7 shows a rather pathological case where the boundary is a closed curve of equilibrium points.

Example 8.5 The second-order system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned}$$

is a Van der Pol equation in reverse time, that is, with t replaced by $-t$. The system has one equilibrium point at the origin and one unstable limit cycle, as determined from the phase portrait shown in Figure 8.2. The phase portrait shows that the

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[9] See [170] for an introduction to the transient stability problem in power systems.

[10] Examples of these methods can be found in [36] and [216].

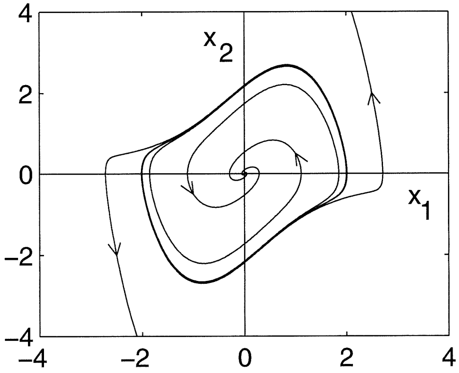

Figure 8.2: Phase portrait for Example 8.5.

origin is a stable focus; hence, it is asymptotically stable. This can be confirmed by linearization, since

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues at $-1/2 \pm j\sqrt{3}/2$. Clearly, the region of attraction is bounded because trajectories starting outside the limit cycle cannot cross it to reach the origin. Because there are no other equilibrium points, the boundary of R_A must be the limit cycle. Inspection of the phase portrait shows that indeed all trajectories starting inside the limit cycle spiral toward the origin. △

Example 8.6 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \tfrac{1}{3}x_1^3 - x_2 \end{aligned}$$

This system has three isolated equilibrium points at $(0,0)$, $(\sqrt{3},0)$, and $(-\sqrt{3},0)$. The phase portrait of the system is shown in Figure 8.3. The phase portrait shows that the origin is a stable focus, and the other two equilibria are saddle points. Thus, the origin is asymptotically stable and the other equilibria are unstable; a fact that can be confirmed by linearization. From the phase portrait, we can also see that the stable trajectories of the saddle points form two separatrices that are the boundaries of the region of attraction. The region is unbounded. △

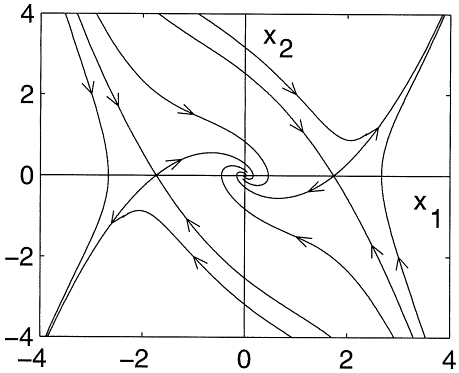

Figure 8.3: Phase portrait for Example 8.6.

Example 8.7 The system

$$\begin{aligned}\dot{x}_1 &= -x_1(1-x_1^2-x_2^2)\\ \dot{x}_2 &= -x_2(1-x_1^2-x_2^2)\end{aligned}$$

has an isolated equilibrium point at the origin and a continuum of equilibrium points on the unit circle; that is, every point on the unit circle is an equilibrium point. Clearly, R_A must be confined to the interior of the unit circle. The trajectories of the system are the radii of the unit circle. This can be seen by transforming the system into polar coordinates. The change of variables

$$x_1 = \rho\cos\theta, \quad x_2 = \rho\sin\theta$$

yields

$$\dot{\rho} = -\rho(1-\rho^2), \quad \dot{\theta} = 0$$

All trajectories starting with $\rho < 1$ approach the origin as $t \to \infty$. Therefore, R_A is the interior of the unit circle. △

Lyapunov's method can be used to find the region of attraction R_A or an estimate of it. The basic tool for finding the boundary of R_A is Zubov's theorem, which is given in Exercise 8.10. The theorem, however, has the character of an existence theorem and requires the solution of a partial differential equation. Via much simpler procedures, we can find estimates of R_A by using Lyapunov's method. By an estimate of R_A, we mean a set $\Omega \subset R_A$ such that every trajectory starting in Ω approaches the origin as $t \to \infty$. For the rest of this section, we will discuss some aspects of estimating R_A. Let us start by showing that the domain D of Theorem 4.1 (or Corollary 4.1) is not an estimate of R_A. We have seen in Theorem 4.1 and Corollary 4.1 that if D is a domain that contains the origin, and if we can find a Lyapunov function $V(x)$ that is positive definite in D and $\dot{V}(x)$ is negative definite in D or negative semidefinite, but no solution can stay identically in the set $\{\dot{V}(x) = 0\}$ except for the zero solution $x = 0$, then the origin is asymptotically stable. One may jump to the conclusion that D is an estimate of R_A. This conjecture is not true, as illustrated by the next example.

Example 8.8 Consider again the system of Example 8.6:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \tfrac{1}{3}x_1^3 - x_2 \end{aligned}$$

This system is a special case of that of Example 4.5 with

$$h(x_1) = x_1 - \tfrac{1}{3}x_1^3 \quad \text{and} \quad a = 1$$

Therefore, a Lyapunov function is given by

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \int_0^{x_1} (y - \tfrac{1}{3}y^3)\, dy \\ &= \tfrac{3}{4}x_1^2 - \tfrac{1}{12}x_1^4 + \tfrac{1}{2}x_1x_2 + \tfrac{1}{2}x_2^2 \end{aligned}$$

and

$$\dot{V}(x) = -\,\tfrac{1}{2}x_1^2(1 - \tfrac{1}{3}x_1^2) - \tfrac{1}{2}x_2^2$$

Defining a domain D by

$$D = \{x \in R^2 \mid -\sqrt{3} < x_1 < \sqrt{3}\}$$

it can be easily seen that $V(x) > 0$ and $\dot{V}(x) < 0$ in $D - \{0\}$. Inspecting the phase portrait in Figure 8.3 shows that D is not a subset of R_A. △

In view of this example, it is not difficult to see why D of Theorem 4.1 or Corollary 4.1 is not an estimate of R_A. Even though a trajectory starting in D will move from one Lyapunov surface $V(x) = c_1$ to an inner Lyapunov surface $V(x) = c_2$, with $c_2 < c_1$, there is no guarantee that the trajectory will remain forever in D. Once the trajectory leaves D, there is no guarantee that $\dot{V}(x)$ will be negative. Consequently, the whole argument about $V(x)$ decreasing to zero falls apart. This problem does not arise when R_A is estimated by a compact positively invariant subset of D; that is, a compact set $\Omega \subset D$ such that every trajectory starting in Ω stays in Ω for all future time. Theorem 4.4 shows that Ω is a subset of R_A. The simplest such estimate is the set[11]

$$\Omega_c = \{x \in R^n \mid V(x) \le c\}$$

---

[11] The set $\{V(x) \le c\}$ may have more than one component, but there can be only one bounded component in D, and that is the component we work with. For example, if $V(x) = x^2/(1 + x^4)$ and $D = \{|x| < 1\}$, the set $\{V(x) \le 1/4\}$ has two components: $\left\{|x| \le \sqrt{2 - \sqrt{3}}\right\}$ and $\left\{|x| \ge \sqrt{2 + \sqrt{3}}\right\}$. We work with $\left\{|x| \le \sqrt{2 - \sqrt{3}}\right\}$.

when Ω_c is bounded and contained in D. For a quadratic Lyapunov function $V(x) = x^TPx$ and $D = \{\|x\|_2 < r\}$, we can ensure that $\Omega_c \subset D$ by choosing

$$c < \min_{\|x\|_2=r} x^TPx = \lambda_{\min}(P)r^2$$

For $D = \{|b^Tx| < r\}$, where $b \in R^n$,[12]

$$\min_{|b^Tx|=r} x^TPx = \frac{r^2}{b^TP^{-1}b}$$

Therefore, $\{x^TPx \le c\}$ will be a subset of $D = \{|b_i^Tx| < r_i,\ i = 1, \dots p\}$, if we choose

$$c < \min_{1 \le i \le p} \frac{r_i^2}{b_i^TP^{-1}b_i}$$

The simplicity of estimating the region of attraction by $\Omega_c = \{x^TPx \le c\}$ has increased significance in view of the linearization results of Section 4.3. There, we saw that if the Jacobian matrix

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

is Hurwitz, then we can always find a quadratic Lyapunov function $V(x) = x^TPx$ by solving the Lyapunov equation $PA + A^TP = -Q$ for any positive definite matrix Q. Putting the pieces together, we see that *whenever A is Hurwitz, we can estimate the region of attraction of the origin.* This is illustrated by the next example.

Example 8.9 The second-order system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned}$$

was treated in Example 8.5. There, we saw that the origin is asymptotically stable since

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

is Hurwitz. A Lyapunov function for the system can be found by taking $Q = I$ and solving the Lyapunov equation

$$PA + A^TP = -I$$

---

[12] Following [122, Section 10.3], the Lagrangian associated with the constrained optimization problem is $\mathcal{L}(x, \lambda) = x^TPx + \lambda[(b^Tx)^2 - r^2]$. The first-order necessary conditions are $2Px + 2\lambda(b^Tx)b = 0$ and $(b^Tx)^2 - r^2 = 0$. It can be verified that the solutions $\lambda = -1/(b^TP^{-1}b)$ and $x = \pm rP^{-1}b/(b^TP^{-1}b)$ yield the minimal value $r^2/(b^TP^{-1}b)$.

for P. The unique solution is the positive definite matrix

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

The quadratic function $V(x) = x^T Px$ is a Lyapunov function for the system in a certain neighborhood of the origin. Because our interest here is in estimating the region of attraction, we need to determine a domain D about the origin where $\dot{V}(x)$ is negative definite and a constant $c > 0$ such that $\Omega_c = \{V(x) \leq c\}$ is a subset of D. We are interested in the largest set Ω_c that we can determine, that is, the largest value for the constant c. Notice that we do not have to worry about checking positive definiteness of $V(x)$ in D because $V(x)$ is positive definite for all x. Moreover, $V(x)$ is radially unbounded; hence Ω_c is bounded for any $c > 0$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2)$$

The right-hand side of $\dot{V}(x)$ is written as the sum of two terms. The first term, $-\|x\|_2^2$, is the contribution of the linear part Ax, while the second term is the contribution of the nonlinear term $g(x) = f(x) - Ax$. Since

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \ \text{ as } \ \|x\|_2 \to 0$$

we know that there is an open ball $D = \{x \in R^2 \mid \|x\|_2 < r\}$ such that $\dot{V}(x)$ is negative definite in D. Once we find such a ball, we can find $\Omega_c \subset D$ by choosing

$$c < \min_{\|x\|_2 = r} V(x) = \lambda_{\min}(P) r^2$$

Thus, to enlarge the estimate of the region of attraction, we need to find the largest ball on which $\dot{V}(x)$ is negative definite. We have

$$\dot{V}(x) \leq -\|x\|_2^2 + |x_1|\ |x_1 x_2|\ |x_1 - 2x_2| \leq -\|x\|_2^2 + \frac{\sqrt{5}}{2}\|x\|_2^4$$

where we used $|x_1| \leq \|x\|_2$, $|x_1 x_2| \leq \|x\|_2^2/2$, and $|x_1 - 2x_2| \leq \sqrt{5}\|x\|_2$. Thus, $\dot{V}(x)$ is negative definite on a ball D of radius given by $r^2 = 2/\sqrt{5} = 0.8944$. In this second-order example, a less conservative estimate of Ω_c can be found by searching for the ball D in polar coordinates. Taking

$$x_1 = \rho \cos\theta, \quad x_2 = \rho \sin\theta$$

we get

$$\begin{aligned} \dot{V} &= -\rho^2 + \rho^4 \cos^2\theta \sin\theta (2\sin\theta - \cos\theta) \\ &\leq -\rho^2 + \rho^4 |\cos^2\theta \sin\theta| \cdot |2\sin\theta - \cos\theta| \\ &\leq -\rho^2 + \rho^4 \times 0.3849 \times 2.2361 \\ &\leq -\rho^2 + 0.861\rho^4 \ < 0, \ \text{ for } \ \rho^2 < \frac{1}{0.861} \end{aligned}$$

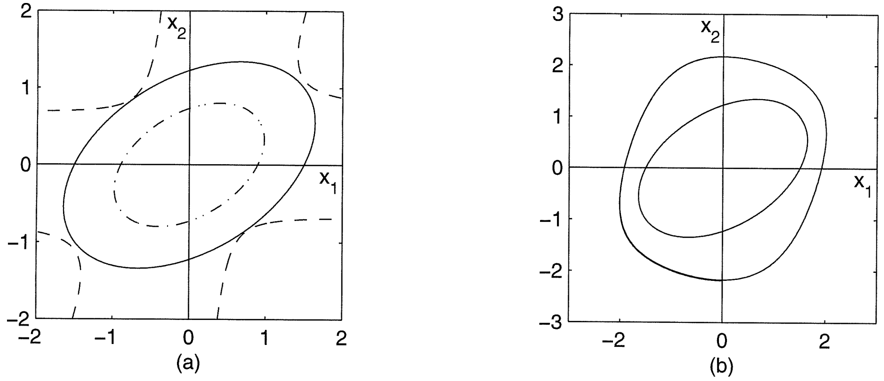

Figure 8.4: (a) Contours of $\dot{V}(x) = 0$ (dashed), $V(x) = 0.8$ (dash-dot), and $V(x) = 2.25$ (solid) for Example 8.9; (b) comparison of the region of attraction with its estimate.

Using this last equation, together with $\lambda_{\min}(P) \geq 0.69$, we choose

$$c = 0.8 < \frac{0.69}{0.861} = 0.801$$

The set Ω_c with $c = 0.8$ is an estimate of the region of attraction. A less conservative (that is, larger) estimate can be obtained by plotting contours of $\dot{V}(x) = 0$ and $V(x) = c$ for increasing values of c until we determine the largest c for which $V(x) = c$ will be in $\{\dot{V}(x) < 0\}$. This is shown in Figure 8.4(a) where c is determined to be $c = 2.25$. Figure 8.4(b) compares this estimate with the region of attraction whose boundary is a limit cycle. △

Estimating the region of attraction by $\Omega_c = \{V(x) \leq c\}$ is simple, but usually conservative. According to LaSalle's theorem (Theorem 4.4), we can work with any compact set $\Omega \subset D$ provided we can show that Ω is positively invariant. It typically requires investigating the vector field at the boundary of Ω to ensure that trajectories starting in Ω cannot leave it. The next example illustrates this idea.

Example 8.10 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

where $h : R \to R$ is a locally Lipschitz function that satisfies

$$h(0) = 0; \quad uh(u) \geq 0, \ \forall \ |u| \leq 1$$

Consider the quadratic function

$$V(x) = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

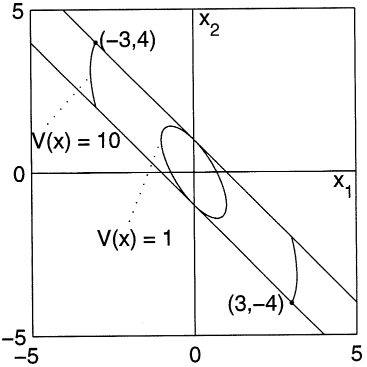

Figure 8.5: Estimates of the region of attraction for Example 8.10.

as a Lyapunov function candidate.[13] The derivative $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\le -2x_1^2 - 6(x_1 + x_2)^2, \;\; \forall\; |x_1 + x_2| \le 1 \\ &= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x\end{aligned}$$

Therefore, $\dot{V}(x)$ is negative definite in the set

$$G = \{x \in R^2 \mid |x_1 + x_2| \le 1\}$$

and we can conclude that the origin is asymptotically stable. To estimate R_A, let us start by an estimate of the form $\Omega_c = \{V(x) \le c\}$. The largest $c > 0$ for which $\Omega_c \subset G$ is given by

$$c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1$$

where $b^T = [1 \;\; 1]$. Hence, Ω_c with $c = 1$ is an estimate of R_A. (See Figure 8.5.) In this example, we can obtain a better estimate of R_A by not restricting ourselves to estimates of the form Ω_c. A key point in the development is to observe that trajectories inside G cannot leave through certain segments of the boundary $|x_1 + x_2| = 1$. This can be seen by examining the vector field at the boundary or by the following analysis: Let

$$\sigma = x_1 + x_2$$

---

[13] This Lyapunov function candidate can be derived by using the variable gradient method or by applying the circle criterion and the Kalman–Yakubovich–Popov lemma.

such that the boundary of G is given by $\sigma = 1$ and $\sigma = -1$. The derivative of σ^2 along the trajectories of the system is given by

$$\frac{d}{dt}\sigma^2 = 2\sigma(\dot{x}_1 + \dot{x}_2) = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \le 2\sigma x_2 - 8\sigma^2, \quad \forall\ |\sigma| \le 1$$

On the boundary $\sigma = 1$,

$$\frac{d}{dt}\sigma^2 \le 2x_2 - 8 \le 0, \quad \forall\ x_2 \le 4$$

This implies that when the trajectory is at any point on the segment of the boundary $\sigma = 1$ for which $x_2 \le 4$, it cannot move outside the set G, because at such point σ^2 is nonincreasing. Similarly, on the boundary $\sigma = -1$,

$$\frac{d}{dt}\sigma^2 \le -2x_2 - 8 \le 0, \quad \forall\ x_2 \ge -4$$

Hence, the trajectory cannot leave the set G through the segment of the boundary $\sigma = -1$ for which $x_2 \ge -4$. This information can be used to form a closed, bounded, positively invariant set Ω that satisfies the conditions of Theorem 4.4. Using the two segments of the boundary of G just identified to define the boundary of Ω, we now need two other segments to close the set. These segments should have the property that trajectories cannot leave the set through them. We can take them as segments of a Lyapunov surface. Let c_1 be such that the Lyapunov surface $V(x) = c_1$ intersects the boundary $x_1 + x_2 = 1$ at $x_2 = 4$, that is, at the point $(-3, 4)$. (See Figure 8.5.) Let c_2 be such that the Lyapunov surface $V(x) = c_2$ intersects the boundary $x_1 + x_2 = -1$ at $x_2 = -4$, that is, at the point $(3, -4)$. The required Lyapunov surface is defined by $V(x) = \min\{c_1, c_2\}$. The constants c_1 and c_2 are given by

$$c_1 = V(x)|_{x_1=-3, x_2=4} = 10, \qquad c_2 = V(x)|_{x_1=3, x_2=-4} = 10$$

Therefore, we take $c = 10$ and define the set Ω by

$$\Omega = \{x \in R^2 \mid V(x) \le 10 \text{ and } |x_1 + x_2| \le 1\}$$

This set is closed, bounded, and positively invariant. Moreover, $\dot{V}(x)$ is negative definite in Ω, since $\Omega \subset G$. Thus, all the conditions of Theorem 4.4 are satisfied and we can conclude that all trajectories starting in Ω approach the origin as $t \to \infty$; that is, $\Omega \subset R_A$. △

8.3 Invariance-like Theorems

In the case of autonomous systems, LaSalle's invariance theorem (Theorem 4.4) shows that the trajectory of the system approaches the largest invariant set in E, where E is the set of all points in Ω where $\dot{V}(x) = 0$. In the case of nonautonomous systems, it may not even be clear how to define a set E, since $\dot{V}(t,x)$ is a function of both t and x. The situation will be simpler if it can be shown that

$$\dot{V}(t,x) \le -W(x) \le 0$$

for, then, a set E may be defined as the set of points where $W(x) = 0$. We may expect that the trajectory of the system approaches E as t tends to ∞. This is, basically, the statement of the next theorem. Before we state the theorem, we state a lemma that will be used in the proof of the theorem. The lemma is interesting in its own sake and is known as *Barbalat's lemma.*

Lemma 8.2 *Let $\phi : R \to R$ be a uniformly continuous function on $[0,\infty)$. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\, d\tau$ exists and is finite. Then,*

$$\phi(t) \to 0 \quad \text{as} \quad t \to \infty$$

◇

Proof: If it is not true, then there is a positive constant k_1 such that for every $T > 0$, we can find $T_1 \ge T$ with $|\phi(T_1)| \ge k_1$. Since $\phi(t)$ is uniformly continuous, there is a positive constant k_2 such that $|\phi(t+\tau) - \phi(t)| < k_1/2$ for all $t \ge 0$ and all $0 \le \tau \le k_2$. Hence,

$$\begin{aligned} |\phi(t)| &= |\phi(t) - \phi(T_1) + \phi(T_1)| \\ &\ge |\phi(T_1)| - |\phi(t) - \phi(T_1)| \\ &> k_1 - \tfrac{1}{2}k_1 = \tfrac{1}{2}k_1, \quad \forall\, t \in [T_1, T_1 + k_2] \end{aligned}$$

Therefore,

$$\left| \int_{T_1}^{T_1+k_2} \phi(t)\, dt \right| = \int_{T_1}^{T_1+k_2} |\phi(t)|\, dt > \tfrac{1}{2}k_1k_2$$

where the equality holds, since $\phi(t)$ retains the same sign for $T_1 \le t \le T_1 + k_2$. Thus, $\int_0^t \phi(\tau)\, d\tau$ cannot converge to a finite limit as $t \to \infty$, a contradiction. □

Theorem 8.4 *Let $D \subset R^n$ be a domain containing $x = 0$ and suppose $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $[0,\infty) \times D$. Furthermore, suppose $f(t,0)$ is uniformly bounded for all $t \ge 0$. Let $V : [0,\infty) \times D \to R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t,x) \le W_2(x)$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W(x)$$

$\forall\, t \geq 0$, $\forall\, x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions and $W(x)$ is a continuous positive semidefinite function on D. Choose $r > 0$ such that $B_r \subset D$ and let $\rho < \min_{\|x\|=r} W_1(x)$. Then, all solutions of $\dot{x} = f(t,x)$ with $x(t_0) \in \{x \in B_r \mid W_2(x) \leq \rho\}$ are bounded and satisfy

$$W(x(t)) \to 0 \quad \text{as} \quad t \to \infty$$

Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, the statement is true for all $x(t_0) \in R^n$. ◇

Proof: Similar to the proof of Theorem 4.8, it can be shown that

$$x(t_0) \in \{x \in B_r \mid W_2(x) \leq \rho\} \Rightarrow x(t) \in \Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \leq \rho\}, \quad \forall\, t \geq t_0$$

since $\dot{V}(t,x) \leq 0$. Hence, $\|x(t)\| < r$ for all $t \geq t_0$. Because $V(t,x(t))$ is monotonically nonincreasing and bounded from below by zero, it converges as $t \to \infty$. Now,

$$\int_{t_0}^{t} W(x(\tau))\; d\tau \leq - \int_{t_0}^{t} \dot{V}(\tau, x(\tau))\; d\tau = V(t_0, x(t_0)) - V(t, x(t))$$

Therefore, $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\; d\tau$ exists and is finite. Since $x(t)$ is bounded, $\dot{x}(t) = f(t,x(t))$ is bounded, uniformly in t, for all $t \geq t_0$. Hence, $x(t)$ is uniformly continuous in t on $[t_0, \infty)$. Consequently, $W(x(t))$ is uniformly continuous in t on $[t_0, \infty)$ because $W(x)$ is uniformly continuous in x on the compact set B_r. Therefore, by Lemma 8.2, we conclude that $W(x(t)) \to 0$ as $t \to \infty$. If all the assumptions hold globally and $W_1(x)$ is radially unbounded, then for any $x(t_0)$, we can choose ρ so large that $x(t_0) \in \{x \in R^n \mid W_2(x) \leq \rho\}$. □

The limit $W(x(t)) \to 0$ implies that $x(t)$ approaches E as $t \to \infty$, where

$$E = \{x \in D \mid W(x) = 0\}$$

Therefore, the positive limit set of $x(t)$ is a subset of E. The mere knowledge that $x(t)$ approaches E is much weaker than the invariance principle for autonomous systems, which states that $x(t)$ approaches the largest invariant set in E. The stronger conclusion in the case of autonomous systems is a consequence of the property of autonomous systems stated in Lemma 4.1, namely the positive limit set is an invariant set. There are some special classes of nonautonomous systems where positive limit sets have some sort of an invariance property.[14] However, for a general nonautonomous system, the positive limit sets are not invariant. The fact that, in the case of autonomous systems, $x(t)$ approaches the largest invariant set in E allowed us to arrive at Corollary 4.1, where asymptotic stability of the origin is established by showing that the set E does not contain an entire trajectory of the system, other than the trivial solution. For a general nonautonomous system, there is no extension of Corollary 4.1 that would show uniform asymptotic stability. However, the next theorem shows that it is possible to conclude uniform asymptotic stability if, in addition to $\dot{V}(t,x) \le 0$, we can show that V decreases over the interval $[t, t+\delta]$.[15]

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[14] Examples are periodic systems, almost-periodic systems, and asymptotically autonomous systems. See [154, Chapter 8] for invariance theorems for these classes of systems. See, also, [136] for a different generalization of the invariance principle.

Theorem 8.5 *Let $D \subset R^n$ be a domain containing $x = 0$ and suppose $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \ge 0$ and $x \in D$. Let $x = 0$ be an equilibrium point for $\dot{x} = f(t,x)$ at $t = 0$. Let $V : [0,\infty) \times D \to R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t,x) \le W_2(x)$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$$

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) \le -\lambda V(t,x), \quad 0 < \lambda < 1^{16}$$

$\forall\ t \ge 0$, $\forall\ x \in D$, for some $\delta > 0$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D and $\phi(\tau; t, x)$ is the solution of the system that starts at (t,x). Then, the origin is uniformly asymptotically stable. If all the assumptions hold globally and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable. If

$$W_1(x) \ge k_1 \|x\|^c, \ W_2(x) \le k_2 \|x\|^c, \ k_1 > 0, \ k_2 > 0, \ c > 0$$

then the origin is exponentially stable. ◇

Proof: Choose $r > 0$ such that $B_r \in D$. Similar to the proof of Theorem 4.8, it can be shown that

$$x(t_0) \in \{x \in B_r \mid W_2(x) \le \rho\} \Rightarrow x(t) \in \Omega_{t,\rho}, \quad \forall\ t \ge t_0$$

where $\rho < \min_{\|x\|=r} W_1(x)$, because $\dot{V}(t,x) \le 0$. Now, for all $t \ge t_0$, we have

$$V(t+\delta, x(t+\delta)) \le V(t, x(t)) - \lambda V(t, x(t)) = (1-\lambda) V(t, x(t))$$

Moreover, since $\dot{V}(t,x) \le 0$,

$$V(\tau, x(\tau)) \le V(t, x(t)), \quad \forall\ \tau \in [t, t+\delta]$$

---

[15] It is shown in [1] that the condition $\dot{V} \le 0$ can be dropped and uniform asymptotic stability can be shown if

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) \le -\gamma(\|x\|)$$

for some class $\mathcal{K}$ function γ.

[16] There is no loss of generality in assuming that $\lambda < 1$, for if the inequality is satisfied with $\lambda_1 \ge 1$, then it is satisfied for any positive $\lambda < 1$, since $-\lambda_1 V \le -\lambda V$. Notice, however, that this inequality could not be satisfied with $\lambda > 1$, since $V(t,x) > 0$, $\forall\ x \ne 0$.

For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + N\delta$. Divide the interval $[t_0, t_0 + (N-1)\delta]$ into $(N-1)$ equal subintervals of length δ each. Then,

$$\begin{aligned} V(t, x(t)) &\leq V(t_0 + (N-1)\delta, x(t_0 + (N-1)\delta)) \\ &\leq (1-\lambda)V(t_0 + (N-2)\delta, x(t_0 + (N-2)\delta)) \\ &\vdots \\ &\leq (1-\lambda)^{(N-1)} V(t_0, x(t_0)) \\ &\leq \frac{1}{(1-\lambda)}(1-\lambda)^{(t-t_0)/\delta} V(t_0, x(t_0)) \\ &= \frac{1}{(1-\lambda)} e^{-b(t-t_0)} V(t_0, x(t_0)) \end{aligned}$$

where

$$b = \frac{1}{\delta} \ln \frac{1}{(1-\lambda)}$$

Taking

$$\sigma(r, s) = \frac{r}{(1-\lambda)} e^{-bs}$$

it can be easily seen that $\sigma(r, s)$ is a class $\mathcal{KL}$ function and $V(t, x(t))$ satisfies

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall\ V(t_0, x(t_0)) \in [0, \rho]$$

From this point on, the rest of the proof is identical to that of Theorem 4.9. The proof of the statements on global uniform asymptotic stability and exponential stability are the same as the proofs of Theorems 4.9 and 4.10. □

Example 8.11 Consider the linear time-varying system

$$\dot{x} = A(t)x$$

where $A(t)$ is continuous for all $t \geq 0$. Suppose there is a continuously differentiable, symmetric matrix $P(t)$ that satisfies

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall\ t \geq 0$$

as well as the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t)$$

where $C(t)$ is continuous in t. The derivative of the quadratic function

$$V(t, x) = x^T P(t) x$$

along the trajectories of the system is

$$\dot{V}(t, x) = -x^T C^T(t) C(t) x \leq 0$$

The solution of the linear system is given by $\phi(\tau; t, x) = \Phi(\tau, t)x$, where $\Phi(\tau, t)$ is the state transition matrix. Therefore,

$$\begin{aligned} V(t+\delta, \phi(t+\delta; t, x)) - V(t, x) &= \int_t^{t+\delta} \dot{V}(\tau, \phi(\tau; t, x))\, d\tau \\ &= -x^T \int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t)\, d\tau\, x \\ &= -x^T W(t, t+\delta) x \end{aligned}$$

where

$$W(t, t+\delta) = \int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t)\, d\tau$$

Suppose there is a positive constant $k < c_2$ such that

$$W(t, t+\delta) \geq kI, \quad \forall\, t \geq 0$$

then

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t, x) \leq -k\|x\|_2^2 \leq -\frac{k}{c_2} V(t, x)$$

Thus, all the assumptions of Theorem 8.5 are satisfied globally with

$$W_i(x) = c_i \|x\|_2^2, \; i = 1, 2, \quad \lambda = \frac{k}{c_2} < 1$$

and we conclude that the origin is globally exponentially stable. Readers familiar with linear system theory will recognize that the matrix $W(t, t+\delta)$ is the observability Gramian of the pair $(A(t), C(t))$ and that the inequality $W(t, t+\delta) \geq kI$ is implied by uniform observability of $(A(t), C(t))$. Comparing this example with Example 4.21 shows that Theorem 8.5 allows us to replace the positive definiteness requirement on the matrix $Q(t)$ of (4.28) by the weaker requirement $Q(t) = C^T(t)C(t)$, where the pair $(A(t), C(t))$ is uniformly observable. △

Theorems 8.4 and 8.5 and their application to linear systems, as in Example 8.11, are extensively used in the analysis of adaptive control systems.[17] As an example, we analyze the adaptive control system of Section 1.2.6.

Example 8.12 In Section 1.2.6, we saw that the closed-loop equation of a model reference adaptive control system, with plant $\dot{y}_p = a_p y_p + k_p u$ and reference model $\dot{y}_m = a_m y_m + k_m r$, is given by

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)] \\ \dot{\phi}_1 &= -\gamma e_o r(t) \\ \dot{\phi}_2 &= -\gamma e_o [e_o + y_m(t)] \end{aligned}$$

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[17] See, for example, [87] and [168].

where $\gamma > 0$ is the adaptation gain, $e_o = y_p - y_m$ is the output error, and ϕ_1 and ϕ_2 are the parameter errors. It was assumed that $k_p > 0$ and, of course, the reference model must have $a_m < 0$. Furthermore, we assume that $r(t)$ is piecewise continuous and bounded. Using

$$V = \tfrac{1}{2}\left[\frac{e_o^2}{k_p} + \frac{1}{\gamma}(\phi_1^2 + \phi_2^2)\right]$$

as a Lyapunov function candidate, we obtain

$$\dot{V} = \frac{a_m}{k_p}e_o^2 + e_o(\phi_1 r + \phi_2 e_o + \phi_2 y_m) - \phi_1 e_o r - \phi_2 e_o(e_o + y_m) = \frac{a_m}{k_p}\, e_o^2 \le 0$$

By applying Theorem 8.4, we conclude that for any $c > 0$ and for all initial states in the set $\{V \le c\}$, all state variables are bounded for all $t \ge t_0$ and $\lim_{t\to\infty} e_o(t) = 0$. This shows that the output of the plant y_p tracks the desired output y_m, but says nothing about the convergence of the parameter errors ϕ_1 and ϕ_2 to zero. In fact, they may not converge to zero. For example, if r and y_m are nonzero constant signals, the closed-loop system will have an equilibrium subspace $\{e_o = 0,\ \phi_2 = (a_m/k_m)\phi_1\}$, which shows clearly that, in general, ϕ_1 and ϕ_2 do not converge to zero. To derive conditions under which ϕ_1 and ϕ_2 will converge to zero, we apply Theorem 8.5. This will yield conditions under which the origin $(e_o = 0,\ \phi_1 = 0,\ \phi_2 = 0)$ is uniformly asymptotically stable. Since we have already shown that all state variables are bounded, we can represent the closed-loop system as the linear time-varying system

$$\dot{x} = \begin{bmatrix} a_m & k_p r(t) & k_p y_p(t) \\ -\gamma r(t) & 0 & 0 \\ -\gamma y_p(t) & 0 & 0 \end{bmatrix} x, \quad \text{where} \quad x = \begin{bmatrix} e_o \\ \phi_1 \\ \phi_2 \end{bmatrix}$$

Suppose the reference signal $r(t)$ has a steady-state value $r_{\rm ss}(t)$; that is, $\lim_{t\to\infty}[r(t) - r_{\rm ss}(t)] = 0$. Then, $\lim_{t\to\infty}[y_m(t) - y_{\rm ss}(t)] = 0$, where $y_{\rm ss}(t)$ is the steady-state response of the reference model. These limits, together with $\lim_{t\to\infty} e_o(t) = 0$, show that the linear system can be represented by

$$\dot{x} = [A(t) + B(t)]x$$

where

$$A(t) = \begin{bmatrix} a_m & k_p r_{\rm ss}(t) & k_p y_{\rm ss}(t) \\ -\gamma r_{\rm ss}(t) & 0 & 0 \\ -\gamma y_{\rm ss}(t) & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \lim_{t\to\infty} B(t) = 0$$

If we can show that the origin of $\dot{x} = A(t)x$ is uniformly asymptotically stable, we can use the property $\lim_{t\to\infty} B(t) = 0$ to show that the origin of $\dot{x} = [A(t) + B(t)]x$ is uniformly asymptotically stable.[18] Therefore, we concentrate our attention on the system $\dot{x} = A(t)x$. Once again, using V as a Lyapunov function candidate, we obtain

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[18] See Example 9.6.

$$\dot{V} = \frac{a_m}{k_p}\, e_o^2 = -x^T C^T C x, \quad \text{where} \quad C = \sqrt{\frac{-a_m}{k_p}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

From Example 8.12, we see that the origin will be uniformly asymptotically stable if the pair $(A(t), C)$ is uniformly observable. Since uniform observability of $(A(t), C)$ is equivalent to uniform observability of $(A(t) - K(t)C, C)$ for any piecewise continuous, bounded matrix $K(t)$,[19] we take

$$K(t) = \sqrt{\frac{k_p}{-a_m}} \begin{bmatrix} a_m & -\gamma r_{ss}(t) & -\gamma y_{ss}(t) \end{bmatrix}^T$$

to simplify the pair to

$$A(t) - K(t)C = \begin{bmatrix} 0 & k_p r_{ss}(t) & k_p y_{ss}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \sqrt{\frac{-a_m}{k_p}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

By investigating observability of this pair for a given reference signal, we can determine whether the conditions of Theorem 8.5 are satisfied. For example, if r is a nonzero constant signal, it can be easily seen that the pair is not observable. This is not surprising, since we have already seen that in this case the origin is not uniformly asymptotically stable. On the other hand, if $r(t) = a \sin \omega t$ with positive a and ω, we have $r_{ss}(t) = r(t)$ and $y_{ss}(t) = aM \sin(\omega t + \delta)$, where M and δ are determined by the transfer function of the reference model. It can be verified that the pair is uniformly observable; hence the origin ($e_o = 0$, $\phi_1 = 0$, $\phi_2 = 0$) is uniformly asymptotically stable and the parameter errors $\phi_1(t)$ and $\phi_2(t)$ converge to zero as t tends to infinity.[20] △

8.4 Stability of Periodic Solutions

In Chapter 4, we developed an extensive theory for the stability of equilibrium points. In this section, we consider the corresponding problem for periodic solutions. If $u(t)$ is a periodic solution of the system

$$\dot{x} = f(t, x) \tag{8.14}$$

what can we say about other solutions that start arbitrarily close to $u(t)$? Will they remain in some neighborhood of $u(t)$ for all t? Will they eventually approach

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[19] See [87, Lemma 4.8.1].

[20] For this example, the reference $r(t) = a \sin \omega t$ is said to be persistently exciting, while a constant reference is not persistently exciting. To read more about persistence of excitation, see [5], [15], [87], [139], [168], or Section 13.4 of the second edition of this book.

$u(t)$? Such stability properties of the periodic solution $u(t)$ can be characterized and investigated in the sense of Lyapunov. Let

$$y = x - u(t)$$

so that the origin $y = 0$ becomes an equilibrium point for the nonautonomous system

$$\dot{y} = f(t, y + u(t)) - f(t, u(t)) \tag{8.15}$$

The behavior of solutions of (8.14) near $u(t)$ is equivalent to the behavior of solutions of (8.15) near $y = 0$. Therefore, we can characterize stability properties of $u(t)$ from those of the equilibrium $y = 0$. In particular, we say that the periodic solution $u(t)$ is uniformly asymptotically stable if $y = 0$ is a uniformly asymptotically stable equilibrium point for the system (8.15). Similar statements can be made for other stability properties, like uniform stability. Thus, investigating the stability of $u(t)$ has been reduced to studying the stability of an equilibrium point of a nonautonomous system, which we studied in Chapter 4. We shall find this notion of uniform asymptotic stability of periodic solutions in the sense of Lyapunov to be useful when we study nonautonomous systems dependent on small parameters in Chapter 10. The notion, however, is too restrictive when we analyze periodic solutions of autonomous systems. The next example illustrates the restrictive nature of this notion.

Example 8.13 Consider the second-order system

$$\begin{aligned}
\dot{x}_1 &= x_1 \left[\frac{\left(1 - x_1^2 - x_2^2\right)^3}{x_1^2 + x_2^2} \right] - x_2 \left[1 + \left(1 - x_1^2 - x_2^2\right)^2 \right] \\
\dot{x}_2 &= x_2 \left[\frac{\left(1 - x_1^2 - x_2^2\right)^3}{x_1^2 + x_2^2} \right] + x_1 \left[1 + \left(1 - x_1^2 - x_2^2\right)^2 \right]
\end{aligned}$$

which is represented in the polar coordinates

$$x_1 = r \cos\theta, \qquad x_2 = r \sin\theta$$

by

$$\dot{r} = \frac{(1 - r^2)^3}{r}, \qquad \dot{\theta} = 1 + (1 - r^2)^2$$

The solution starting at (r_0, θ_0) is given by

$$\begin{aligned}
r(t) &= \left[1 - \frac{1 - r_0^2}{\sqrt{1 + 4t\left(1 - r_0^2\right)^2}} \right]^{1/2} \\
\theta(t) &= \theta_0 + t + \tfrac{1}{4} \ln\left[1 + 4t(1 - r_0^2)^2\right]
\end{aligned}$$

From these expressions, we see that the system has a periodic solution

$$\bar{x}_1(t) = \cos t, \qquad \bar{x}_2(t) = \sin t$$

The corresponding periodic orbit is the unit circle $r = 1$. All nearby solutions spiral toward this periodic orbit as $t \to \infty$. This spiralling is clearly the kind of "asymptotically stable" behavior we expect to see with a periodic orbit. In fact, the periodic orbit has been known classically as a stable limit cycle. However, the periodic solution $\bar{x}(t)$ is not uniformly asymptotically stable in the sense of Lyapunov. Recall that for the solution to be uniformly asymptotically stable, we must have

$$[r(t)\cos\theta(t) - \cos t]^2 + [r(t)\sin\theta(t) - \sin t]^2 \to 0 \ \text{ as } \ t \to \infty$$

for sufficiently small $[r_0 \cos\theta_0 - 1]^2 + [r_0 \sin\theta_0]^2$. Because $r(t) \to 1$ as $t \to \infty$, we must have

$$|1 - \cos(\theta(t) - t)| \to 0 \ \text{ as } \ t \to \infty$$

which clearly is not satisfied when $r_0 \neq 1$, since $(\theta(t) - t)$ is an ever-growing monotonically increasing function of t. △

The point illustrated by this example is true in general. In particular, a nontrivial periodic solution of an autonomous system can never be asymptotically stable in the sense of Lyapunov.[21]

The stability-like properties of the periodic orbit of Example 8.13 can be captured by extending the notion of stability in the sense of Lyapunov from stability of an equilibrium point to stability of an invariant set. Consider the autonomous system

$$\dot{x} = f(x) \tag{8.16}$$

where $f : D \to R^n$ is a continuously differentiable map from a domain $D \subset R^n$ into R^n. Let $M \subset D$ be a closed invariant set of (8.16). Define an ε-neighborhood of M by

$$U_\varepsilon = \{x \in R^n \mid \text{dist}(x, M) < \varepsilon\}$$

where $\text{dist}(x, M)$ is the minimum distance from x to a point in M; that is,

$$\text{dist}(x, M) = \inf_{y \in M} \|x - y\|$$

Definition 8.1 *The closed invariant set M of* (8.16) *is*

- *stable if, for each $\varepsilon > 0$, there is $\delta > 0$ such that*

$$x(0) \in U_\delta \Rightarrow x(t) \in U_\varepsilon, \quad \forall \ t \geq 0$$

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[21] See [72, Theorem 81.1] for a proof of this statement.

- *asymptotically stable if it is stable and δ can be chosen such that*

$$x(0) \in U_\delta \Rightarrow \lim_{t\to\infty} \text{dist}(x(t), M) = 0$$

This definition reduces to Definition 4.1 when M is an equilibrium point. Lyapunov stability theory for equilibrium points, as presented in Chapter 4, can be extended to invariant sets.[22] For example, by repeating the proof of Theorem 4.1, it is not hard to see that if there is a function $V(x)$, which is zero on M and positive in some neighborhood D of M, excluding M itself, and if the derivative $\dot{V}(x) = [\partial V/\partial x]f(x) \leq 0$ in D, then M is stable. Furthermore, if $\dot{V}(x)$ is negative in D, excluding M, then M is asymptotically stable.

Stability and asymptotic stability of invariant sets are interesting concepts in their own sake. We will apply them here to the specific case when the invariant set M is the closed orbit associated with a periodic solution. Let $u(t)$ be a nontrivial periodic solution of the autonomous system (8.16) with period T, and let γ be the closed orbit defined by

$$\gamma = \{x \in R^n \mid x = u(t),\ 0 \leq t \leq T\}$$

The periodic orbit γ is the image of $u(t)$ in the state space. It is an invariant set whose stability properties are characterized by Definition 8.1. It is common, especially for second-order systems, to refer to asymptotically stable periodic orbits as stable limit cycles

Example 8.14 The harmonic oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

has a continuum of periodic orbits, which are concentric circles with a center at the origin. Any one of these periodic orbits is stable. Consider, for example, the periodic orbit γ_c defined by

$$\gamma_c = \{x \in R^2 \mid r = c > 0\}, \quad \text{where } r = \sqrt{x_1^2 + x_2^2}$$

The U_ε neighborhood of γ_c is defined by the annular region

$$U_\varepsilon = \{x \in R^2 \mid c - \varepsilon < r < c + \varepsilon\}$$

This annular region itself is an invariant set. Thus, given $\varepsilon > 0$, we can take $\delta = \varepsilon$ and see that any solution starting in the U_δ neighborhood at $t = 0$ will remain in the U_ε neighborhood for all $t \geq 0$. Hence, the periodic orbit γ_c is stable. However, it is not asymptotically stable, because a solution starting in a U_δ neighborhood of γ_c does not approach γ_c as $t \to \infty$, no matter how small δ is. Stability of the periodic orbit $\{r = c\}$ can be also shown by the Lyapunov function

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[22] See, for example, [213] and [221] for comprehensive coverage and [118] for some interesting results on converse Lyapunov theorems.

$$V(x) = (r^2 - c^2)^2 = (x_1^2 + x_2^2 - c^2)^2$$

whose derivative along the trajectories of the system is

$$\dot{V}(x) = 4(r^2 - c^2)r\dot{r} = 0$$

△

Example 8.15 Consider the system of Example 8.13. It has an isolated periodic orbit

$$\gamma = \{x \in R^2 \mid r = 1\}, \quad \text{where } r = \sqrt{x_1^2 + x_2^2}$$

For $x \notin \gamma$, we have

$$\text{dist}(x, \gamma) = \inf_{y \in \gamma} \|x - y\|_2 = \inf_{y \in \gamma} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = |r - 1|$$

Recalling that

$$r(t) = \left[1 - \frac{1 - r_0^2}{\sqrt{1 + 4t\,(1 - r_0^2)^2}} \right]^{1/2}$$

it can be easily seen that the ε–δ requirement for stability is satisfied and

$$\text{dist}(x(t), \gamma) \to 0, \ \text{ as } t \to \infty$$

Hence, the periodic orbit is asymptotically stable. The same conclusion can be arrived at using the Lyapunov function

$$V(x) = (r^2 - 1)^2 = (x_1^2 + x_2^2 - 1)^2$$

whose derivative along the trajectories of the system is

$$\dot{V}(x) = 4(r^2 - 1)r\dot{r} = -4(r^2 - 1)^4 < 0, \quad \text{for } r \neq 1$$

△

Having defined the stability properties of periodic orbits, we can now define the stability properties of periodic solutions.

Definition 8.2 *A nontrivial periodic solution $u(t)$ of* (8.16) *is*

- *orbitally stable if the closed orbit γ generated by $u(t)$ is stable.*
- *asymptotically orbitally stable if the closed orbit γ generated by $u(t)$ is asymptotically stable.*

Notice that different terminology is used depending on whether we are talking about the periodic solution or the corresponding periodic orbit. In Example 8.15, we say that the unit circle is an asymptotically stable periodic orbit, but we say that the periodic solution $(\cos t, \sin t)$ is orbitally asymptotically stable.

8.5 Exercises

8.1 Prove Corollary 8.1.

8.2 Prove Corollary 8.2.

8.3 Suppose the conditions of Theorem 8.1 are satisfied in a case where $g_1(y, 0) = 0$, $g_2(y, 0) = 0$, and $A_1 = 0$. Show that the origin of the full system is stable.

8.4 Reconsider Example 8.1 with $a = 0$. Show that the origin is stable.

8.5 ([88]) Consider the system

$$\begin{aligned} \dot{x}_a &= f_a(x_a, x_b) \\ \dot{x}_b &= A_b x_b + f_b(x_a, x_b) \end{aligned}$$

where $\dim(x_a) = n_1$, $\dim(x_b) = n_2$, A_b is Hurwitz, f_a and f_b are continuously differentiable, $[\partial f_b/\partial x_b](0, 0) = 0$, and $f_b(x_a, 0) = 0$ in a neighborhood of $x_a = 0$.

(a) Show that if $x_a = 0$ is an exponentially stable equilibrium point of $\dot{x}_a = f_a(x_a, 0)$, then $(x_a, x_b) = (0, 0)$ is an exponentially stable equilibrium point of the full system.

(b) Show that if $x_a = 0$ is an asymptotically (but not exponentially) stable equilibrium point of $\dot{x}_a = f_a(x_a, 0)$, then $(x_a, x_b) = (0, 0)$ is an asymptotically stable equilibrium point of the full system.

8.6 ([70]) For each of the following systems, investigate stability of the origin by using the center manifold theorem:

(1)
$$\begin{aligned} \dot{x}_1 &= -x_2^2 \\ \dot{x}_2 &= -x_2 + x_1^2 + x_1 x_2 \end{aligned}$$

(2)
$$\begin{aligned} \dot{x}_1 &= a x_1^2 - x_2^2, \quad a \neq 0 \\ \dot{x}_2 &= -x_2 + x_1^2 + x_1 x_2 \end{aligned}$$

(3)
$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 x_3 \\ \dot{x}_2 &= x_1 + x_2 x_3 \\ \dot{x}_3 &= -x_3 - (x_1^2 + x_2^2) + x_3^2 \end{aligned}$$

(4)
$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 \end{aligned}$$

(5)
$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 \\ \dot{x}_2 &= -x_2 - x_1^2 + 2x_1^8 \end{aligned}$$

(6)
$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3(x_1 + x_2 - 1) \\ \dot{x}_2 &= x_2^3(x_1 + x_2 - 1) \end{aligned}$$

(7)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + a x_1^3/(1 + x_1^2) \\ & \quad a \neq 0 \end{aligned}$$

(8)
$$\begin{aligned} \dot{x}_1 &= -2x_1 - 3x_2 + x_3 + x_3^2 \\ \dot{x}_2 &= x_1 + x_1^2 + x_2 \\ \dot{x}_3 &= x_1^2 \end{aligned}$$

8.7 ([34]) Consider the system

$$\dot{x}_1 = x_1 x_2 + a x_1^3 + b x_1 x_2^2, \qquad \dot{x}_2 = -x_2 + c x_1^2 + d x_1^2 x_2$$

Investigate stability of the origin by using the center manifold theorem for each of the following cases:

(1) $a+c>0$.
(2) $a+c<0$.
(3) $a+c=0$ and $cd+bc^2<0$.
(4) $a+c=0$ and $cd+bc^2>0$.
(5) $a+c=cd+bc^2=0$.

8.8 ([34]) Consider the system

$$\dot{x}_1 = ax_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

Investigate stability of the origin by using the center manifold theorem for all possible values of the real parameter a.

8.9 ([88]) Consider the system

$$\dot{x}_1 = ax_1 x_2 - x_1^3, \qquad \dot{x}_2 = -x_2 + bx_1 x_2 + cx_1^2$$

Investigate stability of the origin by using the center manifold theorem for all possible values of the real constants a, b, and c.

8.10 (Zubov's Theorem) Consider the system (8.13) and let $G \subset R^n$ be a domain containing the origin. Suppose there exist two functions $V : G \rightarrow R$ and $h : R^n \rightarrow R$ with the following properties:

- V is continuously differentiable and positive definite in G and satisfies

$$0 < V(x) < 1, \quad \forall\ x \in G - \{0\}$$

- As x approaches the boundary of G, or in case of unbounded G as $\|x\| \rightarrow \infty$, $\lim V(x) = 1$.

- h is continuous and positive definite on R^n.

- For $x \in G$, $V(x)$ satisfies the partial differential equation

$$\frac{\partial V}{\partial x} f(x) = -h(x)[1 - V(x)] \tag{8.17}$$

Show that $x = 0$ is asymptotically stable and G is the region of attraction.

8.11 ([72]) Consider the second-order system

$$\dot{x}_1 = -h_1(x_1) + g_2(x_2), \qquad \dot{x}_2 = -g_1(x_1)$$

where

$$h_1(0) = 0, \quad zh_1(z) > 0\ \ \forall\ -a_1 < z < b_1$$

$$g_i(0) = 0, \quad zg_i(z) > 0 \ \forall \ -a_i < z < b_i$$

$$\int_0^z g_i(\sigma)\, d\sigma \to \infty \text{ as } z \to -a_i \text{ or } z \to b_i$$

for some positive constants a_i, b_i ($a_i = \infty$ or $b_i = \infty$ is allowed). Apply Zubov's theorem to show that the region of attraction is $\{x \in R^2 \mid -a_i < x_i < b_i\}$.
Hint: Take $h(x) = g_1(x_1)h_1(x_1)$ and seek a solution of the partial differential equation (8.17) in the form $V(x) = 1 - W_1(x_1)W_2(x_2)$. Note that, with this choice of h, $\dot{V}(x)$ is only negative semidefinite; apply LaSalle's invariance principle.

8.12 Find the region of attraction of the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -\tan(x_1)$$

Hint: Use the previous exercise.

8.13 Let Ω be an open, positively invariant set containing the origin. Suppose every trajectory in Ω approaches the origin as $t \to \infty$. Show that Ω is connected.

8.14 Consider a second-order system $\dot{x} = f(x)$ with asymptotically stable origin. Let $V(x) = x_1^2 + x_2^2$, and $D = \{x \in R^2 \mid |x_2| < 1, \ |x_1 - x_2| < 1\}$. Suppose $[\partial V/\partial x]\ f(x)$ is negative definite in D. Estimate the region of attraction.

8.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 - (2x_2 + x_1)(1 - x_2^2)$$

(a) Using $V(x) = 5x_1^2 + 2x_1x_2 + 2x_2^2$, show that the origin is asymptotically stable.

(b) Let

$$S = \{x \in R^2 \mid V(x) \le 5\} \cap \{x \in R^2 \mid |x_2| \le 1\}$$

Show that S is an estimate of the region of attraction.

8.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - x_1 + x_1^3$$

is asymptotically stable and estimate the region of attraction.

8.17 Consider a second-order system $\dot{x} = f(x)$, together with a Lyapunov function $V(x)$. Suppose that $\dot{V}(x) < 0$ for all $x_1^2 + x_2^2 \ge a^2$. The sketch, given in Figure 8.6, shows four different directions of the vector field at a point on the circle $x_1^2 + x_2^2 = a^2$. Which of these directions are possible and which are not? Justify your answer.

8.18 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - \sin x_1 - 2\ \mathrm{sat}(x_1 + x_2)$$

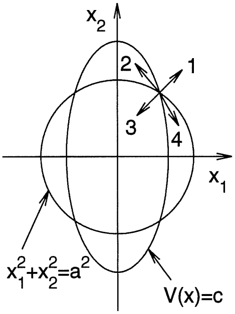

Figure 8.6: Exercise 8.17.

(a) Show that the origin is the unique equilibrium point.

(b) Show, using linearization, that the origin is asymptotically stable.

(c) Let $\sigma = x_1 + x_2$. Show that $\sigma\, \dot{\sigma} \leq -|\sigma|$ for $|\sigma| \geq 1$.

(d) Let $V(x) = x_1^2 + 0.5x_2^2 + 1 - \cos x_1$. Show that

$$M_c = \{x \in R^2 \mid V(x) \leq c\} \cap \{x \in R^2 \mid |\sigma| \leq 1\}, \qquad c > 0$$

is positively invariant and trajectories in M_c approach the origin as $t \to \infty$.

(e) Show that the origin is globally asymptotically stable.

8.19 Consider the synchronous generator model described in Exercise 1.8. Take the state variables and parameters as in parts (a) and (b) of the exercise. Moreover, take $\tau = 6.6$ sec, $M = 0.0147$ (per unit power) × sec^2/rad, and $D/M = 4$ sec^{-1}.

(a) Find all equilibrium points in the region $-\pi \leq x_1 \leq \pi$, and determine the stability properties of each equilibrium by using linearization.

(b) Estimate the region of attraction of each asymptotically stable equilibrium.

8.20 ([113]) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - g(t)x_2$$

where $g(t)$ is continuously differentiable and $0 < k_1 \leq g(t) \leq k_2$ for all $t \geq 0$.

(a) Show that the origin is exponentially stable.

(b) Would (a) be true if $g(t)$ were not bounded? Consider $g(t) = 2 + \exp(t)$.

8.21 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - g(t)x_2$$

where $g(t)$ is continuously differentiable and $0 < k_1 \leq g(t) \leq k_2$ for all $t \geq 0$. Show that the origin is exponentially stable.
Hint: Use the previous exercise.

8.22 Consider the system

$$\dot{x}_1 = -x_1 - x_2 - \alpha(t)x_3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = \alpha(t)x_1$$

where $\alpha(t) = \sin t + \sin 2t$. Show that the origin is exponentially stable.

8.23 Consider the single-input–single-output nonlinear system

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= f_0(x) + (\theta^*)^T f_1(x) + g_0(x)u \end{aligned}$$

where f_0, f_1, and g_0 are known smooth functions of x, defined for all $x \in R^n$, while $\theta^* \in R^p$ is a vector of unknown constant parameters. The function $g_0(x)$ is bounded away from zero; that is, $|g_0(x)| \geq k_0 > 0$, for all $x \in R^n$. We assume that all state variables can be measured. It is desired to design a state feedback adaptive controller such that x_1 asymptotically tracks a desired reference signal $r(t)$, where r and its derivatives up to $r^{(n)}$ are continuous and bounded for all $t \geq 0$.

(a) Taking $e_i = x_i - r^{(i-1)}$ and $e = [e_1, \ \ldots, \ e_n]^T$, show that e satisfies the equation

$$\dot{e} = Ae + B[f_0(x) + (\theta^*)^T f_1(x) + g_0(x)u - r^{(n)}]$$

where (A, B) is a controllable pair.

(b) Design K such that $A - BK$ is Hurwitz and let P be the positive definite solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. Using the Lyapunov function candidate $V = e^T Pe + \phi^T \Gamma^{-1} \phi$, where $\phi = \theta - \theta^*$ and Γ is a symmetric positive definite matrix, show that the adaptive controller

$$\begin{aligned} u &= \frac{1}{g_0(x)} \left[-f_0(x) - \theta^T f_1(x) + r^{(n)} - Ke \right] \\ \dot{\theta} &= \Gamma f_1(x) e^T PB \end{aligned}$$

ensures that all state variables are bounded and $\lim_{t \to \infty} e(t) = 0$.

(c) Let

$$\bar{A}(t) = \begin{bmatrix} 0_{n \times n} & -Bf_1^T(\mathcal{R}) \\ 0_{p \times n} & 0_{p \times p} \end{bmatrix}, \qquad C = \begin{bmatrix} I_n & 0_{n \times p} \end{bmatrix}$$

where $\mathcal{R} = [r, \ \ldots, \ r^{(n-1)}]^T$. Show that if $(\bar{A}(t), C)$ is uniformly observable, then the parameter error ϕ converges to zero as $t \to \infty$.

Chapter 9

Stability of Perturbed Systems

Consider the system

$$\dot{x} = f(t, x) + g(t, x) \tag{9.1}$$

where $f : [0, \infty) \times D \to R^n$ and $g : [0, \infty) \times D \to R^n$ are piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. We think of this system as a perturbation of the nominal system

$$\dot{x} = f(t, x) \tag{9.2}$$

The perturbation term $g(t, x)$ could result from modeling errors, aging, or uncertainties and disturbances, which exist in any realistic problem. In a typical situation, we do not know $g(t, x)$, but we know some information about it, like knowing an upper bound on $\|g(t, x)\|$. Here, we represent the perturbation as an additive term on the right-hand side of the state equation. Uncertainties that do not change the system's order can always be represented in this form. For if the perturbed right-hand side is some function $\tilde{f}(t, x)$, then by adding and subtracting $f(t, x)$, we can rewrite the right-hand side as

$$\tilde{f}(t, x) = f(t, x) + [\tilde{f}(t, x) - f(t, x)]$$

and define

$$g(t, x) = \tilde{f}(t, x) - f(t, x)$$

Suppose the nominal system (9.2) has a uniformly asymptotically stable equilibrium point at the origin, what can we say about the stability behavior of the perturbed system (9.1)? A natural approach to address this question is to use a Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system. This is what we have done in the analysis of the linearization approach in Sections 4.3 and 4.6. The new element here is that the perturbation term could be more general than the perturbation term in the case of linearization. The conclusions we can arrive at depend critically on whether the perturbation term vanishes at the origin. If $g(t, 0) = 0$, the perturbed system (9.2) has an equilibrium point at the origin. In this case, we analyze the stability behavior of the origin as an equilibrium point of the perturbed system. If $g(t, 0) \neq 0$, the origin will not be an equilibrium point of the perturbed system. In this case, we study ultimate boundedness of the solutions of the perturbed system.

The cases of vanishing and nonvanishing perturbations are treated in Sections 9.1 and 9.2, respectively. In Section 9.3, we restrict our attention to the case when the nominal system has an exponentially stable equilibrium point at the origin and use the comparison lemma to derive some sharper results on the asymptotic behavior of the solution of the perturbed system. In Section 9.4, we give a result that establishes continuity of the solution of the state equation on the infinite-time interval.

The last two sections deal with interconnected systems and slowly varying systems, respectively. In both cases, stability analysis is simplified by viewing the system as a perturbation of a simpler system. In the case of interconnected systems, the analysis is simplified by decomposing the system into smaller isolated subsystems, while in the case of slowly varying systems, a nonautonomous system with slowly varying inputs is approximated by an autonomous system where the slowly varying inputs are treated as constant parameters.

9.1 Vanishing Perturbation

Let us start with the case $g(t, 0) = 0$. Suppose $x = 0$ is an exponentially stable equilibrium point of the nominal system (9.2), and let $V(t, x)$ be a Lyapunov function that satisfies

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \tag{9.3}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2 \tag{9.4}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| \tag{9.5}$$

for all $(t, x) \in [0, \infty) \times D$ for some positive constants c_1, c_2, c_3, and c_4. The existence of a Lyapunov function satisfying (9.3) through (9.5) is guaranteed by Theorem 4.14, under some additional assumptions. Suppose the perturbation term $g(t, x)$ satisfies the linear growth bound

$$\|g(t,x)\| \leq \gamma\|x\|, \quad \forall\, t \geq 0, \ \forall\, x \in D \tag{9.6}$$

where γ is a nonnegative constant. This bound is natural in view of the assumptions on $g(t, x)$. In fact, any function $g(t, x)$ that vanishes at the origin and is locally Lipschitz in x, uniformly in t for all $t \geq 0$, in a bounded neighborhood of the origin satisfies (9.6) over that neighborhood.[1] We use V as a Lyapunov function candidate to investigate the stability of the origin as an equilibrium point for the perturbed system (9.1). The derivative of V along the trajectories of (9.1) is given by

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[1] Note, however, that the linear growth bound (9.6) becomes restrictive when required to hold globally, because that would require g to be globally Lipschitz in x.

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x)$$

The first two terms on the right-hand side constitute the derivative of $V(t,x)$ along the trajectories of the nominal system, which is negative definite and satisfies (9.4). The third term, $[\partial V/\partial x]g$, is the effect of the perturbation. Since we do not have complete knowledge of g, we cannot judge whether this term helps or hurts the cause of making $\dot{V}(t,x)$ negative definite. With the growth bound (9.6) as our only information on g, the best we can do is worst case analysis where $[\partial V/\partial x]g$ is bounded by a nonnegative term. Using (9.4) through (9.6), we obtain

$$\dot{V}(t,x) \le -c_3\|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \; \|g(t,x)\| \le -c_3\|x\|^2 + c_4\gamma\|x\|^2$$

If γ is small enough to satisfy the bound

$$\gamma < \frac{c_3}{c_4} \tag{9.7}$$

then

$$\dot{V}(t,x) \le -(c_3 - \gamma c_4)\|x\|^2, \quad (c_3 - \gamma c_4) > 0$$

Therefore, by Theorem 4.10, we conclude the next lemma.

Lemma 9.1 *Let $x = 0$ be an exponentially stable equilibrium point of the nominal system* (9.2). *Let $V(t,x)$ be a Lyapunov function of the nominal system that satisfies* (9.3) *through* (9.5) *in* $[0,\infty) \times D$. *Suppose the perturbation term $g(t,x)$ satisfies* (9.6) *and* (9.7). *Then, the origin is an exponentially stable equilibrium point of the perturbed system* (9.1). *Moreover, if all the assumptions hold globally, then the origin is globally exponentially stable.* ◇

This lemma is conceptually important because it shows that exponential stability of the origin is robust with respect to a class of perturbations that satisfy (9.6) and (9.7). To assert this robustness property, we do not have to know $V(t,x)$ explicitly. It is just enough to know that the origin is an exponentially stable equilibrium of the nominal system. Sometimes, we may be able to show that the origin is exponentially stable without actually finding a Lyapunov function that satisfies (9.3) through (9.5).[2] Irrespective of the method we use to show exponential stability of the origin, we can assert the existence of $V(t,x)$ satisfying (9.3) through (9.5) by application of Theorem 4.14 (provided the Jacobian matrix $[\partial f/\partial x]$ is bounded). However, if we do not know the Lyapunov function $V(t,x)$ we cannot calculate the bound of (9.7). Consequently, our robustness conclusion becomes a qualitative one where we say that the origin is exponentially stable for all perturbations satisfying

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[2] This is the case, for example, when exponential stability of the origin is shown using Theorem 8.5.

$$\|g(t,x)\| \leq \gamma\|x\|$$

with sufficiently small γ. On the other hand, if we know $V(t,x)$, we can calculate the bound of (9.7), which is an additional piece of information. We should be careful not to overemphasize the usefulness of such bounds because they could be conservative for a given perturbation $g(t,x)$. The conservatism is a consequence of the worst case analysis we have adopted from the beginning.

Example 9.1 Consider the system

$$\dot{x} = Ax + g(t,x)$$

where A is Hurwitz and $\|g(t,x)\|_2 \leq \gamma\|x\|_2$ for all $t \geq 0$ and all $x \in R^n$. Let $Q = Q^T > 0$ and solve the Lyapunov equation

$$PA + A^TP = -Q$$

for P. From Theorem 4.6, we know that there is a unique solution $P = P^T > 0$. The quadratic Lyapunov function $V(x) = x^TPx$ satisfies (9.3) through (9.5). In particular,

$$\begin{aligned}
\lambda_{\min}(P)\|x\|_2^2 \leq V(x) &\leq \lambda_{\max}(P)\|x\|_2^2 \\
\frac{\partial V}{\partial x}Ax = -x^TQx &\leq -\lambda_{\min}(Q)\|x\|_2^2 \\
\left\|\frac{\partial V}{\partial x}\right\|_2 = \|2x^TP\|_2 \leq 2\|P\|_2\|x\|_2 &= 2\lambda_{\max}(P)\|x\|_2
\end{aligned}$$

The derivative of $V(x)$ along the trajectories of the perturbed system satisfies

$$\dot{V}(x) \leq -\lambda_{\min}(Q)\|x\|_2^2 + 2\lambda_{\max}(P)\gamma\|x\|_2^2$$

Hence, the origin is globally exponentially stable if $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(P)$. Since this bound depends on the choice of Q, one may wonder how to choose Q to maximize the ratio $\lambda_{\min}(Q)/\lambda_{\max}(P)$. It turns out that this ratio is maximized with the choice $Q = I$ (Exercise 9.1). △

Example 9.2 Consider the second-order system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3
\end{aligned}$$

where the constant $\beta \geq 0$ is unknown. We view the system as a perturbed system of the form (9.1) with

$$f(x) = Ax = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$$

The eigenvalues of A are $-1 \pm j\sqrt{3}$. Hence, A is Hurwitz. The solution of the Lyapunov equation

$$PA + A^T P = -I$$

is given by

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

As we saw in Example 9.1, the Lyapunov function $V(x) = x^T P x$ satisfies inequalities (9.3) through (9.5) with $c_3 = 1$ and

$$c_4 = 2\lambda_{\max}(P) = 2 \times 1.513 = 3.026$$

The perturbation term $g(x)$ satisfies

$$\|g(x)\|_2 = \beta|x_2|^3 \le \beta k_2^2 |x_2| \le \beta k_2^2 \|x\|_2$$

for all $|x_2| \le k_2$. At this point in the analysis, we do not know a bound on $x_2(t)$, although we know that $x_2(t)$ will be bounded whenever the trajectory $x(t)$ is confined to a compact set. We keep k_2 undetermined and proceed with the analysis. Using $V(x)$ as a Lyapunov function candidate for the perturbed system, we obtain

$$\dot{V}(x) \le -\|x\|_2^2 + 3.026\beta k_2^2 \|x\|_2^2$$

Hence, $\dot{V}(x)$ will be negative definite if

$$\beta < \frac{1}{3.026 k_2^2}$$

To estimate the bound k_2, let $\Omega_c = \{x \in R^2 \mid V(x) \le c\}$. For any positive constant c, the set Ω_c is closed and bounded. The boundary of Ω_c is the Lyapunov surface

$$V(x) = \tfrac{3}{2}x_1^2 + \tfrac{1}{4}x_1 x_2 + \tfrac{5}{16}x_2^2 = c$$

The largest value of $|x_2|$ on the surface $V(x) = c$ can be determined by differentiating the surface equation partially with respect to x_1. This results in

$$3x_1 + \tfrac{1}{4}x_2 = 0$$

Therefore, the extreme values of x_2 are obtained at the intersection of the line $x_1 = -x_2/12$ with the Lyapunov surface. Simple calculations show that the largest value of x_2^2 on the Lyapunov surface is $96c/29$. Thus, all points inside Ω_c satisfy the bound

$$|x_2| \le k_2, \quad \text{where} \quad k_2^2 = \frac{96c}{29}$$

Therefore, if

$$\beta < \frac{29}{3.026 \times 96c} \approx \frac{0.1}{c}$$

$\dot{V}(x)$ will be negative definite in Ω_c and we can conclude that the origin $x = 0$ is exponentially stable with Ω_c as an estimate of the region of attraction. The inequality $\beta < 0.1/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on β. The smaller the upper bound on β, the larger the estimate of the region of attraction. This tradeoff is not artificial; it does exist in this example. The change of variables

$$\begin{aligned} z_1 &= \sqrt{\frac{3\beta}{2}}x_2 \\ z_2 &= \sqrt{\frac{3\beta}{8}}(4x_1 + 2x_2 - \beta x_2^3) = -\sqrt{\frac{3\beta}{8}}\dot{x}_2 \\ \tau &= 2t \end{aligned}$$

transforms the state equation into

$$\begin{aligned} \frac{dz_1}{d\tau} &= -z_2 \\ \frac{dz_2}{d\tau} &= z_1 + (z_1^2 - 1)z_2 \end{aligned}$$

which was shown in Example 8.5 to have a bounded region of attraction surrounded by an unstable limit cycle. When transformed into the x-coordinates, the region of attraction will expand with decreasing β and shrink with increasing β. Finally, let us use this example to illustrate our remarks on the conservative nature of the bound of (9.7). Using this bound, we came up with the inequality $\beta < 1/3.026k_2^2$, which allows the perturbation term $g(t,x)$ to be any second-order vector that satisfies $\|g(t,x)\|_2 \le \beta k_2^2 \|x\|_2$. This class of perturbations is more general than the perturbation we have in this specific problem. We have a *structured perturbation* in the sense that the first component of g is always zero, while our analysis allowed for an *unstructured perturbation* where the vector g could change in all directions. Such disregard of the structure of the perturbation will, in general, lead to conservative bounds. Suppose we repeat the analysis, this time taking into consideration the structure of the perturbation. Instead of using the general bound of (9.7), we calculate the derivative of $V(t,x)$ along the trajectories of the perturbed system to obtain

$$\begin{aligned} \dot{V}(x) &= -\|x\|_2^2 + 2x^T P g(x) \\ &= -\|x\|_2^2 + 2\beta x_2^2 \left(\tfrac{1}{8}x_1 x_2 + \tfrac{5}{16}x_2^2\right) \\ &\le -\|x\|_2^2 + 2\beta x_2^2 \left(\tfrac{1}{16}\|x\|_2^2 + \tfrac{5}{16}\|x\|_2^2\right) \\ &\le -\|x\|_2^2 + \tfrac{3}{4}\beta k_2^2 \|x\|_2^2 \end{aligned}$$

Hence, $\dot{V}(x)$ is negative definite for $\beta < 4/3k_2^2$. Using, again, the fact that for all $x \in \Omega_c$, $|x_2|^2 \le k_2^2 = 96c/29$, we arrive at the bound $\beta < 0.4/c$, which is four times the bound we obtained by using (9.7). △

When the origin of the nominal system (9.2) is uniformly asymptotically stable, but not exponentially stable, the stability analysis of the perturbed system is more involved. Suppose the nominal system has a positive definite, decrescent Lyapunov function $V(t, x)$ that satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $(t, x) \in [0, \infty) \times D$, where $W_3(x)$ is positive definite and continuous. The derivative of V along the trajectories of (9.1) is given by

$$\begin{aligned} \dot{V}(t, x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) + \frac{\partial V}{\partial x} g(t, x) \\ &\leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t, x) \right\| \end{aligned}$$

Our task now is to show that

$$\left\| \frac{\partial V}{\partial x} g(t, x) \right\| < W_3(x)$$

for all $(t, x) \in [0, \infty) \times D$, a task that cannot be done by putting a simple order of magnitude bound on $\|g(t, x)\|$, as we have done in the exponential stability case. The growth bound on $\|g(t, x)\|$ will depend on the nature of the Lyapunov function of the nominal system. One class of Lyapunov functions for which the analysis is almost as simple as in exponential stability is the case when $V(t, x)$ is positive definite, decrescent, and satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3 \phi^2(x) \tag{9.8}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x) \tag{9.9}$$

for all $(t, x) \in [0, \infty) \times D$ for some positive constants c_3 and c_4, where $\phi : R^n \rightarrow R$ is positive definite and continuous. A Lyapunov function satisfying (9.8) and (9.9) is usually called a *quadratic-type* Lyapunov function. It is clear that a Lyapunov function satisfying (9.3) through (9.5) is quadratic type, but a quadratic-type Lyapunov function may exist even when the origin is not exponentially stable. We will illustrate this point shortly by an example. If the nominal system (9.2) has a quadratic-type Lyapunov function $V(t, x)$, then its derivative along the trajectories of (9.1) satisfies

$$\dot{V}(t, x) \leq -c_3 \phi^2(x) + c_4 \phi(x) \|g(t, x)\|$$

Suppose now that the perturbation term satisfies the bound

$$\|g(t, x)\| \leq \gamma \phi(x), \quad \gamma < \frac{c_3}{c_4}$$

Then,

$$\dot{V}(t,x) \le -(c_3 - c_4\gamma)\phi^2(x)$$

which shows that $\dot{V}(t,x)$ is negative definite.

Example 9.3 Consider the scalar system

$$\dot{x} = -x^3 + g(t,x)$$

The nominal system

$$\dot{x} = -x^3$$

has a globally asymptotically stable equilibrium point at the origin, but, as we saw in Example 4.23, the origin is not exponentially stable. Thus, there is no Lyapunov function that satisfies (9.3) through (9.5). The Lyapunov function $V(x) = x^4$ satisfies (9.8) and (9.9), with $\phi(x) = |x|^3$, $c_3 = 4$, and $c_4 = 4$. Suppose the perturbation term $g(t,x)$ satisfies the bound $|g(t,x)| \le \gamma|x|^3$ for all x, with $\gamma < 1$. Then, the derivative of V along the trajectories of the perturbed system satisfies

$$\dot{V}(t,x) \le -4(1-\gamma)\phi^2(x)$$

Hence, the origin is a globally uniformly asymptotically stable equilibrium point of the perturbed system. △

In contrast to the case of exponential stability, it is important to notice that a nominal system with uniformly asymptotically stable, but not exponentially stable, origin is not robust to smooth perturbations with arbitrarily small linear growth bounds of the form of (9.6). This point is illustrated by the next example.[3]

Example 9.4 Consider the scalar system of the previous example with perturbation $g = \gamma x$ where $\gamma > 0$; that is,

$$\dot{x} = -x^3 + \gamma x$$

It can be easily seen, via linearization, that for any $\gamma > 0$ the origin is unstable, no matter how small γ is. △

9.2 Nonvanishing Perturbation

Let us turn now to the more general case when we do not know that $g(t,0) = 0$. The origin $x = 0$ may not be an equilibrium point of the perturbed system (9.1). We can no longer study stability of the origin as an equilibrium point, nor should we expect the solution of the perturbed system to approach the origin as $t \to \infty$. The best we can hope for is that $x(t)$ will be ultimately bounded by a small bound, if the perturbation term $g(t,x)$ is small in some sense. We start with the case when the origin of the nominal system (9.2) is exponentially stable.

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[3] See, also, Exercise 9.7.

Lemma 9.2 *Let $x = 0$ be an exponentially stable equilibrium point of the nominal system* (9.2). *Let $V(t, x)$ be a Lyapunov function of the nominal system that satisfies* (9.3) *through* (9.5) *in $[0, \infty) \times D$, where $D = \{x \in R^n \mid \|x\| < r\}$. Suppose the perturbation term $g(t, x)$ satisfies*

$$\|g(t, x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r \tag{9.10}$$

for all $t \geq 0$, all $x \in D$, and some positive constant $\theta < 1$. Then, for all $\|x(t_0)\| < \sqrt{c_1/c_2}r$, the solution $x(t)$ of the perturbed system (9.1) *satisfies*

$$\|x(t)\| \leq k \exp[-\gamma(t - t_0)]\|x(t_0)\|, \quad \forall\ t_0 \leq t < t_0 + T$$

and

$$\|x(t)\| \leq b, \quad \forall\ t \geq t_0 + T$$

for some finite T, where

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1 - \theta)c_3}{2c_2}, \quad b = \frac{c_4}{c_3}\sqrt{\frac{c_2}{c_1}}\frac{\delta}{\theta}$$

◇

Proof: We use $V(t, x)$ as a Lyapunov function candidate for the perturbed system (9.1). The derivative of $V(t, x)$ along the trajectories of (9.1) satisfies

$$\begin{aligned}
\dot{V}(t, x) &\leq -c_3\|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(t, x)\| \\
&\leq -c_3\|x\|^2 + c_4\delta\|x\| \\
&= -(1 - \theta)c_3\|x\|^2 - \theta c_3\|x\|^2 + c_4\delta\|x\|, \quad 0 < \theta < 1 \\
&\leq -(1 - \theta)c_3\|x\|^2, \quad \forall\ \|x\| \geq \delta c_4/\theta c_3
\end{aligned}$$

Applying Theorem 4.18 and Exercise 4.51 completes the proof. □

Note that the ultimate bound b in Lemma 9.2 is proportional to the upper bound on the perturbation δ. Once again, this result can be viewed as a robustness property of nominal systems having exponentially stable equilibria at the origin, because it shows that arbitrarily small (uniformly bounded) perturbations will not result in large steady-state deviations from the origin.

Example 9.5 Consider the second-order system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3 + d(t)
\end{aligned}$$

where $\beta \geq 0$ is unknown and $d(t)$ is a uniformly bounded disturbance that satisfies $|d(t)| \leq \delta$ for all $t \geq 0$. This is the same system we studied in Example 9.2, except for the additional perturbation term $d(t)$. Again, the system can be viewed as a perturbation of a nominal linear system that has a Lyapunov function $V(x) = x^TPx$, where

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

We use $V(x)$ as a Lyapunov function candidate for the perturbed system, but we treat the two perturbation terms βx_2^3 and $d(t)$ differently, since the first term vanishes at the origin while the second one does not. Calculating the derivative of $V(x)$ along the trajectories of the perturbed system, we obtain

$$\begin{aligned} \dot{V}(t,x) &= -\|x\|_2^2 + 2\beta x_2^2\left(\tfrac{1}{8}x_1x_2 + \tfrac{5}{16}x_2^2\right) + 2d(t)\left(\tfrac{1}{8}x_1 + \tfrac{5}{16}x_2\right) \\ &\le -\|x\|_2^2 + \tfrac{3}{4}\beta k_2^2\|x\|_2^2 + \frac{\sqrt{29}\delta}{8}\|x\|_2 \end{aligned}$$

where we have used the inequality

$$|2x_1 + 5x_2| \le \|x\|_2\sqrt{4+25}$$

and k_2 is an upper bound on $|x_2|$. Suppose $\beta \le 4(1-\zeta)/3k_2^2$, where $0 < \zeta < 1$. Then,

$$\dot{V}(t,x) \le -\zeta\|x\|_2^2 + \frac{\sqrt{29}\delta}{8}\|x\|_2 \le -(1-\theta)\zeta\|x\|_2^2, \quad \forall\ \|x\|_2 \ge \mu = \frac{\sqrt{29}\delta}{8\zeta\theta}$$

where $0 < \theta < 1$. As we saw in Example 9.2, $|x_2|^2$ is bounded on Ω_c by $96c/29$. Thus, if $\beta \le 0.4(1-\zeta)/c$ and δ is so small that $\mu^2\lambda_{\max}(P) < c$, then $B_\mu \subset \Omega_c$ and all trajectories starting inside Ω_c remain for all future time in Ω_c. Furthermore, the conditions of Theorem 4.18 are satisfied in Ω_c. Therefore, the solutions of the perturbed system are uniformly ultimately bounded by

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

△

In the more general case when the origin $x = 0$ is a uniformly asymptotically stable equilibrium point of the nominal system (9.2), rather than exponentially stable, the analysis of the perturbed system proceeds in a similar manner.

Lemma 9.3 *Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system* (9.2). *Let $V(t,x)$ be a Lyapunov function of the nominal system that satisfies the inequalities*[4]

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|) \tag{9.11}$$

---

[4] The existence of a Lyapunov function satisfying these inequalities (on a bounded domain) is guaranteed by Theorem 4.16 under some additional assumptions.

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(\|x\|) \tag{9.12}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le \alpha_4(\|x\|) \tag{9.13}$$

in $[0,\infty) \times D$, *where* $D = \{x \in R^n \mid \|x\| < r\}$ *and* $\alpha_i(\cdot)$, $i = 1,2,3,4$, *are class* $\mathcal{K}$ *functions. Suppose the perturbation term* $g(t,x)$ *satisfies the uniform bound*

$$\|g(t,x)\| \le \delta < \frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)} \tag{9.14}$$

for all $t \ge 0$, *all* $x \in D$, *and some positive constant* $\theta < 1$. *Then, for all* $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, *the solution* $x(t)$ *of the perturbed system* (9.1) *satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t_0 \le t < t_0 + T$$

and

$$\|x(t)\| \le \rho(\delta), \quad \forall\ t \ge t_0 + T$$

for some class $\mathcal{KL}$ *function* β *and some finite* T, *where* ρ *is a class* $\mathcal{K}$ *function of* δ *defined by*

$$\rho(\delta) = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{\delta\alpha_4(r)}{\theta}\right)\right)\right)$$

◇

Proof: We use $V(t,x)$ as a Lyapunov function candidate for the perturbed system (9.1). The derivative of $V(t,x)$ along the trajectories of (9.1) satisfies

$$\begin{aligned}
\dot{V}(t,x) &\le -\alpha_3(\|x\|) + \left\| \frac{\partial V}{\partial x} \right\| \ \|g(t,x)\| \\
&\le -\alpha_3(\|x\|) + \delta\alpha_4(\|x\|) \\
&\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \delta\alpha_4(r), \quad 0 < \theta < 1 \\
&\le -(1-\theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \ge \alpha_3^{-1}\left(\frac{\delta\alpha_4(r)}{\theta}\right)
\end{aligned}$$

Applying Theorem 4.18 completes the proof. □

This lemma is similar to the one we arrived at in the special case of exponential stability. However, there is an important feature of our analysis in the case of exponential stability, which has no counterpart in the more general case of uniform asymptotic stability. In the case of exponential stability, δ is required to satisfy (9.10). The right-hand side of (9.10) approaches ∞ as $r \to \infty$. Therefore, if the assumptions hold globally, we can conclude that *for all uniformly bounded disturbances, the solution of the perturbed system will be uniformly bounded.* This is the case because, for any δ, we can choose r large enough to satisfy (9.10). In the case of uniform asymptotic stability, δ is required to satisfy (9.14). Inspection of (9.14) shows that, without further information about the class $\mathcal{K}$ functions, we cannot say anything about the limit of the right-hand side as $r \to \infty$. Thus, we cannot conclude that uniformly bounded perturbations of a nominal system with a uniformly asymptotically stable equilibrium at the origin will have bounded solutions irrespective of the size of the perturbation. Of course the fact that we cannot show it, does not mean it is not true. It turns out, however, that such a statement is not true. It is possible to construct examples (Exercise 9.13) where the origin is globally uniformly asymptotically stable, but a bounded perturbation could drive the solution of the perturbed system to infinity.

9.3 Comparison Method

Consider the perturbed system (9.1). Let $V(t, x)$ be a Lyapunov function for the nominal system (9.2) and suppose the derivative of V along the trajectories of (9.1) satisfies the differential inequality

$$\dot{V} \leq h(t, V)$$

By (the comparison) Lemma 3.4,

$$V(t, x(t)) \leq y(t)$$

where $y(t)$ is the solution of the differential equation

$$\dot{y} = h(t, y), \quad y(t_0) = V(t_0, x(t_0))$$

This approach is particularly useful when the differential inequality is linear, that is, when $h(t, V) = a(t)V + b(t)$, for then we can write down a closed-form expression for the solution of the first-order linear differential equation of y. Arriving at a linear differential inequality is possible when the origin of the nominal system (9.2) is exponentially stable.

Let $V(t, x)$ be a Lyapunov function of the nominal system (9.2) that satisfies (9.3) through (9.5) for all $(t, x) \in [0, \infty) \times D$, where $D = \{x \in R^n \mid \|x\| < r\}$. Suppose the perturbation term $g(t, x)$ satisfies the bound

$$\|g(t, x)\| \leq \gamma(t)\|x\| + \delta(t), \quad \forall\, t \geq 0, \ \forall\, x \in D \tag{9.15}$$

where $\gamma : R \to R$ is nonnegative and continuous for all $t \geq 0$, and $\delta : R \to R$ is nonnegative, continuous, and bounded for all $t \geq 0$. The derivative of V along the trajectories of (9.1) satisfies

$$\begin{aligned} \dot{V}(t, x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) + \frac{\partial V}{\partial x} g(t, x) \\ &\leq -c_3\|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \, \|g(t, x)\| \\ &\leq -c_3\|x\|^2 + c_4\gamma(t)\|x\|^2 + c_4\delta(t)\|x\| \qquad (9.16) \end{aligned}$$

Using (9.3), we can find an upper bound on $\dot{V}$ as

$$\dot{V} \leq -\left[\frac{c_3}{c_2} - \frac{c_4}{c_1}\gamma(t)\right] V + c_4\delta(t)\sqrt{\frac{V}{c_1}}$$

To obtain a linear differential inequality, we take $W(t) = \sqrt{V(t,x(t))}$ and use the fact $\dot{W} = \dot{V}/2\sqrt{V}$, when $V \neq 0$, to obtain

$$\dot{W} \leq -\frac{1}{2}\left[\frac{c_3}{c_2} - \frac{c_4}{c_1}\gamma(t)\right] W + \frac{c_4}{2\sqrt{c_1}}\delta(t) \tag{9.17}$$

When $V = 0$, it can be shown[5] that $D^+W(t) \leq c_4\delta(t)/2\sqrt{c_1}$. Hence, $D^+W(t)$ satisfies (9.17) for all values of V. By the comparison lemma, $W(t)$ satisfies the inequality

$$W(t) \leq \phi(t,t_0)W(t_0) + \frac{c_4}{2\sqrt{c_1}}\int_{t_0}^{t} \phi(t,\tau)\delta(\tau)\; d\tau \tag{9.18}$$

where the transition function $\phi(t,t_0)$ is given by

$$\phi(t,t_0) = \exp\left[-\frac{c_3}{2c_2}(t-t_0) + \frac{c_4}{2c_1}\int_{t_0}^{t}\gamma(\tau)\; d\tau\right]$$

Using (9.3) in (9.18), we obtain

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}}\phi(t,t_0)\|x(t_0)\| + \frac{c_4}{2c_1}\int_{t_0}^{t}\phi(t,\tau)\delta(\tau)\; d\tau \tag{9.19}$$

Suppose now that $\gamma(t)$ satisfies the condition

$$\int_{t_0}^{t}\gamma(\tau)\; d\tau \leq \varepsilon(t-t_0) + \eta \tag{9.20}$$

for some nonnegative constants ε and η, where

$$\varepsilon < \frac{c_1c_3}{c_2c_4} \tag{9.21}$$

Defining the constants α and ρ by

$$\alpha = \frac{1}{2}\left[\frac{c_3}{c_2} - \varepsilon\frac{c_4}{c_1}\right] > 0, \quad \rho = \exp\left(\frac{c_4\eta}{2c_1}\right) \geq 1 \tag{9.22}$$

and using (9.20) and (9.21) in (9.19), we obtain

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}}\rho\|x(t_0)\|e^{-\alpha(t-t_0)} + \frac{c_4\rho}{2c_1}\int_{t_0}^{t} e^{-\alpha(t-\tau)}\delta(\tau)\; d\tau \tag{9.23}$$

---

[5] See Exercise 9.14.

For this bound to be valid, we must ensure that $\|x(t)\| < r$ for all $t \geq t_0$. Noting that[6]

$$\begin{aligned} \|x(t)\| &\leq \sqrt{\frac{c_2}{c_1}}\rho\|x(t_0)\|e^{-\alpha(t-t_0)} + \frac{c_4\rho}{2\alpha c_1}\left[1 - e^{-\alpha(t-t_0)}\right] \sup_{t\geq t_0} \delta(t) \\ &\leq \max\left\{\sqrt{\frac{c_2}{c_1}}\rho\|x(t_0)\|, \ \frac{c_4\rho}{2\alpha c_1} \sup_{t\geq t_0} \delta(t)\right\} \end{aligned}$$

we see that the condition $\|x(t)\| < r$ will be satisfied if

$$\|x(t_0)\| < \frac{r}{\rho}\sqrt{\frac{c_1}{c_2}} \tag{9.24}$$

and

$$\sup_{t\geq t_0} \delta(t) < \frac{2c_1\alpha r}{c_4\rho} \tag{9.25}$$

For easy reference, we summarize our findings in the next lemma.

Lemma 9.4 *Let $x = 0$ be an exponentially stable equilibrium point of the nominal system* (9.2). *Let $V(t, x)$ be a Lyapunov function of the nominal system that satisfies* (9.3) *through* (9.5) *in* $[0, \infty) \times D$, *where* $D = \{x \in R^n \mid \|x\|_2 < r\}$. *Suppose the perturbation term $g(t, x)$ satisfies* (9.15), *where $\gamma(t)$ satisfies* (9.20) *and* (9.21). *Then, provided $x(t_0)$ satisfies* (9.24) *and* $\sup_{t\geq t_0} \delta(t)$ *satisfies* (9.25), *the solution of the perturbed system* (9.1) *satisfies* (9.23). *Furthermore, if all the assumptions hold globally, then* (9.23) *is satisfied for any $x(t_0)$ and any bounded $\delta(t)$.* ◇

Specializing the foregoing lemma to the case of vanishing perturbations; that is, when $\delta(t) \equiv 0$, we obtain the following result:

Corollary 9.1 *Let $x = 0$ be an exponentially stable equilibrium point of the nominal system* (9.2). *Let $V(t, x)$ be a Lyapunov function of the nominal system that satisfies* (9.3) *through* (9.5) *in* $[0, \infty) \times D$. *Suppose the perturbation term $g(t, x)$ satisfies*

$$\|g(t, x)\| \leq \gamma(t)\|x\|$$

where $\gamma(t)$ satisfies (9.20) *and* (9.21). *Then, the origin is an exponentially stable equilibrium point of the perturbed system* (9.1). *Moreover, if all the assumptions hold globally, then the origin is globally exponentially stable.* ◇

If $\gamma(t) \equiv \gamma =$ constant, then Corollary 9.1 requires γ to satisfy the bound $\gamma < c_1c_3/c_2c_4$, which has no advantage over the bound $\gamma < c_3/c_4$ required by Lemma 9.1, since $(c_1/c_2) \leq 1$. In fact, whenever $(c_1/c_2) < 1$, the current bound will be more conservative (that is, smaller) than the bound required by Lemma 9.1. The advantage of Corollary 9.1 is seen in the case when the integral of $\gamma(t)$ satisfies conditions (9.20) and (9.21), even when $\sup_{t\geq t_0} \gamma(t)$ is not small enough to satisfy $\sup_{t\geq t_0} \gamma(t) < c_3/c_4$. Three such cases are given in the next lemma.

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[6] We use the fact that the function $ae^{-\alpha t} + b(1 - e^{-\alpha t})$, with positive a, b, and α, relaxes monotonically from its initial value a to its final value b. Hence, it is bounded by the maximum of the two numbers.

Lemma 9.5

1. *If*
$$\int_0^\infty \gamma(\tau)\ d\tau \le k$$
then (9.20) *is satisfied with* $\varepsilon = 0$ *and* $\eta = k$.

2. *If*
$$\gamma(t) \to 0 \quad \text{as} \quad t \to \infty$$
then for any $\varepsilon > 0$, *there is* $\eta = \eta(\varepsilon) > 0$ *such that* (9.20) *is satisfied.*

3. *If there are constants* $\Delta > 0$, $T \ge 0$, *and* $\varepsilon_1 > 0$ *such that*
$$\frac{1}{\Delta}\int_t^{t+\Delta} \gamma(\tau)\ d\tau \le \varepsilon_1, \quad \forall\, t \ge T$$
then (9.20) *is satisfied with* $\varepsilon = \varepsilon_1$ *and* $\eta = \varepsilon_1\Delta + \int_0^T \gamma(t)\ dt$. ◇

Proof: The first case is obvious. To prove the second case, note that, because $\lim_{t\to\infty}\gamma(t) = 0$, for any $\varepsilon > 0$, there is $T_1 = T_1(\varepsilon) > 0$ such that $\gamma(t) < \varepsilon$ for all $t \ge T_1$. Let $\eta = \int_0^{T_1} \gamma(t)\ dt$. If $t_0 \ge T_1$, then
$$\int_{t_0}^t \gamma(\tau)\ d\tau \le \int_{t_0}^t \varepsilon\ d\tau = \varepsilon(t - t_0)$$
If $t \le T_1$, then
$$\int_{t_0}^t \gamma(\tau)\ d\tau \le \int_0^{T_1} \gamma(\tau)\ d\tau = \eta$$
If $t_0 \le T_1 \le t$, then
$$\begin{aligned}\int_{t_0}^t \gamma(\tau)\ d\tau &= \int_{t_0}^{T_1} \gamma(\tau)\ d\tau + \int_{T_1}^t \gamma(\tau)\ d\tau \\ &\le \int_0^{T_1} \gamma(\tau)\ d\tau + \varepsilon(t - T_1) \ \le\ \eta + \varepsilon(t - t_0)\end{aligned}$$
In the last case, if $t \le T$, then
$$\int_{t_0}^t \gamma(\tau)\ d\tau \le \int_0^T \gamma(\tau)\ d\tau < \eta$$
For $t \ge t_1 \ge T$, let N be the integer for which $(N-1)\Delta \le t - t_1 \le N\Delta$. Then,
$$\begin{aligned}\int_{t_1}^t \gamma(\tau)\ d\tau &= \sum_{i=0}^{i=N-2} \int_{t_1+i\Delta}^{t_1+(i+1)\Delta} \gamma(\tau)\ d\tau + \int_{t_1+(N-1)\Delta}^t \gamma(\tau)\ d\tau \\ &\le \sum_{i=0}^{i=N-2} \varepsilon_1\Delta + \varepsilon_1\Delta \ \le\ \varepsilon_1(t - t_1) + \varepsilon_1\Delta\end{aligned}$$

This inequality is used next with $t_1 = t_0$ when $t \geq t_0 \geq T$, and with $t_1 = T$ when $t_0 \leq T \leq t$. If $t \geq t_0 \geq T$, then

$$\int_{t_0}^{t} \gamma(\tau)\ d\tau \leq \varepsilon_1(t - t_0) + \varepsilon_1\Delta \ < \ \varepsilon_1(t - t_0) + \eta$$

while if $t_0 \leq T \leq t$, then

$$\begin{aligned} \int_{t_0}^{t} \gamma(\tau)\ d\tau &= \int_{t_0}^{T} \gamma(\tau)\ d\tau + \int_{T}^{t} \gamma(\tau)\ d\tau \\ &\leq \int_{0}^{T} \gamma(\tau)\ d\tau + \varepsilon_1(t - T) + \varepsilon_1\Delta \leq \varepsilon_1(t - t_0) + \eta \end{aligned}$$

□

In the first case of the foregoing lemma, condition (9.20) is satisfied with $\varepsilon = 0$, while in the second case, it is satisfied with arbitrarily small ε. Therefore, in both cases, condition (9.21) is always satisfied and the origin of the perturbed system (9.2) is exponentially stable. The third case of the lemma sets a bound on a *moving average* of $\gamma(t)$ as t becomes sufficiently large. The origin of the perturbed system (9.2) will be exponentially stable if this bound is sufficiently small.

Example 9.6 Consider the linear system

$$\dot{x} = [A(t) + B(t)]x$$

where $A(t)$ and $B(t)$ are continuous and $A(t)$ is bounded on $[0, \infty)$. Suppose the origin is an exponentially stable equilibrium point of the nominal system

$$\dot{x} = A(t)x$$

and

$$B(t) \to 0 \text{ as } t \to \infty$$

From Theorem 4.12, we know that there is a quadratic Lyapunov function $V(t, x) = x^T P(t)x$ that satisfies (9.3) through (9.5) globally. The perturbation term $B(t)x$ satisfies the inequality

$$\|B(t)x\| \leq \|B(t)\|\ \|x\|$$

Since $\|B(t)\| \to 0$ as $t \to \infty$, we conclude from Corollary 9.1 and the second case of Lemma 9.5 that the origin is a globally exponentially stable equilibrium point of the perturbed system. △

Similar conclusions can be drawn when $\int_0^\infty \|B(t)\|\ dt < \infty$ (Exercise 9.15) and $\int_0^\infty \|B(t)\|^2\ dt < \infty$ (Exercise 9.16).

In the case of nonvanishing perturbations, that is, when $\delta(t) \not\equiv 0$, the next lemma states a number of conclusions concerning the asymptotic behavior of $x(t)$ as $t \to \infty$.

Lemma 9.6 *Suppose the conditions of Lemma 9.4 are satisfied, and let $x(t)$ denote the solution of the perturbed system (9.1).*

1. *If*

$$\int_{t_0}^{t} e^{-\alpha(t-\tau)}\delta(\tau)\, d\tau \leq \beta, \quad \forall\, t \geq t_0$$

for some positive constant β, then $x(t)$ is uniformly ultimately bounded with the ultimate bound

$$b = \frac{c_4\rho\beta}{2c_1\theta}$$

where $\theta \in (0,1)$ is an arbitrary constant.

2. *If*

$$\lim_{t\to\infty} \delta(t) = \delta_\infty > 0$$

then $x(t)$ is uniformly ultimately bounded with the ultimate bound

$$b = \frac{c_4\rho\delta_\infty}{2\alpha c_1\theta}$$

where $\theta \in (0,1)$ is an arbitrary constant.

3. *If*

$$\lim_{t\to\infty} \delta(t) = 0, \quad \text{then} \quad \lim_{t\to\infty} x(t) = 0$$

If the conditions of Lemma 9.4 are satisfied globally, then the foregoing statements hold for any initial state $x(t_0)$. ◇

Proof: All three cases follow easily from inequality (9.23). In the first two cases, we use the property that if $u(t) = w(t) + a$ with $a > 0$ and $\lim_{t\to\infty} w(t) = 0$, then $u(t)$ is ultimately bounded by a/θ for any positive $\theta < 1$. This is so because there is a finite time T such that $|w(t)| \leq a(1-\theta)/\theta$ for all $t \geq T$. In the last two cases, we use the property that if $u(t) = \int_{t_0}^{t} \exp(-\alpha(t-\tau))w(\tau)\, d\tau$ where $w(t)$ is bounded and $\lim_{t\to\infty} w(t) = w_\infty$, then $\lim_{t\to\infty} u(t) = w_\infty/\alpha$.[7] □

9.4 Continuity of Solutions on the Infinite Interval

In Section 3.2, we studied continuous dependence of the solution of the state equation on initial states and parameters. In particular, in Theorem 3.4, we examined the nominal system

$$\dot{x} = f(t,x) \tag{9.26}$$

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[7] See [33, Theorem 3.3.2.33].

and the perturbed system

$$\dot{x} = f(t, x) + g(t, x) \tag{9.27}$$

under the assumption that $\|g(t, x)\| \leq \delta$ in the domain of interest. Using the Gronwall–Bellman inequality, we found that if $y(t)$ and $z(t)$ are well-defined solutions of the nominal and perturbed systems, respectively, then

$$\|y(t) - z(t)\| \leq \|y(t_0) - z(t_0)\| \exp[L(t - t_0)] + \frac{\delta}{L}\{\exp[L(t - t_0)] - 1\} \tag{9.28}$$

where L is a Lipschitz constant for f. This bound is valid only on compact time intervals, since the exponential term $\exp[L(t - t_0)]$ grows unbounded as $t \to \infty$. In fact, the bound is useful only on an interval $[t_0, t_1]$ where t_1 is reasonably small, for if t_1 is large, the bound will be too large to be of any use. This is not surprising, because in Section 3.2, we did not impose any stability conditions on the system. In this section, we use Lemma 9.4 to calculate a bound on the error between the solutions of (9.26) and (9.27) that is valid uniformly in t for all $t \geq t_0$.

Theorem 9.1 *Let $D \subset R^n$ be domain that contains the origin and suppose*

- *$f(t, x)$ and its first partial derivatives with respect to x are continuous, bounded, and Lipschitz in x, uniformly in t, for all $(t, x) \in [0, \infty) \times D_0$, for every compact set $D_0 \subset D$;*
- *$g(t, x)$ is piecewise continuous in t, locally Lipschitz in x, and*

$$\|g(t, x)\| \leq \delta, \quad \forall\ (t, x) \in [0, \infty) \times D \tag{9.29}$$

- *the origin $x = 0$ is an exponentially stable equilibrium point of the nominal system* (9.26);
- *there is a Lyapunov function $V(t, x)$ that satisfies the conditions of Theorem* 4.9 *for the nominal system* (9.26) *for $(t, x) \in [0, \infty) \times D$ and $\{W_1(x) \leq c\}$ is a compact subset of D.*

Let $y(t)$ and $z(t)$ denote solutions of the nominal system (9.26) *and the perturbed system* (9.27), *respectively. Then, for each compact set $\Omega \subset \{W_2(x) \leq \rho c,\ 0 < \rho < 1\}$, there exist positive constants β, γ, η, μ, and k, independent of δ, such that if $y(t_0) \in \Omega$, $\delta < \eta$, and $\|z(t_0) - y(t_0)\| < \mu$, the solutions $y(t)$ and $z(t)$ will be uniformly bounded for all $t \geq t_0 \geq 0$ and*

$$\|z(t) - y(t)\| \leq k e^{-\gamma(t - t_0)} \|z(t_0) - y(t_0)\| + \beta\delta \tag{9.30}$$

◇

While the origin is exponentially stable, the Lyapunov function V is required to satisfy the conditions of uniform asymptotic stability, rather than (the more stringent) conditions of exponential stability. This provides less conservative estimates of the set Ω. When the nominal system (9.26) is autonomous, the function V is provided by (the converse Lyapunov) Theorem 4.17 and the set Ω can be any compact subset of the region of attraction. Exponential stability is used only locally when the error $z(t) - y(t)$ is sufficiently small.

Proof of Theorem 9.1: The derivative of V along the trajectories of the perturbed system (9.27) satisfies

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \le -W_3(x) + k_1 \delta$$

for all $x \in \{W_1(x) \le c\}$, where k_1 is an upper bound on $\partial V/\partial x$ over $\{W_1(x) \le c\}$. Let $k_2 > 0$ be the minimum of $W_3(x)$ over the compact set $\Lambda = \{W_1(x) \le c \text{ and } W_2(x) \ge c\}$. Then

$$\dot{V} \le -\tfrac{1}{2} W_3(x) - \tfrac{1}{2} k_2 + k_1 \delta \le -\tfrac{1}{2} W_3(x), \quad \forall\, x \in \Lambda, \ \forall\, \delta \le \frac{k_2}{2k_1}$$

This shows that $\dot{V}$ is negative on $V(t,x) = c$; hence the set $\{V(t,x) \le c\}$ is positively invariant. Therefore, for all $z(t_0) \in \{W_2(x) \le c\}$, the solution $z(t)$ of (9.27) is uniformly bounded. Since Ω is in the interior of $\{W_2(x) \le c\}$, there is $\mu_1 > 0$ such that $z(t_0) \in \{W_2(x) \le c\}$ whenever $y(t_0) \in \Omega$ and $\|z(t_0) - y(t_0)\| \le \mu_1$. It is also clear that for $y(t_0) \in \Omega$, $y(t)$ is uniformly bounded and $y(t) \to 0$ as $t \to \infty$, uniformly in t_0. The error $e(t) = z(t) - y(t)$ satisfies the equation

$$\dot{e} = \dot{z} - \dot{y} = f(t,z) + g(t,z) - f(t,y) = f(t,e) + \Delta(t,e) + g(t,z) \tag{9.31}$$

where

$$\Delta(t,e) = f(t, y(t) + e) - f(t, y(t)) - f(t,e)$$

We analyze the error equation (9.31) over the ball $\{\|e\| \le r\} \subset D$. Equation (9.31) can be viewed as a perturbation of the system

$$\dot{e} = f(t,e)$$

whose origin is exponentially stable. By Theorem 4.14, there exists a Lyapunov function $V(t,e)$ that satisfies (9.3) through (9.5) for $\|e\| < r_0 < r$. By the mean value theorem, the error term Δ_i can be written as

$$\Delta_i(t,e) = \left[\frac{\partial f_i}{\partial x}(t, \lambda_1 e + y) - \frac{\partial f_i}{\partial x}(t, \lambda_2 e) \right] e$$

where $0 < \lambda_i < 1$. Since the Jacobian matrix $[\partial f/\partial x]$ is Lipschitz in x, uniformly in t, the perturbation term $(\Delta + g)$ satisfies

$$\|\Delta(t,e) + g(t,z)\| \le L_1 \|e\|^2 + L_2 \|e\| \; \|y(t)\| + \delta$$

where $y(t) \to 0$ as $t \to \infty$, uniformly in t_0. Consequently,

$$\|\Delta(t, e) + g(t, z)\| \le \{L_1 r_1 + L_2 \|y(t)\|\} \, \|e\| + \delta$$

for all $\|e\| \le r_1 < r_0$. This inequality takes the form (9.15) with

$$\gamma(t) = \{L_1 r_1 + L_2 \|y(t)\|\} \quad \text{and} \quad \delta(t) \equiv \delta$$

Given any $\varepsilon_1 > 0$, there is $T_1 > 0$ such that $\|y(t)\| \le \varepsilon_1$ for all $t \ge t_0 + T_1$. Therefore, (9.20) is satisfied with

$$\int_{t_0}^{t} \gamma(\tau) \, d\tau \le (\varepsilon_1 + L_1 r_1)(t - t_0) + T_1 \max_{t \ge t_0} L_2 \|y(t)\|$$

By taking ε_1 and r_1 small enough, we can satisfy (9.21). Thus, all the assumptions of Lemma 9.4 are satisfied and (9.30) follows from (9.23). □

9.5 Interconnected Systems

When we analyze the stability of a nonlinear dynamical system, the complexity of the analysis grows rapidly as the order of the system increases. This situation motivates us to look for ways to simplify the analysis. If the system can be modeled as an interconnection of lower order subsystems, then we may pursue the stability analysis in two steps. In the first step, we decompose the system into smaller isolated subsystems by ignoring interconnections, and analyze the stability of each subsystem. In the second step, we combine our conclusions from the first step with information about the interconnections to draw conclusions about the stability of the interconnected system. In this section, we illustrate how this idea can be utilized in searching for Lyapunov functions for interconnected systems.

Consider the interconnected system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x), \quad i = 1, 2, \ldots, m \tag{9.32}$$

where $x_i \in R^{n_i}$, $n_1 + \cdots + n_m = n$, and $x = \left[x_1^T, \ldots, x_m^T\right]^T$. Suppose f_i and g_i are smooth enough to ensure local existence and uniqueness of the solution for all initial conditions in a domain of interest, and that

$$f_i(t, 0) = 0, \quad g_i(t, 0) = 0, \quad \forall \; i$$

so that the origin $x = 0$ is an equilibrium point of the system. Ignoring the interconnection terms g_i, the system decomposes into m isolated subsystems:

$$\dot{x}_i = f_i(t, x_i) \tag{9.33}$$

with each one having an equilibrium point at its origin $x_i = 0$. We start by searching for Lyapunov functions that establish uniform asymptotic stability of the origin for each isolated subsystem. Suppose this search has been successful and that, for each subsystem, we have a positive definite decrescent Lyapunov function $V_i(t, x_i)$ whose derivative along the trajectories of the isolated subsystem (9.33) is negative definite. The function

$$V(t,x) = \sum_{i=1}^{m} d_i V_i(t, x_i), \quad d_i > 0$$

is a *composite Lyapunov function* for the collection of the m isolated subsystems for all values of the positive constants d_i. Viewing the interconnected system (9.32) as a perturbation of the isolated subsystems (9.33), it is reasonable to try $V(t,x)$ as a Lyapunov function candidate for (9.32). The derivative of $V(t,x)$ along the trajectories of (9.32) is given by

$$\dot{V}(t,x) = \sum_{i=1}^{m} d_i \left[\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i)\right] + \sum_{i=1}^{m} d_i \frac{\partial V_i}{\partial x_i} g_i(t,x)$$

The first term on the right-hand side is negative definite by virtue of the fact that V_i is a Lyapunov function for the ith isolated subsystem, but the second term is, in general, indefinite. The situation is similar to our earlier investigation of perturbed systems in Section 9.1. Therefore, we may approach the problem by performing worst case analysis where the term $[\partial V_i/\partial x_i] g_i$ is bounded by a nonnegative upper bound. Let us illustrate the idea by using quadratic-type Lyapunov functions, introduced in Section 9.1. Suppose that, for $i = 1, 2, \ldots, m$, $V_i(t, x_i)$ satisfies

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i) \le -\alpha_i \phi_i^2(x_i) \tag{9.34}$$

$$\left\| \frac{\partial V_i}{\partial x_i} \right\| \le \beta_i \phi_i(x_i) \tag{9.35}$$

for all $t \ge 0$ and $\|x\| < r$ for some positive constants α_i and β_i, where $\phi_i : R^{n_i} \to R$ are positive definite and continuous. Furthermore, suppose that the interconnection terms $g_i(t,x)$ satisfy the bound

$$\|g_i(t,x)\| \le \sum_{j=1}^{m} \gamma_{ij} \phi_j(x_j) \tag{9.36}$$

for all $t \ge 0$ and $\|x\| < r$ for some nonnegative constants γ_{ij}. Then, the derivative of $V(t,x) = \sum_{i=1}^{m} d_i V_i(t, x_i)$ along the trajectories of the interconnected system (9.32) satisfies the inequality

$$\dot{V}(t,x) \le \sum_{i=1}^{m} d_i \left[-\alpha_i \phi_i^2(x_i) + \sum_{j=1}^{m} \beta_i \gamma_{ij} \phi_i(x_i) \phi_j(x_j) \right]$$

The right-hand side is a quadratic form in $\phi_1, \ldots, \phi_m$, which we rewrite as

$$\dot{V}(t,x) \leq -\ \tfrac{1}{2}\phi^T(DS + S^T D)\phi$$

where

$$\phi = [\phi_1, \ldots, \phi_m]^T, \qquad D = \text{diag}\ (d_1, \ldots, d_m)$$

and S is an $m \times m$ matrix whose elements are defined by

$$s_{ij} = \begin{cases} \alpha_i - \beta_i \gamma_{ii}, & i = j \\ \\ -\beta_i \gamma_{ij}, & i \neq j \end{cases} \tag{9.37}$$

If there is a positive diagonal matrix D such that

$$DS + S^T D > 0$$

then $\dot{V}(t,x)$ is negative definite, since $\phi(x) = 0$ if and only if $x = 0$; recall that $\phi_i(x_i)$ is a positive definite function of x_i. Thus, a sufficient condition for uniform asymptotic stability of the origin as an equilibrium point of the interconnected system is the existence of a positive diagonal matrix D such that $DS + S^T D$ is positive definite. The matrix S is special in that its off-diagonal elements are nonpositive. The next lemma applies to this class of matrices.

Lemma 9.7 *There exists a positive diagonal matrix D such that $DS + S^T D$ is positive definite if and only if S is an M-matrix; that is, the leading principal minors of S are positive:*

$$\mathit{det} \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1k} \\ s_{21} & & & \\ \vdots & & & \\ s_{k1} & \cdots & \cdots & s_{kk} \end{bmatrix} > 0, \quad k = 1, 2, \ldots, m$$

◇

Proof: See [57].

The M-matrix condition can be interpreted as a requirement that the diagonal elements of S be "larger as a whole" than the off-diagonal elements. It can be seen (Exercise 9.22) that diagonally dominant matrices with nonpositive off-diagonal elements are M-matrices. The diagonal elements of S are measures of the "degree of stability" for the isolated subsystems in the sense that the constant α_i gives a lower bound on the rate of decrease of the Lyapunov function V_i with respect to $\phi_i^2(x_i)$. The off-diagonal elements of S represent the "strength of the interconnections" in the sense that they give an upper bound on $g_i(t,x)$ with respect to $\phi_j(x_j)$ for $j = 1, \ldots, m$. Thus, the M-matrix condition says that *if the degrees of stability for the isolated subsystems are larger as a whole than the strength of the interconnections, then the interconnected system has a uniformly asymptotically stable equilibrium at the origin.* We summarize our conclusion in the next theorem.

Theorem 9.2 *Consider the system* (9.32) *and suppose there are positive definite decrescent Lyapunov functions* $V_i(t, x_i)$ *that satisfy* (9.34) *and* (9.35) *and that* $g_i(t, x)$ *satisfies* (9.36) *for all* $t \geq 0$ *and* $\|x\| < r$. *Suppose the matrix* S *defined by* (9.37) *is an* M*-matrix. Then, the origin is uniformly asymptotically stable. Moreover, if all the assumptions hold globally and* $V_i(t, x_i)$ *are radially unbounded, it will be globally uniformly asymptotically stable.* ◇

Example 9.7 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= -x_1 - 1.5x_1^2x_2^3 \\ \dot{x}_2 &= -x_2^3 + 0.5x_1^2x_2^2 \end{aligned}$$

The system can be represented in the form (9.32) with

$$f_1(x_1) = -x_1, \quad g_1(x) = -1.5x_1^2x_2^3, \quad f_2(x_2) = -x_2^3, \quad \text{and} \quad g_2(x) = 0.5x_1^2x_2^2$$

The first isolated subsystem $\dot{x}_1 = -x_1$ has a Lyapunov function $V_1(x_1) = x_1^2/2$, which satisfies

$$\frac{\partial V_1}{\partial x_1} f_1(x_1) = -x_1^2 = -\alpha_1\phi_1^2(x_1)$$

where $\alpha_1 = 1$ and $\phi_1(x_1) = |x_1|$. The second isolated subsystem $\dot{x}_2 = -x_2^3$ has a Lyapunov function $V_2(x_2) = x_2^4/4$, which satisfies

$$\frac{\partial V_2}{\partial x_2} f_2(x_2) = -x_2^6 = -\alpha_2\phi_2^2(x_2)$$

where $\alpha_2 = 1$ and $\phi_2(x_2) = |x_2|^3$. The Lyapunov functions satisfy (9.35) with $\beta_1 = \beta_2 = 1$. The interconnection term $g_1(x)$ satisfies the inequality

$$|g_1(x)| = 1.5x_1^2|x_2|^3 \leq 1.5c_1^2\phi_2(x_2)$$

for all $|x_1| \leq c_1$. The interconnection term $g_2(x)$ satisfies the inequality

$$|g_2(x)| = 0.5x_1^2x_2^2 \leq 0.5c_1c_2^2\phi_1(x_1)$$

for all $|x_1| \leq c_1$ and $|x_2| \leq c_2$. Thus, if we restrict our attention to the set

$$G = \{x \in R^2 \mid |x_1| \leq c_1, \ |x_2| \leq c_2\}$$

we can conclude that the interconnection terms satisfy (9.36) with

$$\gamma_{11} = 0, \quad \gamma_{12} = 1.5c_1^2, \quad \gamma_{21} = 0.5c_1c_2^2, \quad \text{and} \quad \gamma_{22} = 0$$

The matrix

$$S = \begin{bmatrix} 1 & -1.5c_1^2 \\ -0.5c_1c_2^2 & 1 \end{bmatrix}$$

is an M-matrix if $0.75c_1^3c_2^2 < 1$. This will be the case, for example, when $c_1 = c_2 = 1$. Thus, the origin is asymptotically stable. If we are interested in estimating the region of attraction, we need to know the composite Lyapunov function $V = d_1V_1 + d_2V_2$; that is, we need to know a positive diagonal matrix D such that $DS + S^TD > 0$. Taking $c_1 = c_2 = 1$, we have

$$DS + S^TD = \begin{bmatrix} 2d_1 & -1.5d_1 - 0.5d_2 \\ -1.5d_1 - 0.5d_2 & 2d_2 \end{bmatrix}$$

which is positive definite for $1 < d_2/d_1 < 9$. Since there is no loss of generality in multiplying a Lyapunov function by a positive constant, we take $d_1 = 1$ and write the composite Lyapunov function as

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}d_2x_2^4, \quad 1 < d_2 < 9$$

An estimate of the region of attraction is given by

$$\Omega_c = \{x \in R^2 \mid V(x) \le c\}$$

where $c \le \min\{1/2, d_2/4\}$ to ensure that Ω_c is inside the rectangle $|x_i| \le 1$. Noting that the surface $V(x) = c$ intersects the x_1-axis and the x_2-axis at $\sqrt{2c}$ and $(4c/d_2)^{1/4}$, respectively, we maximize these distances by choosing $d_2 = 2$ and $c = 0.5$. △

Example 9.8 The mathematical model of an artificial neural network was presented in Section 1.2.5, and its stability properties were analyzed in Example 4.11 by using LaSalle's invariance principle. A key assumption in Example 4.11 is the symmetry requirement $T_{ij} = T_{ji}$, which allows us to represent the right-hand side of the state equation as the gradient of a scalar function. Let us relax this requirement and allow $T_{ij} \ne T_{ji}$. We will analyze the stability properties of the network by viewing it as an interconnection of subsystems; each subsystem corresponds to one neuron. We find it convenient here to work with the voltages at the amplifier inputs u_i. The equations of motion are

$$C_i\dot{u}_i = \sum_j T_{ij}g_j(u_j) - \frac{1}{R_i}u_i + I_i$$

for $i = 1, 2, \ldots, n$, where $g_i(\cdot)$ are sigmoid functions, I_i are constant current inputs, $R_i > 0$, and $C_i > 0$. We assume that the system has a finite number of isolated equilibrium points. Each equilibrium point u^* satisfies the equation

$$0 = \sum_j T_{ij}g_j(u_j^*) - \frac{1}{R_i}u_i^* + I_i$$

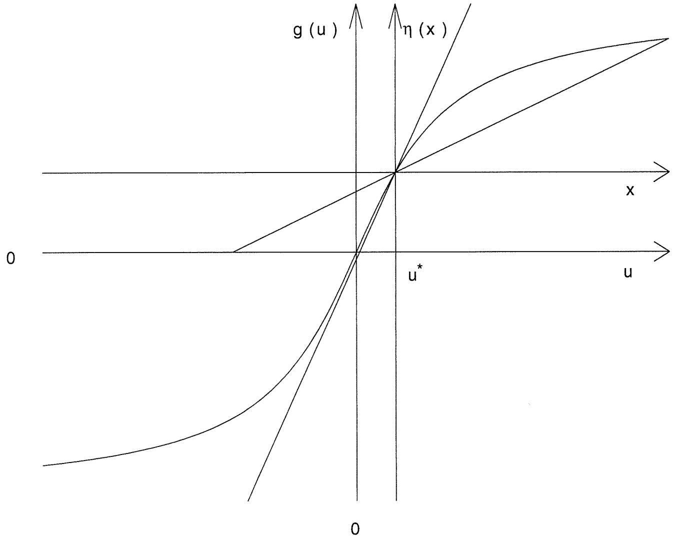

Figure 9.1: The sector nonlinearity $\eta_i(x_i)$ of Example 9.8.

To analyze the stability properties of a given equilibrium point u^*, we shift it to the origin. Let $x_i = u_i - u_i^*$. Then,

$$\begin{aligned} \dot{x}_i &= \frac{1}{C_i}\dot{u}_i = \frac{1}{C_i}\left[\sum_j T_{ij}g_j(x_j + u_j^*) - \frac{1}{R_i}(x_i + u_i^*) + I_i\right] \\ &= \frac{1}{C_i}\left[\sum_j T_{ij}\eta_j(x_j) - \frac{1}{R_i}x_i\right] \end{aligned}$$

where

$$\eta_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$$

Assume that $\eta_i(\cdot)$ satisfies the sector condition

$$\sigma^2 k_{i1} \le \sigma\eta_i(\sigma) \le \sigma^2 k_{i2}, \quad \text{for } \sigma \in [-r_i, r_i]$$

where k_{i1} and k_{i2} are positive constants. Figure 9.1 shows that such condition is indeed satisfied when $g_i(u_i) = (2V_M/\pi)\tan^{-1}(\lambda\pi u_i/2V_M)$, $\lambda > 0$. We can recast this system in the form (9.32) with

$$f_i(x_i) = -\frac{1}{C_iR_i}x_i + \frac{1}{C_i}T_{ii}\eta_i(x_i), \qquad g_i(x) = \frac{1}{C_i}\sum_{j\neq i} T_{ij}\eta_j(x_j)$$

Using

$$V_i(x_i) = \tfrac{1}{2}C_ix_i^2$$

as a Lyapunov function candidate for the ith isolated subsystem, we obtain

$$\frac{\partial V_i}{\partial x_i} f_i(x_i) = -\frac{1}{R_i} x_i^2 + T_{ii} x_i \eta_i(x_i)$$

If $T_{ii} \leq 0$, then

$$T_{ii} x_i \eta_i(x_i) \leq -|T_{ii}| k_{i1} x_i^2$$

and

$$\frac{\partial V_i}{\partial x_i} f_i(x_i) \leq -\left(\frac{1}{R_i} + |T_{ii}| k_{i1}\right) x_i^2$$

which is negative definite. If $T_{ii} > 0$, then

$$T_{ii} x_i \eta_i(x_i) \leq T_{ii} k_{i2} x_i^2$$

and

$$\frac{\partial V_i}{\partial x_i} f_i(x_i) \leq -\left(\frac{1}{R_i} - T_{ii} k_{i2}\right) x_i^2$$

In this case, we assume that $T_{ii} k_{i2} < 1/R_i$, so that the derivative of V_i is negative definite. To simplify the notation, let

$$\delta_i = \begin{cases} |T_{ii}| k_{i1}, & \text{if } T_{ii} \leq 0 \\ -T_{ii} k_{i2}, & \text{if } T_{ii} > 0 \end{cases}$$

Then, $V_i(x_i)$ satisfies (9.34) and (9.35) on the interval $[-r_i, r_i]$ with

$$\alpha_i = \left(\frac{1}{R_i} + \delta_i\right), \quad \beta_i = C_i, \quad \phi_i(x_i) = |x_i|$$

where α_i is positive by assumption. The interconnection term $g_i(x)$ satisfies the inequality

$$|g_i(x)| \leq \frac{1}{C_i} \sum_{j \neq i} |T_{ij}| |\eta_j(x_j)| \leq \frac{1}{C_i} \sum_{j \neq i} |T_{ij}| k_{j2} |x_j|$$

Thus, $g_i(x)$ satisfies (9.36) with $\gamma_{ii} = 0$ and $\gamma_{ij} = k_{j2} |T_{ij}| / C_i$ for $i \neq j$. Now we can form the matrix S as

$$s_{ij} = \begin{cases} \delta_i + 1/R_i, & \text{for } i = j \\ -|T_{ij}| k_{j2} & \text{for } i \neq j \end{cases}$$

The equilibrium point u^* is asymptotically stable if S is an M-matrix. We may estimate the region of attraction by the set

$$\Omega_c = \left\{ x \in R^n \;\middle|\; \sum_{i=1}^{n} d_i V_i(x_i) \leq c \right\}$$

where $c \leq 0.5 \min_i\{d_i C_i r_i^2\}$ to ensure that Ω_c is inside the set $|x_i| \leq r_i$. This analysis is repeated for each asymptotically stable equilibrium point. The conclusions we could arrive at in this example are more conservative compared with the conclusions we arrived at using LaSalle's invariance principle. First, the interconnection coefficients T_{ij} must be restricted to satisfy the M-matrix condition. Second, we obtain only local estimates of the regions of attractions for the isolated equilibrium points. The union of these estimates does not cover the whole domain of interest. On the other hand, we do not have to assume that $T_{ij} = T_{ji}$. △

9.6 Slowly Varying Systems

The system

$$\dot{x} = f(x, u(t)) \tag{9.38}$$

where $x \in R^n$ and $u(t) \in \Gamma \subset R^m$ for all $t \geq 0$ is considered to be slowly varying if $u(t)$ is continuously differentiable and $\|\dot{u}(t)\|$ is "sufficiently" small. The components of $u(t)$ could be input variables or time-varying parameters. In the analysis of (9.38), one usually treats u as a "frozen" parameter and assumes that for each fixed $u = \alpha \in \Gamma$, the frozen system has an isolated equilibrium point defined by $x = h(\alpha)$. If a property of $x = h(\alpha)$ is uniform in α, then it is reasonable to expect that the slowly varying system (9.38) will possess a similar property. The underlying characteristic of such systems is that the motion caused by changes of initial conditions is much faster than that caused by inputs or time-varying parameters. In this section, we will see how Lyapunov stability can be used to analyze slowly varying systems.

Suppose $f(x, u)$ is locally Lipschitz on $R^n \times \Gamma$, and for every $u \in \Gamma$ the equation

$$0 = f(x, u)$$

has a continuously differentiable isolated root $x = h(u)$; that is,

$$0 = f(h(u), u)$$

Furthermore, suppose

$$\left\| \frac{\partial h}{\partial u} \right\| \leq L, \quad \forall\, u \in \Gamma \tag{9.39}$$

To analyze the stability properties of the frozen equilibrium point $x = h(\alpha)$, we shift it to the origin via the change of variables $z = x - h(\alpha)$ to obtain the equation

$$\dot{z} = f(z + h(\alpha), \alpha) \overset{\text{def}}{=} g(z, \alpha) \tag{9.40}$$

Now we search for a Lyapunov function to show that $z = 0$ is asymptotically stable. Since $g(z, \alpha)$ depends on the parameter α, a Lyapunov function for the system may depend, in general, on α. Suppose we can find a Lyapunov function $V(z, \alpha)$ that satisfies the conditions

$$c_1\|z\|^2 \le V(z, \alpha) \le c_2\|z\|^2 \tag{9.41}$$

$$\frac{\partial V}{\partial z} g(z, \alpha) \le -c_3\|z\|^2 \tag{9.42}$$

$$\left\|\frac{\partial V}{\partial z}\right\| \le c_4\|z\| \tag{9.43}$$

$$\left\|\frac{\partial V}{\partial \alpha}\right\| \le c_5\|z\|^2 \tag{9.44}$$

for all $z \in D = \{z \in R^n \mid \|z\| < r\}$ and $\alpha \in \Gamma$, where c_i, $i = 1, 2, \ldots, 5$ are positive constants independent of α. Inequalities (9.41) and (9.42) state the usual requirements that V be positive definite and decrescent and has a negative definite derivative along the trajectories of the system (9.40). Furthermore, they show that the origin $z = 0$ is exponentially stable. The special requirement here is that these inequalities hold uniformly in α. Inequalities (9.43) and (9.44) are needed to handle the perturbations of (9.40), which will result from the fact that $u(t)$ is not constant, but a time-varying function. With $V(z, u)$ as a Lyapunov function candidate, the analysis of (9.38) proceeds as follows: The change of variables $z = x - h(u)$ transforms (9.38) into the form

$$\dot{z} = g(z, u) - \frac{\partial h}{\partial u}\dot{u} \tag{9.45}$$

where the effect of the time variation of u appears as a perturbation of the frozen system (9.40). The derivative of $V(z, u)$ along the trajectories (9.45) is given by

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial u}\dot{u}(t) \\ &= \frac{\partial V}{\partial z} g(z, u) + \left[\frac{\partial V}{\partial u} - \frac{\partial V}{\partial z}\frac{\partial h}{\partial u}\right]\dot{u}(t) \\ &\le -c_3\|z\|^2 + c_5\|z\|^2\|\dot{u}(t)\| + c_4 L\|z\|\ \|\dot{u}(t)\| \end{aligned}$$

Setting

$$\gamma(t) = \frac{c_5}{c_4}\|\dot{u}(t)\| \quad \text{and} \quad \delta(t) = L\|\dot{u}(t)\|$$

we can rewrite the last inequality as

$$\dot{V} \le -c_3\|z\|^2 + c_4\gamma(t)\|z\|^2 + c_4\delta(t)\|z\|$$

which takes the form of inequality (9.16) of Section 9.3. Therefore, by applying the comparison lemma, as in Section 9.3, it can be shown that, if $\dot{u}(t)$ satisfies

$$\int_{t_0}^{t} \|\dot{u}(\tau)\|\ d\tau \le \varepsilon_1(t - t_0) + \eta_1, \quad \text{where } \varepsilon_1 < \frac{c_1 c_3}{c_2 c_5} \tag{9.46}$$

and

$$\|z(0)\| < \frac{r}{\rho_1}\sqrt{\frac{c_1}{c_2}}; \quad \sup_{t\geq 0}\|\dot{u}(t)\| \leq \frac{2c_1\alpha_1 r}{c_4\rho_1 L}$$

in which α_1 and ρ_1 are defined by

$$\alpha_1 = \frac{1}{2}\left[\frac{c_3}{c_2} - \varepsilon_1\frac{c_5}{c_1}\right] > 0, \quad \rho_1 = \exp\left(\frac{c_5\eta_1}{2c_1}\right) \geq 1$$

then $z(t)$ satisfies the inequality

$$\|z(t)\| \leq \sqrt{\frac{c_2}{c_1}}\rho_1\|z(0)\|e^{-\alpha_1 t} + \frac{c_4\rho_1 L}{2c_1}\int_0^t e^{-\alpha_1(t-\tau)}\|\dot{u}(\tau)\|\, d\tau \tag{9.47}$$

Depending upon the assumptions for $\|\dot{u}\|$, several conclusions can be drawn from the foregoing inequality. Some of these conclusions are stated in the next theorem.

Theorem 9.3 *Consider the system* (9.45). *Suppose that* $[\partial h/\partial u]$ *satisfies* (9.39), $\|\dot{u}(t)\| \leq \varepsilon$ *for all* $t \geq 0$, *and there is a Lyapunov function* $V(z,u)$ *that satisfies* (9.41) *through* (9.44). *If*

$$\varepsilon < \frac{c_1c_3}{c_2c_5} \times \frac{r}{r + c_4L/c_5}$$

then for all $\|z(0)\| < r\sqrt{c_1/c_2}$, *the solutions of* (9.45) *are uniformly bounded for all* $t \geq 0$ *and uniformly ultimately bounded by*

$$b = \frac{c_2c_4L\varepsilon}{\theta(c_1c_3 - \varepsilon c_2c_5)}$$

where $\theta \in (0,1)$ *is an arbitrary constant. If, in addition,* $\dot{u}(t) \to 0$ *as* $t \to \infty$, *then* $z(t) \to 0$ *as* $t \to \infty$. *Finally, if* $h(u) = 0$ *for all* $u \in \Gamma$ *and* $\varepsilon < c_3/c_5$, *then* $z = 0$ *is an exponentially stable equilibrium point of* (9.45). *Equivalently,* $x = 0$ *is an exponentially stable equilibrium point of* (9.38). ◇

Proof: Since $\|\dot{u}(t)\| \leq \varepsilon < c_1c_3/c_2c_5$, inequality (9.46) is satisfied with $\varepsilon_1 = \varepsilon$ and $\eta_1 = 0$. Hence,

$$\alpha_1 = \frac{1}{2}\left[\frac{c_3}{c_2} - \varepsilon\frac{c_5}{c_1}\right], \quad \rho_1 = 1$$

Using the given upper bound on ε, we have

$$\begin{aligned}
\frac{2c_1\alpha_1 r}{c_4L} &= \frac{c_1 r}{c_4L}\left[\frac{c_3}{c_2} - \varepsilon\frac{c_5}{c_1}\right] \\
&> \frac{c_1 r}{c_4L}\left[\frac{c_3}{c_2} - \frac{c_3}{c_2} \times \frac{r}{r + c_4L/c_5}\right] \\
&= \frac{c_1c_3}{c_2c_5} \times \frac{r}{r + c_4L/c_5} > \varepsilon
\end{aligned}$$

Hence, the inequality $\sup_{t\geq 0}\|\dot{u}(t)\| < 2c_1\alpha_1 r/c_4 L$ is satisfied and from (9.47), we obtain

$$\begin{aligned}\|z(t)\| &\leq \sqrt{\frac{c_2}{c_1}}\|z(0)\|e^{-\alpha_1 t} + \frac{c_4 L}{2c_1}\int_0^t e^{-\alpha_1(t-\tau)}\varepsilon \, d\tau \\ &\leq \sqrt{\frac{c_2}{c_1}}\|z(0)\|e^{-\alpha_1 t} + \frac{c_4 L\varepsilon}{2c_1\alpha_1} = \sqrt{\frac{c_2}{c_1}}\|z(0)\|e^{-\alpha_1 t} + b\theta\end{aligned}$$

After a finite time, the exponentially decaying term will be less than $(1-\theta)b$, which shows that $z(t)$ will be ultimately bounded by b. If, in addition, $\dot{u}(t) \to 0$ as $t \to \infty$, then it is clear from (9.47) that $z(t) \to 0$ as $t \to \infty$. If $h(u) = 0$ for all $u \in \Gamma$, we can take $L = 0$. Consequently, the upper bound on $\dot{V}$ simplifies to

$$\dot{V} \leq -(c_3 - c_5\varepsilon)\|z\|^2$$

which shows that $z = 0$ will be exponentially stable if $\varepsilon < c_3/c_5$. □

Theorem 9.3 requires the existence of a Lyapunov function $V(z,\alpha)$ for the frozen system (9.40), which satisfies inequalities (9.41) through (9.44). Lemma 9.8 shows that such Lyapunov function will exist, under some mild smoothness requirements, if the equilibrium point $z = 0$ of the frozen system is exponentially stable uniformly in α. This is done by deriving a converse Lyapunov function for the system, as in the converse Lyapunov theorems of Section 4.7.

Lemma 9.8 *Consider the system* (9.40) *and suppose $g(z,\alpha)$ is continuously differentiable and the Jacobian matrices $[\partial g/\partial z]$ and $[\partial g/\partial \alpha]$ satisfy*

$$\left\|\frac{\partial g}{\partial z}(z,\alpha)\right\| \leq L_1, \qquad \left\|\frac{\partial g}{\partial \alpha}(z,\alpha)\right\| \leq L_2\|z\|$$

for all $(z,\alpha) \in D \times \Gamma$, where $D = \{z \in R^n \mid \|z\| < r\}$. Let k, γ, and r_0 be positive constants with $r_0 < r/k$, and define $D_0 = \{z \in R^n \mid \|z\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|z(t)\| \leq k\|z(0)\|e^{-\gamma t}, \quad \forall \; z(0) \in D_0, \; \alpha \in \Gamma, \; t \geq 0$$

Then, there is a function $V : D_0 \times \Gamma \to R$ that satisfies (9.41) *through* (9.44). *Moreover, if all the assumptions hold globally (in z), then $V(z,\alpha)$ is defined and satisfies* (9.41) *through* (9.44) *on $R^n \times \Gamma$.* ◇

Proof: Owing to the equivalence of norms, it is sufficient to prove the lemma for the 2-norm. Let $\phi(t; z, \alpha)$ be the solution of (9.40) that starts at $(0, z)$; that is, $\phi(0; z, \alpha) = z$. The notation emphasizes the dependence of the solution on the parameter α. Let

$$V(z,\alpha) = \int_0^T \phi^T(t; z, \alpha)\phi(t; z, \alpha) \, dt$$

where $T = \ln(2k^2)/2\gamma$. Similar to the proof of Theorem 4.14, it can be shown that $V(z,\alpha)$ satisfies (9.41) through (9.43) with $c_1 = [1 - \exp(-2L_1T)]/2L_1$, $c_2 = k^2[1 - \exp(-2\gamma T)]/2\gamma$, $c_3 = 1/2$, and $c_4 = 2k\{1 - \exp[-(\gamma - L_1)T]\}/(\gamma - L_1)$. To show that $V(z,\alpha)$ satisfies (9.44), note that the sensitivity function $\phi_\alpha(t;z,\alpha)$ satisfies the sensitivity equation

$$\frac{\partial}{\partial t}\phi_\alpha = \frac{\partial g}{\partial z}(\phi(t;z,\alpha),\alpha)\phi_\alpha + \frac{\partial g}{\partial \alpha}(\phi(t;z,\alpha),\alpha), \quad \phi_\alpha(0;z,\alpha) = 0$$

from which we obtain

$$\begin{aligned}
\|\phi_\alpha(t;z,\alpha)\|_2 &\le \int_0^t L_1\|\phi_\alpha(\tau;z,\alpha)\|_2 \, d\tau + \int_0^t L_2\|\phi(\tau;z,\alpha)\|_2 \, d\tau \\
&\le \int_0^t L_1\|\phi_\alpha(\tau;z,\alpha)\|_2 \, d\tau + \int_0^t L_2ke^{-\gamma\tau} \, d\tau\|z\|_2 \\
&\le \int_0^t L_1\|\phi_\alpha(\tau;z,\alpha)\|_2 \, d\tau + \frac{L_2k}{\gamma}\|z\|_2
\end{aligned}$$

Use of the Gronwall–Bellman inequality yields

$$\|\phi_\alpha(t;z,\alpha)\|_2 \le \frac{L_2k}{\gamma}\|z\|_2e^{L_1t}$$

Hence,

$$\begin{aligned}
\left\|\frac{\partial V}{\partial \alpha}\right\|_2 &= \left\|\int_0^T 2\phi^T(t;z,\alpha)\phi_\alpha(t;z,\alpha) \, dt\right\|_2 \\
&\le \int_0^T 2ke^{-\gamma t}\|z\|_2\left(\frac{L_2k}{\gamma}\right)e^{L_1t}\|z\|_2 \, dt \\
&\le \frac{2k^2L_2}{\gamma(\gamma - L_1)}\left[1 - e^{-(\gamma - L_1)T}\right]\|z\|_2^2 \overset{\text{def}}{=} c_5\|z\|_2^2
\end{aligned}$$

which completes the proof of the lemma. □

When the frozen system (9.40) is linear, a Lyapunov function satisfying (9.41) through (9.44) can be explicitly determined by solving a parameterized Lyapunov equation. This fact is stated in the next lemma.

Lemma 9.9 *Consider the system $\dot{z} = A(\alpha)z$, where $\alpha \in \Gamma$ and $A(\alpha)$ is continuously differentiable. Suppose the elements of A and their first partial derivatives with respect to α are uniformly bounded; that is,*

$$\|A(\alpha)\|_2 \le c, \quad \left\|\frac{\partial}{\partial \alpha_i}A(\alpha)\right\|_2 \le b_i, \quad \forall\ \alpha \in \Gamma, \ \forall\ 1 \le i \le m$$

Suppose further that $A(\alpha)$ *is Hurwitz uniformly in* α*; that is,*

$$\text{Re}[\lambda(A(\alpha))] \leq -\sigma < 0, \ \ \forall\ \alpha \in \Gamma$$

Then, the Lyapunov equation

$$PA(\alpha) + A^T(\alpha)P = -I \tag{9.48}$$

has a unique positive definite solution $P(\alpha)$ *for every* $\alpha \in \Gamma$. *Moreover,* $P(\alpha)$ *is continuously differentiable and satisfies*

$$c_1 z^T z \leq z^T P(\alpha) z \leq c_2 z^T z$$

$$\left\| \frac{\partial}{\partial \alpha_i} P(\alpha) \right\|_2 \leq \mu_i, \ \ \forall\ 1 \leq i \leq m$$

for all $(z, \alpha) \in R^n \times \Gamma$*, where* c_1*,* c_2*, and* μ_i *are positive constants independent of* α*. Consequently,* $V(z,\alpha) = z^T P(\alpha) z$ *satisfies* (9.42) *through* (9.44) *in the* 2*-norm with* $c_3 = 1$*,* $c_4 = 2c_2$*, and* $c_5 = \sqrt{\sum_{i=1}^m \mu_i^2}$*.* ◇

Proof: The uniform Hurwitz property of $A(\alpha)$ implies that the exponential matrix $\exp[tA(\alpha)]$ satisfies

$$\|\exp[tA(\alpha)]\| \leq k(A)e^{-\beta t}, \ \ \forall\ t \geq 0, \ \forall\ \alpha \in \Gamma$$

where $\beta > 0$ is independent of α, but $k(A) > 0$ depends on α. For the exponentially decaying bound to hold uniformly in α, we need to use the property that $\|A(\alpha)\|$ is bounded. The set of matrices satisfying $\text{Re}[\lambda(A(\alpha))] \leq -\sigma$ and $\|A(\alpha)\| \leq c$ is a compact set, which we denote by S. Let A and B be any two elements of S. Consider[8]

$$\exp[t(A+B)] = \exp[tA] + \int_0^t \exp[(t-\tau)A]B\exp[\tau(A+B)]\ d\tau$$

Using the exponentially decaying bound on $\exp[tA]$, we get

$$\|\exp[t(A+B)]\| \leq k(A)e^{-\beta t} + \int_0^t k(A)e^{-\beta(t-\tau)}\|B\|\ \|\exp[\tau(A+B)]\|\ d\tau$$

Multiply through by $e^{\beta t}$,

$$e^{\beta t}\|\exp[t(A+B)]\| \leq k(A) + k(A)\|B\| \int_0^t e^{\beta\tau}\|\exp[\tau(A+B)]\|\ d\tau$$

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[8] This matrix identity follows by writing $\dot{x} = (A+B)x$ as $\dot{x} = Ax + Bx$ and viewing Bx as an input term. Substituting $x(t) = \exp[t(A+B)]x_0$ into the input term yields

$$\exp[t(A+B)]x_0 = \exp[tA]x_0 + \int_0^t \exp[(t-\tau)A]B\exp[\tau(A+B)]x_0\ d\tau$$

Since this expression holds for all $x_0 \in R^n$, we arrive at the matrix identity.

Applying the Gronwall–Bellman inequality yields

$$\| \exp[t(A+B)]\| \leq k(A)e^{-(\beta - k(A)\|B\|)t}, \ \ \forall\ t \geq 0$$

Hence, there exists a positive constant $\gamma < \beta$ and a neighborhood $\mathcal{N}(A)$ of A such that if $C \in \mathcal{N}(A)$, then

$$\| \exp[tC]\| \leq k(A)e^{-\gamma t}, \ \ \forall\ t \geq 0$$

Since S is compact, it is covered by a finite number of these neighborhoods. Therefore, we can find a positive constant k independent of α such that

$$\| \exp[tA(\alpha)]\| \leq ke^{-\gamma t}, \ \ \forall\ t \geq 0, \ \forall\ \alpha \in \Gamma$$

Consider now the Lyapunov equation (9.48). Existence of a unique positive definite solution for every $\alpha \in \Gamma$ follows from Theorem 4.6. Moreover, the proof of that theorem shows that

$$P(\alpha) = \int_0^\infty \left[e^{tA(\alpha)}\right]^T \left[e^{tA(\alpha)}\right]\ dt$$

Since $A(\alpha)$ is continuously differentiable, so is $P(\alpha)$. We have

$$z^T P(\alpha) z \leq \int_0^\infty k^2 e^{-2\gamma t}\|z\|_2^2\ dt = \frac{k^2}{2\gamma}\|z\|_2^2 \ \Rightarrow\ c_2 = \frac{k^2}{2\gamma}$$

Let $y(t) = e^{tA(\alpha)}z$. Then, $\dot{y} = A(\alpha)y$,

$$-y^T(t)\dot{y}(t) = -y^T(t)A(\alpha)y(t) \leq \|A(\alpha)\|_2 y^T(t)y(t) \leq cy^T(t)y(t)$$

and

$$\begin{aligned}
z^T P(\alpha) z &= \int_0^\infty y^T(t)y(t)\ dt \ \geq\ \int_0^\infty \frac{-1}{c} y^T(t)\dot{y}(t)\ dt \\
&= \frac{1}{2c}\int_0^\infty \frac{d}{dt}[-y^T(t)y(t)]\ dt \ =\ \frac{1}{2c}\left[-y^T(t)y(t)\right]\Big|_0^\infty \\
&= \frac{1}{2c}y^T(0)y(0) \ =\ \frac{1}{2c}z^T z \ \Rightarrow\ c_1 = \frac{1}{2c}
\end{aligned}$$

Differentiate $P(\alpha)A(\alpha) + A^T(\alpha)P(\alpha) = -I$ partially with respect to any component α_i of α, and denote the derivative of $P(\alpha)$ by $P'(\alpha)$. Then,

$$P'(\alpha)A(\alpha) + A^T(\alpha)P'(\alpha) = -\{P(\alpha)A'(\alpha) + [A'(\alpha)]^T P(\alpha)\}$$

Thus, $P'(\alpha)$ is given by

$$P'(\alpha) = \int_0^\infty \left[e^{tA(\alpha)}\right]^T \{P(\alpha)A'(\alpha) + [A'(\alpha)]^T P(\alpha)\} \left[e^{tA(\alpha)}\right]\ dt$$

It follows that

$$\|P'(\alpha)\|_2 \leq \int_0^\infty k^2 e^{-2\gamma t} 2\frac{k^2}{2\gamma} b_i \; dt = \frac{b_i k^4}{2\gamma^2} \; \Rightarrow \; \mu_i = \frac{b_i k^4}{2\gamma^2}$$

which completes the proof of the lemma. □

It should be noted that the set Γ in Lemma 9.9 is not necessarily compact. When Γ is compact, the boundedness of $A(\alpha)$ and its partial derivatives follow from the assumption that $A(\alpha)$ is continuously differentiable.

Example 9.9 Consider the system

$$\dot{x} = A(\varepsilon t)x$$

where $\varepsilon > 0$. When ε is sufficiently small, we can treat this system as a slowly varying system. It is in the form of (9.38) with $u = \varepsilon t$ and $\Gamma = [0, \infty)$. For all $u \in \Gamma$, the origin $x = 0$ is an equilibrium point. Hence, this is a special case where $h(u) = 0$. Suppose $\text{Re}[\lambda(A(\alpha))] \leq -\sigma < 0$, and $A(\alpha)$ and $A'(\alpha)$ are uniformly bounded for all $\alpha \in \Gamma$. Then, the solution of the Lyapunov equation (9.48) holds the properties stated in Lemma 9.9. Using $V(x, u) = x^T P(u) x$ as a Lyapunov function candidate for $\dot{x} = A(u)x$, we obtain

$$\begin{aligned} \dot{V}(t,x) &= x^T[P(u(t))A(u(t)) + A^T(u(t))P(u(t))]x + x^T P'(u(t))\dot{u}(t)x \\ &\leq -x^T x + \varepsilon c_5 \|x\|_2^2 \; = \; -(1 - \varepsilon c_5)\|x\|_2^2 \end{aligned}$$

where c_5 is an upper bound on $\|P'(\alpha)\|_2$. Therefore, for all $\varepsilon < 1/c_5$, the origin $x = 0$ is an exponentially stable equilibrium point of $\dot{x} = A(\varepsilon t)x$. △

9.7 Exercises

9.1 ([150]) Consider the Lyapunov equation $PA + A^T P = -Q$, where $Q = Q^T > 0$ and A is Hurwitz. Let $\mu(Q) = \lambda_{\min}(Q)/\lambda_{\max}(P)$.

(a) Show that $\mu(kQ) = \mu(Q)$ for any positive constant k.

(b) Let $\hat{Q} = \hat{Q}^T > 0$ have $\lambda_{\min}(\hat{Q}) = 1$. Show that $\mu(I) \geq \mu(\hat{Q})$.

(c) Show that $\mu(I) \geq \mu(Q)$, $\forall \; Q = Q^T > 0$.

Hint: In part (b), let P_1 and P_2 be the solutions of the Lyapunov equation for $Q = I$ and $Q = \hat{Q}$, respectively. Show that

$$P_1 - P_2 = \int_0^\infty \exp(A^T t)(I - \hat{Q}) \exp(At) \; dt \leq 0$$

9.2 Consider the system $\dot{x} = Ax + Bu$ and let $u = -Fx$ be a stabilizing state feedback control; that is, the matrix $(A - BF)$ is Hurwitz. Suppose that, due to physical limitations, we have to use a limiter to limit the value of u_i to $|u_i(t)| \leq L$. The closed-loop system can be represented by $\dot{x} = Ax - BL\ \text{sat}(Fx/L)$, where $\text{sat}(v)$ is a vector whose ith component is the saturation function. By adding and subtracting the term BFx, we can rewrite the closed-loop state equation as $\dot{x} = (A - BF)x - Bh(Fx)$, where $h(v) = L\ \text{sat}(v/L) - v$. Thus, the effect of the limiter can be viewed as a perturbation of the nominal system without the limiter.

(a) Show that

$$|h_i(v)| \leq \frac{\delta}{(1+\delta)}|v_i|, \ \forall\ |v_i| \leq L(1+\delta)$$

where $\delta > 0$.

(b) Let P be the solution of

$$P(A - BF) + (A - BF)^T P = -I$$

Show that the derivative of $V(x) = x^T Px$ along the trajectories of the closed-loop system will be negative definite over the region $|(Fx)_i| \leq L(1+\delta),\ \forall\ i$, provided $\delta/(1+\delta) < 1/(2\|PB\|_2\ \|F\|_2)$.

(c) Show that the origin is asymptotically stable and discuss how you would estimate the region of attraction.

(d) Apply the result obtained in part (c) to the case

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \text{and} \quad L = 1$$

and estimate the region of attraction.

9.3 Consider the system

$$\dot{x} = f(t,x) + Bu, \quad y = Cx, \quad \text{and} \quad u = -g(t,y)$$

where $f(t,0) = 0$, $g(t,0) = 0$, and $\|g(t,y)\| \leq \gamma\|y\|$ for all $t \geq 0$. Suppose that the origin of $\dot{x} = f(t,x)$ is globally exponentially stable and let $V(t,x)$ be a Lyapunov function that satisfies (9.3) through (9.5) globally. Find a bound γ^* on γ such that the origin of the given system is globally exponentially stable for $\gamma < \gamma^*$.

9.4 Consider the perturbed system

$$\dot{x} = Ax + B[u + g(t,x)]$$

where $g(t,x)$ is continuously differentiable and satisfies $\|g(t,x)\|_2 \leq k\|x\|_2,\ \forall\ t \geq 0,\ \forall\ x \in B_r$ for some $r > 0$. Let $P = P^T > 0$ be the solution of the Riccati equation

$$PA + A^T P + Q - PBB^T P + 2\alpha P = 0$$

where $Q \geq k^2 I$ and $\alpha > 0$. Show that $u = -B^T P x$ stabilizes the origin of the perturbed system.

9.5 (**[101]**) Consider the perturbed system

$$\dot{x} = Ax + Bu + Dg(t, y), \qquad y = Cx$$

where $g(t, y)$ is continuously differentiable and satisfies $\|g(t, y)\|_2 \leq k\|y\|_2$, $\forall\ t \geq 0$, $\forall\ \|y\|_2 \leq r$ for some $r > 0$. Suppose the equation

$$PA + A^T P + \varepsilon Q - \frac{1}{\varepsilon} PBB^T P + \frac{1}{\gamma} PDD^T P + \frac{1}{\gamma} C^T C = 0$$

where $Q = Q^T > 0$, $\varepsilon > 0$, and $0 < \gamma < 1/k$ has a positive definite solution $P = P^T > 0$. Show that $u = -(1/2\varepsilon)B^T P x$ stabilizes the origin of the perturbed system.

9.6 Consider the system

$$\begin{aligned} \dot{x}_1 &= -\alpha x_1 - \omega x_2 + (\beta x_1 - \gamma x_2)(x_1^2 + x_2^2) \\ \dot{x}_2 &= \omega x_1 - \alpha x_2 + (\gamma x_1 + \beta x_2)(x_1^2 + x_2^2) \end{aligned}$$

where $\alpha > 0$, β, γ, and $\omega > 0$ are constants.

(a) By viewing this system as a perturbation of the linear system

$$\dot{x}_1 = -\alpha x_1 - \omega x_2, \qquad \dot{x}_2 = \omega x_1 - \alpha x_2$$

show that the origin of the perturbed system is exponentially stable with $\{\|x\|_2 \leq r\}$ included in the region of attraction, provided $|\beta|$ and $|\gamma|$ are sufficiently small. Find upper bounds on $|\beta|$ and $|\gamma|$ in terms of r.

(b) Using $V(x) = x_1^2 + x_2^2$ as a Lyapunov function candidate for the perturbed system, show that the origin is globally exponentially stable when $\beta \leq 0$ and exponentially stable with $\{\|x\|_2 < \sqrt{\alpha/\beta}\}$ included in the region of attraction when $\beta > 0$.

(c) Compare the results of (a) and (b) and comment on the conservative nature of the result of (a).

9.7 Consider the perturbed system

$$\dot{x} = f(x) + g(x)$$

Suppose the origin of the nominal system $\dot{x} = f(x)$ is asymptotically (but not exponentially) stable. Show that, for any $\gamma > 0$, there is a function $g(x)$ satisfying $\|g(x)\| \leq \gamma\|x\|$ in some neighborhood of the origin such that the origin of the perturbed system is unstable.

9.8 (**[66]**) Consider the perturbed system

$$\dot{x} = f(x) + g(x)$$

where $f(x)$ and $g(x)$ are continuously differentiable and $\|g(x)\| \leq \gamma\|x\|$ for all $\|x\| < r$. Suppose the origin of the nominal system $\dot{x} = f(x)$ is asymptotically stable and there is a Lyapunov function $V(x)$ that satisfies inequalities (9.11) through (9.13) for all $\|x\| < r$. Let $\Omega = \{V(x) \leq c\}$, with $c < \alpha_1(r)$.

(a) Show that there is a positive constant γ^* such that, for $\gamma < \gamma^*$, the solutions of the perturbed system starting in Ω stay in Ω for all $t \geq 0$ and are ultimately bounded by a class $\mathcal{K}$ function of γ.

(b) Suppose the nominal system has the additional property that $A = [\partial f/\partial x](0)$ is Hurwitz. Show that there is γ_1^* such that, for $\gamma < \gamma_1^*$, the solutions of the perturbed system starting in Ω converge to the origin as $t \to \infty$.

(c) Would (b) hold if A was not Hurwitz? Consider

$$f(x) = \begin{bmatrix} -x_2 - (2x_1 + x_3)^3 \\ x_1 \\ x_2 \end{bmatrix}, \quad g(x) = a \begin{bmatrix} x_1 - x_3 - (2x_1 + x_3)^3 \\ 0 \\ 0 \end{bmatrix}, \quad a \neq 0$$

Hint: For the example of part (c), use

$$V(x) = x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2 + x_1 x_3$$

to show that the origin of $\dot{x} = f(x)$ is asymptotically stable and then apply Theorem 4.16 to obtain a Lyapunov function that satisfies (9.11) through (9.13).

9.9 Consider the system

$$\dot{x}_1 = -x_1^3 + x_2^5 - \gamma x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^5 + \gamma(x_1 + x_2), \quad 0 \leq \gamma \leq \tfrac{1}{2}$$

(a) With $\gamma = 0$, show that the origin is globally asymptotically stable. Is it exponentially stable?

(b) With $0 < \gamma \leq 1/2$, show that the origin is unstable and the solutions of the system are globally ultimately bounded by an ultimate bound that is a class $\mathcal{K}$ function of γ.

9.10 (**[19]**) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a \sin x_1 - b x_1 - c x_2 - \gamma(c x_1 + 2x_2) + q(t) \cos x_1$$

where a, $b > a$, c, and γ are positive constants and $q(t)$ is a continuous function.

(a) With $q(t) \equiv 0$, use

$$V(x) = \left(b + \tfrac{1}{2}c^2\right) x_1^2 + c x_1 x_2 + x_2^2 + 2a(1 - \cos x_1)$$

to show that the origin is globally exponentially stable.

(b) Study the stability of the system when $q(t) \neq 0$ and $|q(t)| \leq k$ for all $t \geq 0$.

9.11 Consider the system

$$\dot{x}_1 = \left[(\sin x_2)^2 - 1\right] x_1, \qquad \dot{x}_2 = -b x_1 - (1 + b) x_2$$

(a) With $b = 0$, show that the origin is exponentially stable and globally asymptotically stable.

(b) With $b \neq 0$, show that the origin is exponentially stable for sufficiently small $|b|$, but not globally asymptotically stable, no matter how small $|b|$ is.

(c) Discuss the results of parts (a) and (b) in view of the robustness results of Section 9.1, and show that when $b = 0$ the origin is not globally exponentially stable.

9.12 ([8]) Consider the system

$$\dot{x}_1 = -x_1 + (x_1 + a) x_2, \qquad \dot{x}_2 = -x_1(x_1 + a) + b x_2, \qquad a \neq 0$$

(a) Let $b = 0$. Show that the origin is globally asymptotically stable. Is it exponentially stable?

(b) Let $b > 0$. Show that the origin is exponentially stable for $b < \min\{1, a^2\}$.

(c) Show that the origin is not globally asymptotically stable for any $b > 0$.

(d) Discuss the results of parts (a) through (c) in view of the robustness results of Section 9.1, and show that when $b = 0$ the origin is not globally exponentially stable.

Hint: In part (d), note that the Jacobian matrix of the nominal system is not globally bounded.

9.13 Consider the scalar system $\dot{x} = -x/(1 + x^2)$ and $V(x) = x^4$.

(a) Show that inequalities (9.11) through (9.13) are satisfied globally with

$$\alpha_1(r) = \alpha_2(r) = r^4; \quad \alpha_3(r) = \frac{4r^4}{1 + r^2}; \quad \alpha_4(r) = 4r^3$$

(b) Verify that these functions belong to class $\mathcal{K}_\infty$.

(c) Show that the right-hand side of (9.14) approaches zero as $r \to \infty$.

(d) Consider the perturbed system $\dot{x} = -x/(1+x^2) + \delta$, where δ is a positive constant. Show that whenever $\delta > 1/2$, the solution $x(t)$ escapes to ∞ for any initial state $x(0)$.

9.14 Verify that $D^+W(t)$ satisfies (9.17) when $V = 0$.
Hint: Show that $V(t+h, x(t+h)) \leq 0.5c_4h^2\|g(t,0)\|^2 + h\ o(h)$, where $o(h)/h \to 0$ as $h \to 0$. Then, use the fact that $\sqrt{c_4/2c_1} \geq 1$.

9.15 Consider the linear system of Example 9.6, but change the assumption on $B(t)$ to $\int_0^\infty \|B(t)\|\ dt < \infty$. Show that the origin is exponentially stable.

9.16 Consider the linear system of Example 9.6, but change the assumption on $B(t)$ to $\int_0^\infty \|B(t)\|^2\ dt < \infty$. Show that the origin is exponentially stable.
Hint: Use the inequality

$$\int_a^b v(t)\ dt \leq \sqrt{(b-a)\int_a^b v^2(t)\ dt}, \quad \forall\ v(t) \geq 0$$

which follows from the Cauchy–Schwartz inequality.

9.17 Consider the system $\dot{x} = A(t)x$ where $A(t)$ is continuous. Suppose $\lim_{t\to\infty} A(t) = \bar{A}$ exists and $\bar{A}$ is Hurwitz. Show that the origin is exponentially stable.

9.18 Repeat part(b) of Exercise 9.10 when $q(t)$ is bounded and $q(t) \to 0$ as $t \to \infty$.

9.19 Consider the system $\dot{x} = f(t,x)$, where $\|f(t,x) - f(0,x)\|_2 \leq \gamma(t)\|x\|_2$ for all $t \geq 0$, $x \in R^2$, $\gamma(t) \to 0$ as $t \to \infty$,

$$f(0,x) = Ax - (x_1^2 + x_2^2)Bx, \quad A = \begin{bmatrix} -\alpha & -\omega \\ \omega & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} \beta & \Omega \\ -\Omega & \beta \end{bmatrix}$$

and α, β, ω, Ω are positive constants. Show that the origin is globally exponentially stable.

9.20 Consider the system $\dot{x} = f(x) + G(x)u + w(t)$, where $\|w(t)\|_2 \leq a + c\ e^{-t}$. Suppose there exist a symmetric positive definite matrix P, a positive semidefinite function $W(x)$, and positive constants γ and σ such that

$$2x^TPf(x) + \gamma x^TPx + W(x) - 2\sigma x^TPG(x)G^T(x)Px \leq 0, \quad \forall\ x \in R^n$$

Show that with $u = -\sigma G^T(x)Px$, the trajectories of the closed-loop system are uniformly ultimately bounded by $2ak\lambda_{\max}(P)/\gamma\lambda_{\min}(P)$, for some $k > 1$.

9.21 Consider the perturbed system (9.1). Suppose there is a Lyapunov function $V(t,x)$ that satisfies (9.11) through (9.13), and the perturbation term satisfies $\|g(t,x)\| \leq \delta(t)$, $\forall\ t \geq 0$, $\forall\ x \in D$. Show that for any $\varepsilon > 0$ and $\Delta > 0$, there exist $\eta > 0$ and $\rho > 0$ such that whenever $(1/\Delta)\int_t^{t+\Delta} \delta(\tau)\ d\tau < \eta$, every solution of the perturbed system with $\|x(t_0)\| < \rho$ will satisfy $\|x(t)\| < \varepsilon$, $\forall\ t \geq t_0$.
(This result is known as total stability in the presence of perturbation that is bounded in the mean [107].)
Hint: Choose $W = \sqrt{V}$, discretize the time interval with sampling points at $t_0 + i\Delta$ for $i = 0, 1, 2, \ldots$, and show that $W(t_0 + i\Delta)$ satisfies the difference inequality

$$W(t_0 + (i+1)\Delta) \leq e^{-\sigma\Delta} W(t_0 + i\Delta) + k\eta\Delta$$

9.22 Let A be an $n \times n$ matrix with $a_{ij} \leq 0$ for all $i \neq j$ and $a_{ii} > \sum_{j \neq i} |a_{ij}|$, $i = 1, 2, \ldots, n$. Show that A is an M-matrix.
Hint: Show that $\sum_{j=1}^{n} a_{ij} > 0$ for $i = 1, \ldots, n$, and use mathematical induction to show that all the leading principal minors are positive.

9.23 Suppose the conditions of Theorem 9.3 are satisfied with

$$\phi_i(x_i) = \|x_i\| \quad \text{and} \quad c_{i1}\|x_i\|^2 \leq V_i(t, x_i) \leq c_{i2}\|x_i\|^2$$

Show that the origin is exponentially stable.

9.24 ([132]) Study the stability of the origin of the system

$$\dot{x}_1 = -x_1^3 - 1.5x_1|x_2|^3, \qquad \dot{x}_2 = -x_2^5 + x_1^2 x_2^2$$

by using composite Lyapunov analysis.

9.25 Study the stability of the origin of the system

$$\dot{x}_1 = x_2 + x_2 x_3^3, \quad \dot{x}_2 = -x_1 - x_2 + x_1^2, \quad \dot{x}_3 = x_1 + x_2 - x_3^3$$

by using composite Lyapunov analysis.

9.26 Consider the linear interconnected system

$$\dot{x}_i = A_{ii} x_i + \sum_{j=1;\, j \neq i}^{m} A_{ij} x_j, \quad i = 1, 2, \ldots, m$$

where, for each i, x_i is an n_i-dimensional vector and A_{ii} is a Hurwitz matrix. Study the stability of the origin by using composite Lyapunov analysis.

9.27 ([175]) Complex interconnected systems could be subject to structural perturbations that cause groups of subsystems to be connected or disconnected from each other during operation. Such structural perturbations can be represented as

$$\dot{x}_i = f_i(t, x_i) + g_i(t, e_{i1}x_1, \ldots, e_{im}x_m), \quad i = 1, 2, \ldots, m$$

where e_{ij} is a binary variable that takes the value 1 when the jth subsystem acts on the ith subsystem and the value 0 otherwise. The origin of the interconnected system is said to be connectively asymptotically stable if it is asymptotically stable for all interconnection patterns, that is, for all possible values of the binary variables e_{ij}. Suppose that all the assumptions of Theorem 9.2 are satisfied, with (9.36) taking the form

$$\|g_i(t, e_{i1}x_1, \dots, e_{im}x_m)\| \le \sum_{i=1}^{m} e_{ij}\gamma_{ij}\phi_j(x_j)$$

Show that the origin is connectively asymptotically stable.

9.28 (**[49]**) The output $y(t)$ of the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is required to track a reference input r. Consider the integral controller

$$\dot{z} = r - Cx, \qquad u = -F_1x - F_2z$$

where we have assumed that the state x can be measured and the matrices F_1 and F_2 can be designed such that the matrix

$$\begin{bmatrix} A - BF_1 & -BF_2 \\ -C & 0 \end{bmatrix}$$

is Hurwitz.

(a) Show that if $r =$ constant, then $y(t) \to r$ as $t \to \infty$.

(b) Study the tracking properties of the system when $r(t)$ is a slowly varying input.

9.29 (**[86]**) The output $y(t)$ of the nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

is required to track a reference input r. Consider the integral controller

$$\dot{z} = r - h(x), \qquad u = \gamma(x, z, r)$$

where we have assumed that the state x can be measured, the function γ can be designed such that the closed-loop system

$$\dot{x} = f(x, \gamma(x, z, r)), \qquad \dot{z} = r - h(x)$$

has an exponentially stable equilibrium point $(\bar{x}, \bar{z})$, and the functions f, h, and γ are twice continuously differentiable in their arguments.

(a) Show that if $r =$ constant and the initial state $(x(0), z(0))$ is sufficiently close to $(\bar{x}, \bar{z})$, then $y(t) \to r$ as $t \to \infty$.

(b) Study the tracking properties of the system when $r(t)$ is a slowly varying input.

9.30 ([86]) Consider the tracking problem of Exercise 9.29, but assume that we can only measure $y = h(x)$. Consider the observer-based integral controller

$$\dot{z}_1 = f(z_1, u) + G(r)[y - h(z_1)], \quad \dot{z}_2 = r - y, \quad \text{and} \quad u = \gamma(z_1, z_2, r)$$

Suppose γ and G can be designed such that the closed-loop system has an exponentially stable equilibrium point $(\bar{x}, \bar{z}_1, \bar{z}_2)$. Study the tracking properties of the system when

(1) $r =$ constant. **(2)** $r(t)$ is slowly varying.

9.31 Consider the linear system $\dot{x} = A(t)x$ where $\|A(t)\| \leq k$ and the eigenvalues of $A(t)$ satisfy $\text{Re}[\lambda(t)] \leq -\sigma$ for all $t \geq 0$. Suppose that $\int_0^\infty \|\dot{A}(t)\|^2 \, dt \leq \rho$. Show that the origin of $\dot{x} = A(t)x$ is exponentially stable.

Chapter 10

Perturbation Theory and Averaging

Exact closed-form analytic solutions of nonlinear differential equations are possible only for a limited number of special classes of differential equations. In general, we have to resort to approximate solutions. There are two distinct categories of approximation methods that engineers and scientists should have at their disposal as they analyze nonlinear systems: (1) numerical solution methods and (2) asymptotic methods. In this and the next chapter, we introduce the reader to some asymptotic methods for the analysis of nonlinear differential equations.[1]

Suppose we are given the state equation

$$\dot{x} = f(t, x, \varepsilon)$$

where ε is a "small" scalar parameter, and, under certain conditions, the equation has an exact solution $x(t, \varepsilon)$. Equations of this type are encountered in many applications. The goal of an asymptotic method is to obtain an approximate solution $\tilde{x}(t, \varepsilon)$ such that the approximation error $x(t, \varepsilon) - \tilde{x}(t, \varepsilon)$ is small, in some norm, for small $|\varepsilon|$ and the approximate solution $\tilde{x}(t, \varepsilon)$ is expressed in terms of equations simpler than the original equation. The practical significance of asymptotic methods is in revealing underlying structural properties possessed by the original state equation for small $|\varepsilon|$. We will see, in Section 10.1, examples where asymptotic methods reveal a weak coupling structure among isolated subsystems or the structure of a weakly nonlinear system. More important, asymptotic methods reveal multiple-time-scale structures inherent in many practical problems. Quite often, the solution of the state equation exhibits the phenomenon that some variables move in time faster than other variables, leading to the classification of variables as "slow" and "fast." Both the averaging method of this chapter and the singular perturbation method of the next chapter deal with the interaction of slow and fast variables.

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[1] Numerical solution methods are not studied in this textbook on the premise that most students are introduced to them in elementary differential equation courses and they get their in-depth study of the subject in numerical analysis courses.

Section 10.1 presents the classical perturbation method of seeking an approximate solution as a finite Taylor expansion of the exact solution. The asymptotic validity of the approximation is established in Section 10.1 on finite time intervals and in Section 10.2 on the infinite-time interval. Section 10.3 examines an autonomous system under the influence of a weak periodic perturbation. While the results of the first three sections are interesting in their own sake, they provide the technical basis for the averaging method. In Section 10.4, we introduce the averaging method in its simplest form, which is sometimes called "periodic averaging" since the right-hand side function is periodic in time. Section 10.5 gives an application of the averaging method to the study of periodic solutions of weakly nonlinear second-order systems. Finally, we present a more general form of the averaging method in Section 10.6.

10.1 The Perturbation Method

Consider the system

$$\dot{x} = f(t, x, \varepsilon) \tag{10.1}$$

where $f : [t_0, t_1] \times D \times [-\varepsilon_0, \varepsilon_0] \to R^n$ is "sufficiently smooth" in its arguments over a domain $D \subset R^n$. The required smoothness conditions will be spelled out as we proceed. Suppose we want to solve the state equation (10.1) for a given initial state

$$x(t_0) = \eta(\varepsilon) \tag{10.2}$$

where, for more generality, we allow the initial state to depend "smoothly" on ε. The solution of (10.1) and (10.2) depends on the parameter ε, a point that we emphasize by writing the solution as $x(t, \varepsilon)$. The goal of the perturbation method is to exploit the "smallness" of the perturbation parameter ε to construct approximate solutions that are valid for sufficiently small $|\varepsilon|$. The simplest approximation results by setting $\varepsilon = 0$ in (10.1) and (10.2) to obtain the nominal or unperturbed problem

$$\dot{x} = f(t, x, 0), \qquad x(t_0) = \eta_0 \tag{10.3}$$

where $\eta_0 = \eta(0)$. Suppose this problem has a unique solution $x_0(t)$ defined on $[t_0, t_1]$ and $x_0(t) \in D$ for all $t \in [t_0, t_1]$. Suppose further that f is continuous in (t, x, ε) and locally Lipschitz in (x, ε), uniformly in t, and η is locally Lipschitz in ε for (t, x, ε) in $[t_0, t_1] \times D \times [-\varepsilon_0, \varepsilon_0]$. The closeness of the solutions of the perturbed and unperturbed problems follows from continuity of solutions with respect to initial states and parameters. In particular, Theorem 3.5 shows that there is a positive constant $\varepsilon_1 \leq \varepsilon_0$ such that for all $|\varepsilon| \leq \varepsilon_1$, the problem of (10.1) and (10.2) has a unique solution $x(t, \varepsilon)$ defined on $[t_0, t_1]$. Furthermore, Theorem 3.4 shows that there is a positive constant k such that

$$\|x(t, \varepsilon) - x_0(t)\| \leq k|\varepsilon|, \quad \forall\ |\varepsilon| < \varepsilon_1,\ \forall\ t \in [t_0, t_1] \tag{10.4}$$

When the approximation error satisfies the bound of (10.4), we say that the error is of order $O(\varepsilon)$ and write

$$x(t,\varepsilon) - x_0(t) = O(\varepsilon)$$

This order of magnitude notation will be used frequently in this chapter and the next one. It is defined next.

Definition 10.1 $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ *if there exist positive constants* k *and* c *such that*

$$|\delta_1(\varepsilon)| \leq k|\delta_2(\varepsilon)|, \quad \forall\ |\varepsilon| < c$$

Example 10.1

- $\varepsilon^n = O(\varepsilon^m)$ for all $n \geq m$, since

$$|\varepsilon|^n = |\varepsilon|^m |\varepsilon|^{n-m} < |\varepsilon|^m, \quad \forall\ |\varepsilon| < 1$$

- $\varepsilon^2/(0.5+\varepsilon) = O(\varepsilon^2)$, since

$$\left|\frac{\varepsilon^2}{0.5+\varepsilon}\right| < \frac{1}{0.5-a}|\varepsilon|^2, \quad \forall\ |\varepsilon| < a < 0.5$$

- $1 + 2\varepsilon = O(1)$, since

$$|1+2\varepsilon| < 1 + 2a, \quad \forall\ |\varepsilon| < a$$

- $\exp(-a/\varepsilon)$ with positive a and ε is $O(\varepsilon^n)$ for any positive integer n, since

$$\frac{e^{-a/\varepsilon}}{\varepsilon^n} \leq \left(\frac{n}{a}\right)^n e^{-n}, \quad \forall\ \varepsilon > 0$$

△

What can we say about the numerical value of the approximation error $x(t,\varepsilon) - x_0(t)$ for a given numerical value of ε when the error is $O(\varepsilon)$? Unfortunately, we cannot translate the $O(\varepsilon)$ order of magnitude statement into a numerical bound on the error. Knowing that the error is $O(\varepsilon)$ means that its norm is less than $k|\varepsilon|$ for some positive constant k that is independent of ε. However, we do not know the value of k, which might be 1, 10, or any positive number.[2] The fact that k is independent of ε guarantees that the bound $k|\varepsilon|$ decreases monotonically as $|\varepsilon|$ decreases. Therefore, for sufficiently small $|\varepsilon|$, the error will be small. More precisely, given any tolerance δ, we know that the norm of the error will be less than δ for all $|\varepsilon| < \delta/k$. If this range is too small to cover the numerical values of interest for ε, we then need to extend the range of validity by obtaining a higher order approximation. An $O(\varepsilon^2)$ approximation will meet the same δ tolerance for all $|\varepsilon| < \sqrt{\delta/k_2}$, an $O(\varepsilon^3)$ approximation will do it for all $|\varepsilon| < (\delta/k_3)^{1/3}$, and so on. Although the constants $k, k_2, k_3, \ldots$ are not necessarily equal, these intervals are increasing in length, since the tolerance δ is typically much smaller than one. Another way to look at higher order approximations is to see that, for a given "sufficiently small" value of ε, an $O(\varepsilon^n)$ error will be smaller than an $O(\varepsilon^m)$ error for $n > m$, since

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[2] It should be noted, however, that in a well-formulated perturbation problem where variables are normalized to have dimensionless state variables, time, and perturbation parameter, one should expect the numerical value of k not to be much larger than one. See Example 10.4 for further discussion of normalization, or consult [98] and [141] for more examples.

$$\frac{k_1|\varepsilon|^n}{k_2|\varepsilon|^m} < 1, \quad \forall\ |\varepsilon| < \left(\frac{k_2}{k_1}\right)^{1/(n-m)}$$

Higher-order approximations for solutions of (10.1) and (10.2) can be obtained in a straightforward manner, provided the functions f and η are sufficiently smooth. Suppose f and η have continuous partial derivatives with respect to (x, ε) up to order N for $(t, x, \varepsilon) \in [t_0, t_1] \times D \times [-\varepsilon_0, \varepsilon_0]$. To obtain a higher order approximation of $x(t, \varepsilon)$, we construct a finite Taylor series

$$x(t,\varepsilon) = \sum_{k=0}^{N-1} x_k(t)\varepsilon^k + \varepsilon^N R_x(t,\varepsilon) \tag{10.5}$$

Two things need to be done here. First, we need to calculate the terms x_0, x_1, $\ldots$, x_{N-1}; in the process of doing that, it will be shown that these terms are well defined. Second, we need to show that the remainder term R_x is well defined and bounded on $[t_0, t_1]$, which will establish that $\sum_{k=0}^{N-1} x_k(t)\varepsilon^k$ is an $O(\varepsilon^N)$ (Nth-order) approximation of $x(t, \varepsilon)$. By Taylor's theorem,[3] the smoothness requirement on the initial state $\eta(\varepsilon)$ guarantees the existence of a finite Taylor series for $\eta(\varepsilon)$; that is,

$$\eta(\varepsilon) = \sum_{k=0}^{N-1} \eta_k\varepsilon^k + \varepsilon^N R_\eta(\varepsilon)$$

Therefore,

$$x_k(t_0) = \eta_k, \quad k = 0, 1, 2, \ldots, N-1$$

Substituting (10.5) into (10.1) yields

$$\begin{aligned} \sum_{k=0}^{N-1} \dot{x}_k(t)\varepsilon^k + \varepsilon^N \dot{R}_x(t,\varepsilon) &= f(t, x(t,\varepsilon), \varepsilon) \overset{\text{def}}{=} h(t,\varepsilon) \\ &= \sum_{k=0}^{N-1} h_k(t)\varepsilon^k + \varepsilon^N R_h(t,\varepsilon) \end{aligned} \tag{10.6}$$

where the coefficients of the Taylor series of $h(t, \varepsilon)$ are functions of the coefficients of the Taylor series of $x(t, \varepsilon)$. Since (10.6) holds for all sufficiently small ε, it must hold as an identity in ε. Hence, coefficients of like powers of ε must be equal. Matching those coefficients, we can derive the equations that must be satisfied by x_0, x_1, and so on. Before we do that, we have to generate the coefficients of the Taylor series of $h(t, \varepsilon)$. The zeroth-order term $h_0(t)$ is given by

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[3] See [10, Theorem 5-14].

$$h_0(t) = f(t, x_0(t), 0)$$

Consequently, matching coefficients of ε^0 in (10.6), we determine that $x_0(t)$ satisfies

$$\dot{x}_0 = f(t, x_0, 0), \qquad x_0(t_0) = \eta_0$$

which, not surprisingly, is the unperturbed problem (10.3). The first-order term $h_1(t)$ is given by

$$\begin{aligned} h_1(t) &= \left.\frac{\partial}{\partial \varepsilon} f(t, x(t, \varepsilon), \varepsilon)\right|_{\varepsilon=0} \\ &= \left.\left\{\frac{\partial f}{\partial x}(t, x(t, \varepsilon), \varepsilon)\frac{\partial x}{\partial \varepsilon}(t, \varepsilon) + \frac{\partial f}{\partial \varepsilon}(t, x(t, \varepsilon), \varepsilon)\right\}\right|_{\varepsilon=0} \\ &= \frac{\partial f}{\partial x}(t, x_0(t), 0)\ x_1(t) + \frac{\partial f}{\partial \varepsilon}(t, x_0(t), 0) \end{aligned}$$

Matching coefficients of ε in (10.6), we find that $x_1(t)$ satisfies

$$\dot{x}_1 = \frac{\partial f}{\partial x}(t, x_0(t), 0)\ x_1 + \frac{\partial f}{\partial \varepsilon}(t, x_0(t), 0), \qquad x_1(t_0) = \eta_1$$

Define

$$A(t) = \frac{\partial f}{\partial x}(t, x_0(t), 0), \qquad g_1(t, x_0(t)) = \frac{\partial f}{\partial \varepsilon}(t, x_0(t), 0)$$

and rewrite the equation for x_1 as

$$\dot{x}_1 = A(t)x_1 + g_1(t, x_0(t)), \qquad x_1(t_0) = \eta_1$$

This linear equation has a unique solution defined on $[t_0, t_1]$.

The process can be continued to derive the equations satisfied by x_2, x_3, and so on. This, however, will involve higher order differentials of f with respect to x, which makes the notation cumbersome. There is no point in writing the equations in a general form. Once the idea is clear, we can generate the equations for the specific problem of interest. Nevertheless, to set the pattern that these equations take, we will, at the risk of boring some readers, derive the equation for x_2. The second-order coefficient in the Taylor series of $h(t, \varepsilon)$ is given by

$$h_2(t) = \left.\frac{1}{2}\,\frac{\partial^2}{\partial \varepsilon^2} h(t, \varepsilon)\right|_{\varepsilon=0}$$

Now,

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}h(t,\varepsilon) &= \frac{\partial f}{\partial x}(t,x,\varepsilon)\ \frac{\partial x}{\partial \varepsilon}(t,\varepsilon) + \frac{\partial f}{\partial \varepsilon}(t,x,\varepsilon)\\ &= \frac{\partial f}{\partial x}(t,x,\varepsilon)[x_1(t) + 2\varepsilon x_2(t) + \cdots\ ] + \frac{\partial f}{\partial \varepsilon}(t,x,\varepsilon)\end{aligned}$$

To simplify the notation, let

$$\psi(t,x,\varepsilon) = \frac{\partial f}{\partial x}(t,x,\varepsilon)\ x_1(t)$$

and continue to calculate the second derivative of h with respect to ε:

$$\begin{aligned}\frac{\partial^2}{\partial \varepsilon^2}h(t,\varepsilon) &= \frac{\partial \psi}{\partial x}(t,x,\varepsilon)\ \frac{\partial x}{\partial \varepsilon}(t,\varepsilon) + \frac{\partial}{\partial \varepsilon}\frac{\partial f}{\partial x}(t,x,\varepsilon)\ x_1(t)\\ &\quad + 2\frac{\partial f}{\partial x}(t,x,\varepsilon)\ x_2(t) + \frac{\partial}{\partial x}\frac{\partial f}{\partial \varepsilon}(t,x,\varepsilon)\ \frac{\partial x}{\partial \varepsilon}(t,\varepsilon)\\ &\quad + \frac{\partial^2 f}{\partial \varepsilon^2}(t,x,\varepsilon) + \varepsilon[\ \cdot\ ]\end{aligned}$$

Thus,

$$h_2(t) = A(t)x_2(t) + g_2(t,x_0(t),x_1(t))$$

where

$$\begin{aligned}g_2(t,x_0(t),x_1(t)) &= \frac{1}{2}\frac{\partial \psi}{\partial x}(t,x_0(t),0)\ x_1(t) + \frac{\partial}{\partial \varepsilon}\frac{\partial f}{\partial x}(t,x_0(t),0)\ x_1(t)\\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial \varepsilon^2}(t,x_0(t),0)\end{aligned}$$

Matching coefficients of ε^2 in (10.6) yields

$$\dot{x}_2 = A(t)x_2 + g_2(t,x_0(t),x_1(t)), \qquad x_2(t_0) = \eta_2$$

In summary, the Taylor series coefficients x_0, x_1, $\ldots$, x_{N-1} are obtained by solving the equations

$$\dot{x}_0 = f(t,x_0,0), \qquad x_0(t_0) = \eta_0 \tag{10.7}$$

$$\dot{x}_k = A(t)x_k + g_k(t,x_0(t),\ldots,x_{k-1}(t)), \qquad x_k(t_0) = \eta_k \tag{10.8}$$

for $k = 1,2,\ldots,N-1$, where $A(t)$ is the Jacobian $[\partial f/\partial x]$ evaluated at $x = x_0(t)$ and $\varepsilon = 0$, and the term $g_k(t,x_0(t),x_1(t),\ldots,x_{k-1}(t))$ is a polynomial in $x_1,\ldots,x_{k-1}$ with coefficients depending continuously on t and $x_0(t)$. The assumption that $x_0(t)$ is defined on $[t_0,t_1]$ implies that $A(t)$ is defined on the same interval; hence, the linear equations (10.8) have unique solutions defined on $[t_0,t_1]$. Let us now illustrate the calculation of the Taylor series coefficients by a second-order example.

Example 10.2 Consider the Van der Pol state equation

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= \eta_1(\varepsilon) \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2, & x_2(0) &= \eta_2(\varepsilon) \end{aligned}$$

Suppose we want to construct a finite Taylor series with $N = 3$. Let

$$x_i = x_{i0} + \varepsilon x_{i1} + \varepsilon^2 x_{i2} + \varepsilon^3 R_{x_i}, \quad i = 1, 2$$

and

$$\eta_i = \eta_{i0} + \varepsilon \eta_{i1} + \varepsilon^2 \eta_{i2} + \varepsilon^3 R_{\eta_i}, \quad i = 1, 2$$

Substituting the series for x_1 and x_2 into the state equation results in

$$\begin{aligned} \dot{x}_{10} + \varepsilon \dot{x}_{11} + \varepsilon^2 \dot{x}_{12} + \varepsilon^3 \dot{R}_{x_1} &= x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 R_{x_2} \\ \dot{x}_{20} + \varepsilon \dot{x}_{21} + \varepsilon^2 \dot{x}_{22} + \varepsilon^3 \dot{R}_{x_2} &= -x_{10} - \varepsilon x_{11} - \varepsilon^2 x_{12} - \varepsilon^3 R_{x_1} \\ &\quad + \varepsilon \left[1 - (x_{10} + \varepsilon x_{11} + \varepsilon^2 x_{12} + \varepsilon^3 R_{x_1})^2\right] \\ &\quad \times (x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 R_{x_2}) \end{aligned}$$

Matching coefficients of ε^0, we obtain

$$\begin{aligned} \dot{x}_{10} &= x_{20}, & x_{10}(0) &= \eta_{10} \\ \dot{x}_{20} &= -x_{10}, & x_{20}(0) &= \eta_{20} \end{aligned}$$

which is the unperturbed problem at $\varepsilon = 0$. Matching coefficients of ε, we obtain

$$\begin{aligned} \dot{x}_{11} &= x_{21}, & x_{11}(0) &= \eta_{11} \\ \dot{x}_{21} &= -x_{11} + (1 - x_{10}^2)x_{20}, & x_{21}(0) &= \eta_{21} \end{aligned}$$

while matching coefficients of ε^2 results in

$$\begin{aligned} \dot{x}_{12} &= x_{22}, & x_{12}(0) &= \eta_{12} \\ \dot{x}_{22} &= -x_{12} + (1 - x_{10}^2)x_{21} - 2x_{10}x_{11}x_{20}, & x_{22}(0) &= \eta_{22} \end{aligned}$$

The latter two sets of equations are in the form of (10.8) for $k = 1, 2$. △

Having calculated the terms $x_0, x_1, \ldots, x_{N-1}$, our task now is to show that $\sum_{k=0}^{N-1} x_k(t)\varepsilon^k$ is indeed an $O(\varepsilon^N)$ approximation of $x(t, \varepsilon)$. Consider the approximation error

$$e = x - \sum_{k=0}^{N-1} x_k(t)\varepsilon^k \tag{10.9}$$

Differentiating both sides of (10.9) with respect to t and substituting for the derivatives of x and x_k from (10.1), (10.7), and (10.8), it can be shown that e satisfies the equation

$$\dot{e} = A(t)e + \rho_1(t,e,\varepsilon) + \rho_2(t,\varepsilon), \quad e(t_0) = \varepsilon^N R_\eta(\varepsilon) \tag{10.10}$$

where

$$\begin{aligned} \rho_1(t,e,\varepsilon) &= f(t, e + \sum_{k=0}^{N-1} x_k(t)\varepsilon^k, \varepsilon) - f(t, \sum_{k=0}^{N-1} x_k(t)\varepsilon^k, \varepsilon) - A(t)e \\ \rho_2(t,\varepsilon) &= f(t, \sum_{k=0}^{N-1} x_k(t)\varepsilon^k, \varepsilon) - f(t, x_0(t), 0) - \sum_{k=1}^{N-1} [A(t)x_k(t) + g_k(\cdot)]\,\varepsilon^k \end{aligned}$$

By assumption, $x_0(t)$ is bounded and belongs to D for all $t \in [t_0, t_1]$. Hence, there exist $\lambda > 0$ and $\varepsilon_1 > 0$ such that for all $\|e\| \le \lambda$ and $|\varepsilon| \le \varepsilon_1$, the functions $x_0(t)$, $\sum_{k=0}^{N-1} x_k(t)\varepsilon^k$, and $e + \sum_{k=0}^{N-1} x_k(t)\varepsilon^k$ belong to a compact subset of D. It can be easily verified that

$$\rho_1(t,0,\varepsilon) = 0 \tag{10.11}$$

$$\|\rho_1(t,e_2,\varepsilon) - \rho_1(t,e_1,\varepsilon)\| \le k_1\|e_2 - e_1\| \tag{10.12}$$

$$\|\rho_2(t,\varepsilon)\| \le k_2|\varepsilon|^N \tag{10.13}$$

for all $t \in [t_0, t_1]$, $e_1, e_2 \in B_\lambda$, $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$, for some positive constants k_1 and k_2. Equation (10.10) can be viewed as a perturbation of

$$\dot{e}_0 = A(t)e_0 + \rho_1(t,e_0,\varepsilon), \quad e_0(t_0) = 0 \tag{10.14}$$

which has the unique solution $e_0(t,\varepsilon) \equiv 0$ for $t \in [t_0, t_1]$. Applying Theorem 3.5 shows that (10.10) has a unique solution defined on $[t_0, t_1]$ for sufficiently small $|\varepsilon|$. Applying Theorem 3.4 shows that

$$\|e(t,\varepsilon)\| = \|e(t,\varepsilon) - e_0(t,\varepsilon)\| = O(\varepsilon^N)$$

We summarize our conclusion in the next theorem.

Theorem 10.1 *Suppose*

- *f and its partial derivatives with respect to (x,ε) up to order N are continuous in (t,x,ε) for $(t,x,\varepsilon) \in [t_0,t_1] \times D \times [-\varepsilon_0, \varepsilon_0]$;*
- *η and its derivatives up to order N are continuous for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$;*
- *the nominal problem given in (10.3) has a unique solution $x_0(t)$ defined on $[t_0, t_1]$ and $x_0(t) \in D$ for all $t \in [t_0, t_1]$.*

Then, there exists $\varepsilon^* > 0$ *such that* $\forall\ |\varepsilon| < \varepsilon^*$, *the problem given by* (10.1) *and* (10.2) *has a unique solution* $x(t, \varepsilon)$, *defined on* $[t_0, t_1]$, *which satisfies*

$$x(t,\varepsilon) - \sum_{k=0}^{N-1} x_k(t)\varepsilon^k = O(\varepsilon^N)$$

◇

When we approximate $x(t, \varepsilon)$ by $x_0(t)$, we need not know the value of the parameter ε, which could be an unknown parameter that represents deviations of the system's parameters from their nominal values. When we use a higher order approximation $\sum_{k=0}^{N-1} x_k(t)\varepsilon^k$ for $N \geq 2$, we need to know the value of ε to construct the series, even though we do not need it to calculate the terms x_1, x_2, and so on. If we have to know ε to construct the Taylor series approximation, we must then compare the computational effort needed to approximate the solution via a Taylor series with the effort needed to calculate the exact solution. The exact solution $x(t, \varepsilon)$ can be obtained by solving the nonlinear state equation (10.1), while the approximate solution is obtained by solving the nonlinear state equation (10.7) and a number of linear state equations (10.8), depending on the order of the approximation. Since, in both cases, we have to solve a nonlinear state equation of order n, we must ask ourselves, What do we gain by solving (10.7) instead of (10.1)? One situation where the Taylor series approximation will be clearly preferable is the case when the solution is sought for several values of ε. In the Taylor series approximation, equations (10.7) and (10.8) will be solved only once; then, different Taylor expansions will be constructed for different values of ε. Aside from this special (repeated values of ε) case, we find the Taylor series approximation to be effective when

- the unperturbed state equation (10.7) is considerably simpler than the ε-dependent state equation (10.1), and
- ε is reasonably small that an "acceptable" approximation can be achieved with a few terms in the series.

In most engineering applications of the perturbation method, adequate approximations are achieved with $N = 2$ or 3, and the process of setting $\varepsilon = 0$ simplifies the state equation considerably. In the next two examples, we look at two typical cases where setting $\varepsilon = 0$ reduces the complexity of the state equation. In the first example, we consider again the Van der Pol equation of Example 10.2, which represents a wide class of "weakly nonlinear systems" that become linear at $\varepsilon = 0$. To construct a Taylor series approximation, we only solve linear equations. In the second example, we look at a system formed of interconnected subsystems with "weak" or ε-coupling. At $\varepsilon = 0$, the system decomposes into lower order decoupled subsystems. To construct a Taylor series approximation, we always solve lower order decoupled equations as opposed to solving the original higher order equation (10.1).

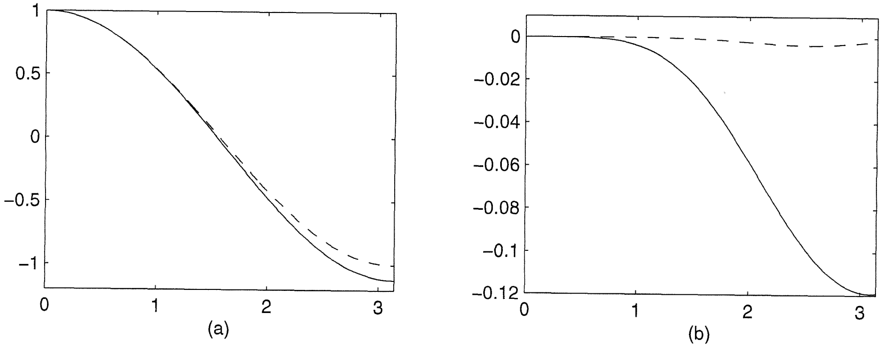

Figure 10.1: Example 10.3 at $\varepsilon = 0.1$: (a) $x_1(t,\varepsilon)$ (solid) and $x_{10}(t)$ (dashed); (b) $x_1(t,\varepsilon) - x_{10}(t)$ (solid) and $x_1(t,\varepsilon) - x_{10}(t) - \varepsilon x_{11}(t)$ (dashed).

Example 10.3 Suppose we want to solve the Van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= 1 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2, & x_2(0) &= 0 \end{aligned}$$

over the time interval $[0, \pi]$. We start by setting $\varepsilon = 0$ to obtain the linear unperturbed equation

$$\begin{aligned} \dot{x}_{10} &= x_{20}, & x_{10}(0) &= 1 \\ \dot{x}_{20} &= -x_{10}, & x_{20}(0) &= 0 \end{aligned}$$

whose solution is

$$x_{10}(t) = \cos t, \quad x_{20}(t) = -\sin t$$

Clearly, all the assumptions of Theorem 10.1 are satisfied, and we conclude that the approximation error $x(t,\varepsilon) - x_0(t)$ is $O(\varepsilon)$. Calculating $x(t,\varepsilon)$ numerically at three different values of ε and using

$$E_0 = \max_{0 \le t \le \pi} \|x(t,\varepsilon) - x_0(t)\|_2$$

as a measure of the approximation error, we find that $E_0 = 0.0112$, 0.0589, and 0.1192 for $\varepsilon = 0.01$, 0.05, and 0.1, respectively. These numbers show that the error is bounded by 1.2ε for $\varepsilon \le 0.1$. Figure 10.1(a) shows the exact and approximate trajectories of the first component of the state vector when $\varepsilon = 0.1$. Suppose we want to improve the approximation at $\varepsilon = 0.1$. From Example 10.2, we know that x_{11} and x_{21} satisfy the equation

$$\begin{aligned} \dot{x}_{11} &= x_{21}, & x_{11}(0) &= 0 \\ \dot{x}_{21} &= -x_{11} - (1 - \cos^2 t)\sin t, & x_{21}(0) &= 0 \end{aligned}$$

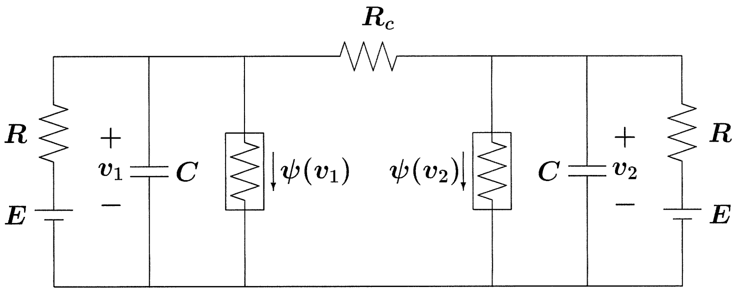

Figure 10.2: Electric circuit of Example 10.4.

whose solution is

$$\begin{aligned} x_{11}(t) &= -\tfrac{9}{32}\sin t - \tfrac{1}{32}\sin 3t + \tfrac{3}{8}t\cos t \\ x_{21}(t) &= \tfrac{3}{32}\cos t - \tfrac{3}{32}\cos 3t - \tfrac{3}{8}t\sin t \end{aligned}$$

By Theorem 10.1, the second-order approximation $x_0(t) + \varepsilon x_1(t)$ is $O(\varepsilon^2)$ close to the exact solution for sufficiently small ε. To compare the approximate solution with the exact one at $\varepsilon = 0.1$, we calculate

$$E_1 = \max_{0\le t\le\pi} \|x(t,0.1) - x_0(t) - 0.1x_1(t)\|_2 = 0.0057$$

which shows a reduction in the approximation error by almost an order of magnitude. Figure 10.1(b) shows the approximation errors in the first component of the state vector for the first-order approximation x_0 and the second-order approximation $x_0 + \varepsilon x_1$ at $\varepsilon = 0.1$. △

Example 10.4 The circuit shown in Figure 10.2 contains nonlinear resistors whose I–V characteristics are given by $i = \psi(v)$. The differential equations for the voltages across the capacitors are

$$\begin{aligned} C\frac{dv_1}{dt} &= \frac{1}{R}(E - v_1) - \psi(v_1) - \frac{1}{R_c}(v_1 - v_2) \\ C\frac{dv_2}{dt} &= \frac{1}{R}(E - v_2) - \psi(v_2) - \frac{1}{R_c}(v_2 - v_1) \end{aligned}$$

The circuit has two similar RC sections connected through the resistor R_c. When R_c is "relatively large," the connection between the two sections becomes "weak." In particular, when $R_c = \infty$, the connection is open circuit and the two sections are decoupled from each other. This circuit lends itself to an ε-coupling representation where the coupling between the two sections may be parameterized by a small parameter ε. At first glance, it appears that a reasonable choice of ε is $\varepsilon = 1/R_c$. Indeed, with this choice, the coupling terms in the foregoing equations will be multiplied by ε. However, such a choice makes ε dependent on the absolute value of a physical parameter whose value, no matter how small or large, has no significance by itself without considering the values of other physical parameters in the system. In a well-formulated perturbation problem, the parameter ε would be chosen as a ratio between physical parameters that reflects the "true smallness" of ε in a relative sense. To choose ε this way, we usually start by choosing the state variables or the time variable (or both) as dimensionless quantities. In our circuit, the clear choice of state variables is v_1 and v_2. Instead of working with v_1 and v_2, we scale them in such a way that the typical extreme values of the scaled variables would be close to ± 1. Due to the weak coupling between the two identical sections, it is reasonable to use the same scaling factor α for both state variables. Define the state variables as $x_1 = v_1/\alpha$ and $x_2 = v_2/\alpha$. Taking a dimensionless time $\tau = t/RC$ and writing $dx/d\tau = \dot{x}$, we obtain the state equation

$$\begin{aligned}\dot{x}_1 &= \frac{E}{\alpha} - x_1 - \frac{R}{\alpha}\psi(\alpha x_1) - \frac{R}{R_c}(x_1 - x_2)\\ \dot{x}_2 &= \frac{E}{\alpha} - x_2 - \frac{R}{\alpha}\psi(\alpha x_2) - \frac{R}{R_c}(x_2 - x_1)\end{aligned}$$

It appears now that a reasonable choice of ε is R/R_c. Suppose that $R = 1.5 \times 10^3$ Ω, $E = 1.2$ V, and the nonlinear resistors are tunnel diodes with

$$\psi(v) = 10^{-3} \times \left(17.76v - 103.79v^2 + 229.62v^3 - 226.31v^4 + 83.72v^5\right)$$

Take $\alpha = 1$ and rewrite the state equation as

$$\begin{aligned}\dot{x}_1 &= 1.2 - x_1 - h(x_1) - \varepsilon(x_1 - x_2)\\ \dot{x}_2 &= 1.2 - x_2 - h(x_2) - \varepsilon(x_2 - x_1)\end{aligned}$$

where $h(v) = 1.5 \times 10^3 \times \psi(v)$. Suppose we want to solve this equation for the initial state

$$x_1(0) = 0.15; \quad x_2(0) = 0.6$$

Setting $\varepsilon = 0$, we obtain the decoupled equations

$$\begin{aligned}\dot{x}_1 &= 1.2 - x_1 - h(x_1), \quad x_1(0) &= 0.15\\ \dot{x}_2 &= 1.2 - x_2 - h(x_2), \quad x_2(0) &= 0.6\end{aligned}$$

which are solved independently of each other. Let $x_{10}(t)$ and $x_{20}(t)$ be the solutions. According to Theorem 10.1, they provide an $O(\varepsilon)$ approximation of the exact solution for sufficiently small ε. To obtain an $O(\varepsilon^2)$ approximation, we set up the equations for x_{11} and x_{21} as

$$\begin{aligned}\dot{x}_{11} &= -[1 + h'(x_{10}(t))]x_{11} - [x_{10}(t) - x_{20}(t)], \quad x_{11}(0) &= 0\\ \dot{x}_{21} &= -[1 + h'(x_{20}(t))]x_{21} - [x_{20}(t) - x_{10}(t)], \quad x_{21}(0) &= 0\end{aligned}$$

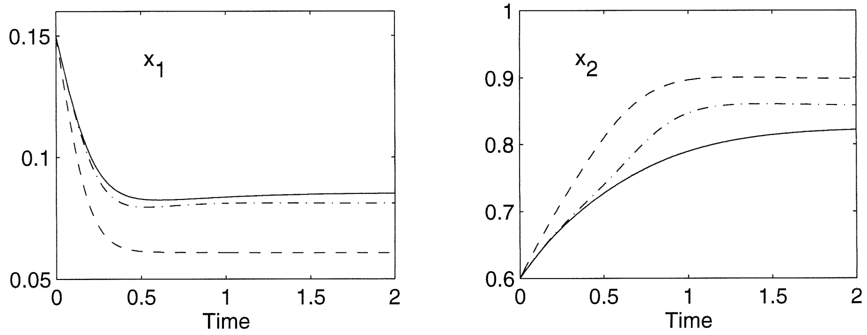

Figure 10.3: Exact solution (solid), first-order approximation (dashed), and second-order approximation (dash-dot) for Example 10.4 at $\varepsilon = 0.3$.

where $h'(\cdot)$ is the derivative of $h(\cdot)$. Figure 10.3 shows the exact solution as well as the first-order and second-order approximations for $\varepsilon = 0.3$. △

A serious limitation of Theorem 10.1 is that the $O(\varepsilon^N)$ error bound is valid only on finite (order $O(1)$) time intervals $[t_0, t_1]$. It does not hold on intervals like $[t_0, T/\varepsilon]$ nor on the infinite-time interval $[t_0, \infty)$. The reason is that the constant k in the bound $k|\varepsilon|^N$ depends on t_1 in such a way that it grows unbounded as t_1 increases. In particular, since the constant k results from application of Theorem 3.4, it has a component of the form $\exp(Lt_1)$. In the next section, we will see how to employ stability conditions to extend Theorem 10.1 to the infinite interval. In the lack of such stability conditions, the approximation may not be valid for large t, even though it is valid on $O(1)$ time intervals. Figure 10.4 shows the exact and approximate solutions for the Van der Pol equation of Example 10.3, at $\varepsilon = 0.1$, over a large time interval. For large t, the error $x_1(t,\varepsilon) - x_{10}(t)$ is no longer $O(\varepsilon)$. More seriously, the error $x_1(t,\varepsilon) - x_{10}(t) - \varepsilon x_{11}(t)$ grows unbounded, which is a consequence of the term $t\cos t$ in $x_{11}(t)$.

10.2 Perturbation on the Infinite Interval

The perturbation result of Theorem 10.1 can be extended to the infinite time interval $[t_0, \infty)$ under some additional stability conditions. In the next theorem, we require the nominal system (10.7) to have an exponentially stable equilibrium point at the origin and use a Lyapunov function to estimate its region of attraction. There is no loss of generality in taking the equilibrium point at the origin, since any equilibrium point can be shifted to the origin by a change of variables.

Theorem 10.2 *Let $D \subset R^n$ be a domain that contains the origin and suppose*

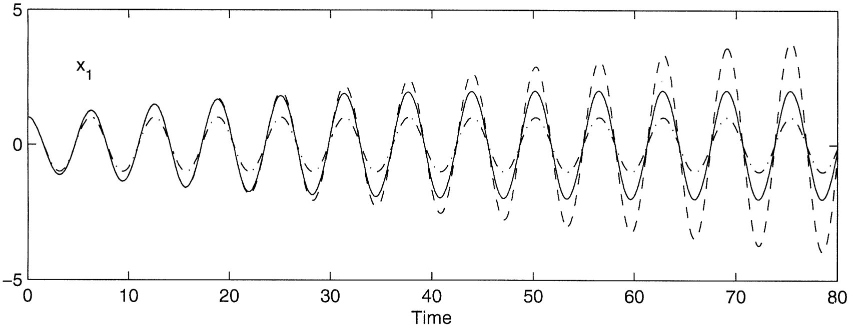

Figure 10.4: Exact solution (solid), first-order approximation (dash-dot), and second-order approximation (dashed) for the Van der Pol equation over a large time interval.

- *f and its partial derivatives with respect to (x, ε) up to order N are continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [-\varepsilon_0, \varepsilon_0]$, for every compact set $D_0 \subset D$; if $N = 1$, $[\partial f/\partial x](t, x, \varepsilon)$ is Lipschitz in (x, ε), uniformly in t;*
- *η and its derivatives up to order N are continuous for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$;*
- *the origin is an exponentially stable equilibrium point of the nominal system (10.7);*
- *there is a Lyapunov function $V(t, x)$ that satisfies the conditions of Theorem 4.9 for the nominal system (10.7) for $(t, x) \in [0, \infty) \times D$ and $\{W_1(x) \le c\}$ is a compact subset of D.*

Then, for each compact set $\Omega \subset \{W_2(x) \le \rho c,\ 0 < \rho < 1\}$, there is a positive constant ε^* such that for all $t_0 \ge 0$, $\eta_0 \in \Omega$, and $|\varepsilon| < \varepsilon^*$, equations (10.1) and (10.2) have a unique solution $x(t, \varepsilon)$, uniformly bounded on $[t_0, \infty)$, and

$$x(t, \varepsilon) - \sum_{k=0}^{N-1} x_k(t)\varepsilon^k = O(\varepsilon^N)$$

where $O(\varepsilon^N)$ holds uniformly in t for all $t \ge t_0$. ◇

If the nominal system (10.7) is autonomous, the set Ω in Theorem 10.2 can be any compact subset of the region of attraction of the origin. This is a consequence of (the converse Lyapunov) Theorem 4.17, since the Lyapunov function $V(x)$ provided by the theorem has the property that any compact subset of the region of attraction can be included in the interior of a compact set of the form $\{V(x) \le c\}$.

Proof of Theorem 10.2: Application of Theorem 9.1 shows that there is $\varepsilon_1 > 0$

such that for all $|\varepsilon| < \varepsilon_1$, $x(t, \varepsilon)$ is uniformly bounded and $x(t, \varepsilon) - x_0(t)$ is $O(\varepsilon)$, uniformly in t, for all $t \geq t_0$. It is also clear that for $\eta_0 \in \Omega$, $x_0(t)$ is uniformly bounded and $\lim_{t\to\infty} x_0(t) = 0$. Consider the linear equations (10.8). We know from bounded-input–bounded-output stability (Theorem 5.1) that the solution of (10.8) will be uniformly bounded if the origin of $\dot{z} = A(t)z$ is exponentially stable and the input term g_k is bounded. The input g_k is a polynomial in $x_1, \ldots, x_{k-1}$ with coefficients depending on t and $x_0(t)$. The dependence on t comes through the partial derivatives of f, which are bounded on compact subsets of D. Since $x_0(t)$ is bounded, the polynomial coefficients are bounded for all $t \geq t_0$. Hence, boundedness of g_k will follow from boundedness of $x_1, \ldots, x_{k-1}$. The matrix $A(t)$ is given by

$$A(t) = \frac{\partial f}{\partial x}(t, x_0(t), 0)$$

where $x_0(t)$ is the solution of the nominal system (10.7). It turns out that exponential stability of the origin as an equilibrium for (10.7) ensures that the origin of $\dot{z} = A(t)z$ will be exponentially stable for every solution $x_0(t)$ that starts in the set Ω. To see this point, let

$$A_0(t) = \frac{\partial f}{\partial x}(t, 0, 0)$$

and write

$$A(t) = A_0(t) + [A(t) - A_0(t)] \overset{\text{def}}{=} A_0(t) + B(t)$$

so that the linear system $\dot{z} = A(t)z$ can be viewed as a linear perturbation of $\dot{y} = A_0(t)y$. Since $[\partial f/\partial x](t, x, 0)$ is Lipschitz in x, uniformly in t,

$$\|B(t)\| = \left\| \frac{\partial f}{\partial x}(t, x_0(t), 0) - \frac{\partial f}{\partial x}(t, 0, 0) \right\| \leq L\|x_0(t)\|$$

On the other hand, by exponential stability of the origin of (10.7) and Theorem 4.15, we know that the origin of the linear system $\dot{y} = A_0(t)y$ is exponentially stable. Therefore, similar to Example 9.6, we can use $\lim_{t\to\infty} x_0(t) = 0$ to show that the origin of the linear system $\dot{z} = A(t)z$ is exponentially stable.

Since $\|x_0(t)\|$ is bounded and $g_1(t, x_0(t)) = [\partial f/\partial \varepsilon](t, x_0(t), 0)$, we see that g_1 is bounded for all $t \geq t_0$. Hence, by Theorem 5.1, we conclude that $x_1(t)$ is bounded. By a simple induction argument, we can see that $x_2(t), \ldots, x_{k-1}(t)$ are bounded.

So far, we have verified that the exact solution $x(t, \varepsilon)$ and the approximate solution $\sum_{k=0}^{N-1} x_k(t)\varepsilon^k$ are uniformly bounded on $[t_0, \infty)$ for sufficiently small $|\varepsilon|$. All that remains now is to analyze the approximation error $e = x - \sum_{k=0}^{N-1} x_k(t)\varepsilon^k$. The error analysis is quite similar to what we have done in Section 10.1. The error satisfies (10.10), where ρ_1 and ρ_2 satisfy (10.11), (10.13) and

$$\left\| \frac{\partial \rho_1}{\partial e}(t, e, \varepsilon) \right\| \leq k_1(\|e\| + |\varepsilon|)$$

for all $(t, e, \varepsilon) \in [t_0, \infty) \times B_\lambda \times [-\varepsilon_1, \varepsilon_1]$ for sufficiently small ε_1. The error equation (10.10) can be viewed as a perturbation of $\dot{e} = A(t)e$, where the perturbation term satisfies

$$\|\rho_1(t, e, \varepsilon) + \rho_2(t, \varepsilon)\| \leq k_1(\|e\| + |\varepsilon|)\|e\| + k_2|\varepsilon|^N \leq k_1(\lambda + |\varepsilon|)\|e\| + k_2|\varepsilon|^N$$

Noting that $\|e(t_0, \varepsilon)\| = O(\varepsilon^N)$, we conclude from Lemma 9.4 that, for sufficiently small $|\varepsilon|$, $\|e(t, \varepsilon)\| = O(\varepsilon^N)$ for all $t \geq t_0$. □

Example 10.5 The electric circuit of Example 10.4 is represented by

$$\begin{aligned} \dot{x}_1 &= 1.2 - x_1 - h(x_1) - \varepsilon(x_1 - x_2) \\ \dot{x}_2 &= 1.2 - x_2 - h(x_2) - \varepsilon(x_2 - x_1) \end{aligned}$$

where

$$h(v) = 1.5\left(17.76v - 103.79v^2 + 229.62v^3 - 226.31v^4 + 83.72v^5\right)$$

At $\varepsilon = 0$, the unperturbed system comprises two isolated first-order subsystems:

$$\begin{aligned} \dot{x}_1 &= 1.2 - x_1 - h(x_1) \\ \dot{x}_2 &= 1.2 - x_2 - h(x_2) \end{aligned}$$

It can be verified that each of the two systems has three equilibrium points at 0.063, 0.285, and 0.884. The Jacobian $-1 + h'(x_i)$ is negative at $x_i = 0.063$ and $x_i = 0.884$ and positive at $x_i = 0.285$. Hence, the equilibrium points at 0.063 and 0.884 are exponentially stable, while the equilibrium point at 0.285 is unstable. When the two first-order systems are put together, the composite second-order system will have nine equilibrium points; only four of them will be exponentially stable. These are the equilibrium points $(0.063, 0.063)$, $(0.063, 0.884)$, $(0.884, 0.063)$, and $(0.884, 0.884)$. Theorem 10.2 says that if the initial state $x(0)$ belongs to a compact subset of the region of attraction of any one of these equilibrium points, the approximation calculated in Example 10.4 will be valid for all $t \geq 0$. The simulation shown in Figure 10.3 was taken over a time interval long enough for the solution to reach steady state. In this particular case, the initial state $(0.15, 0.6)$ belongs to the region of attraction of $(0.063, 0.884)$. △

The $O(\varepsilon^N)$ estimate of Theorem 10.2 is valid only when the origin is exponentially stable. It does not necessarily hold if it is asymptotically, but not exponentially, stable, as illustrated by the next example.

Example 10.6 Consider the first-order system

$$\dot{x} = -x^3 + \varepsilon x$$

and suppose $\varepsilon > 0$. The origin of the unperturbed system

$$\dot{x} = -x^3$$

is globally asymptotically stable, but not exponentially stable. (See Example 4.23.) The perturbed system has three equilibrium points at $x = 0$ and $x = \pm\sqrt{\varepsilon}$. The equilibrium $x = 0$ is unstable, while the other two equilibria are asymptotically stable. Solving both systems with the same positive initial condition $x(0) = a$, it can be easily seen that

$$x(t, \varepsilon) \to \sqrt{\varepsilon} \ \text{ and } \ x_0(t) \to 0 \ \text{ as } \ t \to \infty$$

Because $\sqrt{\varepsilon}$ is not $O(\varepsilon)$, it is clear that the approximation error $x(t, \varepsilon) - x_0(t)$ is not $O(\varepsilon)$ for all $t \geq 0$. Nevertheless, since the origin is asymptotically stable, we should be able to make a statement about the asymptotic behavior of the approximation as $t \to \infty$. Indeed, we can make a statement, although it will be weaker than the statement of Theorem 10.2. Because the origin of the unperturbed system is asymptotically stable, the solution $x_0(t)$ tends to zero as $t \to \infty$; equivalently, given any $\delta > 0$, there is $T_1 > 0$ such that

$$\|x_0(t)\| < \delta/2, \quad \forall \, t \geq T_1$$

The solutions of the perturbed system are ultimately bounded by a bound that shrinks with ε. Therefore, given any $\delta > 0$, there is $T_2 > 0$ and $\varepsilon^* > 0$ such that

$$\|x(t, \varepsilon)\| < \delta/2, \quad \forall \, t \geq T_2, \ \forall \, \varepsilon < \varepsilon^*$$

Combining these two estimates, we can say that for any $\delta > 0$ the approximation error satisfies

$$\|x(t, \varepsilon) - x_0(t)\| < \delta, \quad \forall \, t \geq T, \ \forall \, \varepsilon < \varepsilon^*$$

where $T = \max\{T_1, T_2\}$. On the order $O(1)$ time interval $[0, T]$, we know from the finite time result of Theorem 10.1 that the approximation error is $O(\varepsilon)$. Therefore, we can say that for any $\delta > 0$, there is $\varepsilon^{**} > 0$ such that

$$\|x(t, \varepsilon) - x_0(t)\| < \delta, \quad \forall \, t \in [0, \infty), \ \forall \, \varepsilon < \varepsilon^{**}$$

The last inequality is equivalent to saying that the approximation error tends to zero as $\varepsilon \to 0$, uniformly in t for all $t \geq 0$, which is the best we can show, in general, in the lack of exponential stability. Of course, in this particular example, we can obtain both $x_0(t)$ and $x(t, \varepsilon)$ in closed form, and we can actually show that the approximation error is $O(\sqrt{\varepsilon})$. △

10.3 Periodic Perturbation of Autonomous Systems

Consider the system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon) \tag{10.15}$$

where f, g, and their first partial derivatives with respect to x are continuous and bounded for all $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [-\varepsilon_0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain that contains the origin. Suppose the origin is an exponentially stable equilibrium point of the autonomous system

$$\dot{x} = f(x) \tag{10.16}$$

Equivalently,[4] the matrix $A = [\partial f / \partial x](0)$ is Hurwitz. Due to boundedness of g, we can use Theorem 4.14 and Lemma 9.2 to show that there exist $r > 0$ and $\varepsilon_1 > 0$ such that for all $\|x(0)\| \leq r$ and $|\varepsilon| \leq \varepsilon_1$, the solution of (10.15) is uniformly ultimately bounded with ultimate bound proportional to $|\varepsilon|$. In other words, all solutions approach an $O(\varepsilon)$ neighborhood of the origin as $t \to \infty$. This is true for any bounded g. In this section, we are interested in what happens inside that $O(\varepsilon)$ neighborhood when g is T-periodic in t; that is,

$$g(t + T, x, \varepsilon) = g(t, x, \varepsilon), \quad \forall\ (t, x, \varepsilon) \in [0, \infty) \times D \times [-\varepsilon_0, \varepsilon_0]$$

In particular, we are interested in the possibility that a T-periodic solution might exist within an $O(\varepsilon)$ neighborhood of the origin.

Let $\phi(t; t_0, x_0, \varepsilon)$ be the solution of (10.15) that starts at (t_0, x_0); that is, $x_0 = \phi(t_0; t_0, x_0, \varepsilon)$. For all $\|x\| < r$, define a map $P_\varepsilon(x)$ by

$$P_\varepsilon(x) = \phi(T; 0, x, \varepsilon)$$

That is, $P_\varepsilon(x)$ is the state of the system at time T when the initial state at time zero is x. This map plays a key role in studying the existence of periodic solutions of (10.15).[5]

Lemma 10.1 *Under the foregoing conditions, equation* (10.15) *has a T-periodic solution if and only if the equation*

$$x = P_\varepsilon(x) \tag{10.17}$$

has a solution. ◇

Proof: Since g is T-periodic in t, the solution of (10.15) is invariant to time shifts that are integer multiples of T. In particular,

$$\phi(t + T; T, x, \varepsilon) = \phi(t; 0, x, \varepsilon), \quad \forall\ t \geq 0 \tag{10.18}$$

This can be seen by changing the time variable from t to $\tau = t - T$, which yields

$$\frac{dx}{d\tau} = f(x) + \varepsilon g(\tau + T, x, \varepsilon) = f(x) + \varepsilon g(\tau, x, \varepsilon)$$

---

[4] The equivalence follows from Theorem 4.15.

[5] This map can be interpreted [70, Section 4.1] as a Poincaré map of the $(n + 1)$-dimensional autonomous system

$$\dot{x} = f(x) + \varepsilon g(\theta, x, \varepsilon), \qquad \dot{\theta} = 1$$

On the other hand, by uniqueness of solution, we have

$$\phi(t+T;0,x,\varepsilon) = \phi(t+T;T,\phi(T;0,x,\varepsilon),\varepsilon), \quad \forall\, t \geq 0 \tag{10.19}$$

To prove sufficiency, let

$$p_\varepsilon = P_\varepsilon(p_\varepsilon) = \phi(T;0,p_\varepsilon,\varepsilon)$$

Then

$$\begin{aligned} \phi(t+T;0,p_\varepsilon,\varepsilon) &= \phi(t+T;T,\phi(T;0,p_\varepsilon,\varepsilon),\varepsilon) \\ &= \phi(t+T;T,p_\varepsilon,\varepsilon) \\ &= \phi(t;0,p_\varepsilon,\varepsilon) \end{aligned} \tag{10.20}$$

where the first equality follows from (10.19) and the last equality from (10.18). Equation (10.20) shows that the solution starting at $(0, p_\varepsilon)$ is T-periodic. To prove necessity, let $\bar{x}(t)$ be a T-periodic solution of (10.15). Set $y = \bar{x}(0)$. Then

$$\phi(t+T;0,y,\varepsilon) = \phi(t;0,y,\varepsilon), \quad \forall\, t \geq 0$$

Taking $t = 0$ yields

$$\phi(T;0,y,\varepsilon) = \phi(0;0,y,\varepsilon) = y$$

which shows that y is a solution of (10.17). □

Lemma 10.2 *Under the foregoing conditions, there exist positive constants k and ε_2 such that* (10.17) *has a unique solution in $\|x\| < k|\varepsilon|$, for all $|\varepsilon| < \varepsilon_2$.* ◇

Proof: At $\varepsilon = 0$, $\phi(t;0,x,0)$ is the solution of the unperturbed system (10.16) that starts at $(0,x)$. Since $x = 0$ is an equilibrium point for (10.16), $0 = \phi(t;0,0,0)$ for all $t \geq 0$. Hence,

$$P_0(0) = \phi(T;0,0,0) = 0$$

From the implicit function theorem, it follows that if the Jacobian matrix

$$J = I - \left.\frac{\partial P_\varepsilon}{\partial x}\right|_{x=0,\varepsilon=0}$$

is nonsingular, then there is a positive constant ε_2 such that equation (10.17) has a unique solution p_ε in $|\varepsilon| < \varepsilon_2$. To check nonsingularity of the Jacobian matrix, recall that the solution $\phi(t;0,x,\varepsilon)$ is given by

$$\phi(t;0,x,\varepsilon) = x + \int_0^t \left[f(\phi(\tau;0,x,\varepsilon)) + \varepsilon g(\tau,\phi(\tau;0,x,\varepsilon),\varepsilon)\right]\, d\tau$$

Differentiating with respect to x yields

$$\frac{\partial}{\partial x}\phi(t;0,x,\varepsilon) = I + \int_0^t \left[\frac{\partial f}{\partial x}(\cdot)\ \frac{\partial \phi}{\partial x}(\cdot) + \varepsilon\ \frac{\partial g}{\partial x}(\cdot)\ \frac{\partial \phi}{\partial x}(\cdot)\right]\ d\tau$$

Let

$$U(t) = \left.\frac{\partial}{\partial x}\phi(t;0,x,\varepsilon)\right|_{x=0,\varepsilon=0}$$

Then,

$$U(t) = I + \int_0^t \frac{\partial f}{\partial x}(0)U(\tau)\ d\tau\ = \ I + \int_0^t AU(\tau)\ d\tau$$

and

$$\frac{d}{dt}U(t) = AU(t),\ \ U(0) = I$$

Thus, $U(t) = \exp(At)$. Consequently,

$$I - \left.\frac{\partial P_\varepsilon}{\partial x}\right|_{x=0,\varepsilon=0} = I - \exp(AT)$$

Because A is Hurwitz, all eigenvalues of $\exp(AT)$ are strictly inside the unit circle.[6] Consequently, J is nonsingular. Hence, (10.17) has a unique solution p_ε, $\forall\ |\varepsilon| < \varepsilon_2$. On the other hand, since all solutions of (10.15) approach an $O(\varepsilon)$ neighborhood of the origin as $t \to \infty$, it must be true that p_ε is $O(\varepsilon)$, because the corresponding periodic solution passes through p_ε infinitely many times as $t \to \infty$. □

It is now clear that, for sufficiently small ε, the perturbed system (10.15) has a T-periodic solution in an $O(\varepsilon)$ neighborhood of the origin. In fact, this periodic solution has to be unique due to the uniqueness of the solution of equation (10.17). Using the Hurwitz property of A, we can go further to show that the periodic solution is exponentially stable.

Lemma 10.3 *Under the foregoing conditions, if $\bar{x}(t,\varepsilon)$ is a T-periodic solution of* (10.15) *such that $\|\bar{x}(t,\varepsilon)\| \leq k|\varepsilon|$, then $\bar{x}(t,\varepsilon)$ is exponentially stable.* ◇

Proof: A systematic procedure to study the stability of $\bar{x}(t,\varepsilon)$ is to apply the change of variables $z = x - \bar{x}(t,\varepsilon)$ and study the stability of the equilibrium point at $z = 0$. The new variable z satisfies the equation

$$\begin{aligned} \dot{z} &= f(z + \bar{x}(t,\varepsilon)) - f(\bar{x}(t,\varepsilon)) + \varepsilon\left[g(t, z + \bar{x}(t,\varepsilon), \varepsilon) - g(t, \bar{x}(t,\varepsilon), \varepsilon)\right] \\ &\overset{\text{def}}{=} \hat{f}(t,z) \end{aligned}$$

---

[6] This is a well-known fact in sampled-data control theory. It can be proved by transforming A into its Jordan form.

Linearization about $z = 0$ yields

$$\begin{aligned}\left.\frac{\partial \hat{f}}{\partial z}\right|_{z=0} &= \left.\frac{\partial f}{\partial x}\right|_{z=0} + \varepsilon \left.\frac{\partial g}{\partial x}\right|_{z=0} \\ &= A + \left[\frac{\partial f}{\partial x}(\bar{x}(t,\varepsilon)) - A\right] + \varepsilon \frac{\partial g}{\partial x}(t, \bar{x}(t,\varepsilon), \varepsilon)\end{aligned}$$

By continuity of $[\partial f/\partial x]$, we know that for any $\delta > 0$, there is $\varepsilon^* > 0$ such that

$$\left\| \frac{\partial f}{\partial x}(\bar{x}(t,\varepsilon)) - \frac{\partial f}{\partial x}(0) \right\| < \delta$$

for $\varepsilon < \varepsilon^*$. Since A is Hurwitz and $[\partial g/\partial x](t, \bar{x}, \varepsilon)$ is $O(1)$, we conclude from Lemma 9.1 that, for sufficiently small ε, the linear system

$$\dot{y} = \left[A + \left(\frac{\partial f}{\partial x}(\bar{x}(t,\varepsilon)) - A\right) + \varepsilon \frac{\partial g}{\partial x}(t, \bar{x}(t,\varepsilon), \varepsilon)\right] y$$

has an exponentially stable equilibrium point at $y = 0$. Therefore, by Theorem 4.13, $z = 0$ is an exponentially stable equilibrium point. □

We summarize our findings in the next theorem.

Theorem 10.3 *Suppose*

- *f, g, and their first partial derivatives with respect to x are continuous and bounded for all $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [-\varepsilon_0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain that contains the origin;*
- *The origin is an exponentially stable equilibrium point of the autonomous system* (10.16);
- *$g(t, x, \varepsilon)$ is T-periodic in t.*

Then, there exist positive constants ε^* and k such that for all $|\varepsilon| < \varepsilon^*$, equation (10.15) *has a unique T-periodic solution $\bar{x}(t,\varepsilon)$ with the property that $\|\bar{x}(t,\varepsilon)\| \leq k|\varepsilon|$. Moreover, this solution is exponentially stable.* ◇

If $g(t, 0, \varepsilon) = 0$, the origin will be an equilibrium point of the perturbed system (10.15). By uniqueness of the periodic solution $\bar{x}(t,\varepsilon)$, it follows that $\bar{x}(t,\varepsilon)$ is the trivial solution $x = 0$. In this case, the theorem ensures that the origin is an exponentially stable equilibrium point of the perturbed system (10.15).

10.4 Averaging

The averaging method applies to a system of the form

$$\dot{x} = \varepsilon f(t, x, \varepsilon)$$

where ε is a small positive parameter and $f(t, x, \varepsilon)$ is T-periodic in t; that is,

$$f(t+T, x, \varepsilon) = f(t, x, \varepsilon), \quad \forall\ (t, x, \varepsilon) \in [0, \infty) \times D \times [0, \varepsilon_0]$$

for some domain $D \subset R^n$. The method approximates the solution of this system by the solution of an "average system," obtained by averaging $f(t, x, \varepsilon)$ at $\varepsilon = 0$. To motivate the averaging method, let us start by examining a scalar example.

Example 10.7 Consider the first-order linear system

$$\dot{x} = \varepsilon a(t, \varepsilon)x, \quad x(0) = \eta \tag{10.21}$$

where ε is a positive parameter, a is sufficiently smooth in its arguments, and $a(t+T, \varepsilon) = a(t, \varepsilon)$ for all $t \geq 0$. To obtain an approximate solution that is valid for small ε, we may apply the perturbation method of Section 10.1. Setting $\varepsilon = 0$ results in the unperturbed system

$$\dot{x} = 0, \quad x(0) = \eta$$

which has a constant solution $x_0(t) = \eta$. According to Theorem 10.1, the error of this approximation will be $O(\varepsilon)$ on $O(1)$ time intervals. The unperturbed system does not satisfy the conditions of Theorem 10.2. Therefore, it is not clear whether this approximation is valid on time intervals larger than $O(1)$. Because in this example we can write down a closed-form expression for the exact solution, we will examine the approximation error by direct calculations. The solution of (10.21) is given by

$$x(t, \varepsilon) = \exp\left[\varepsilon \int_0^t a(\tau, \varepsilon)\ d\tau\right] \eta$$

Hence, the approximation error is

$$x(t, \varepsilon) - x_0(t) = \left\{\exp\left[\varepsilon \int_0^t a(\tau, \varepsilon)\ d\tau\right] - 1\right\} \eta$$

To see how the approximation error behaves as t increases, we need to evaluate the integral term in the foregoing expression. The function $a(t, \varepsilon)$ is periodic in t. Let its mean be

$$\bar{a}(\varepsilon) = \frac{1}{T} \int_0^T a(\tau, \varepsilon)\ d\tau$$

We can write $a(t, \varepsilon)$ as

$$a(t, \varepsilon) = \bar{a}(\varepsilon) + [a(t, \varepsilon) - \bar{a}(\varepsilon)]$$

The term inside the bracket is a T-periodic function of t with zero mean. Therefore, the integral

$$\int_0^t [a(\tau,\varepsilon) - \bar{a}(\varepsilon)] \; d\tau \stackrel{\text{def}}{=} \Delta(t,\varepsilon)$$

is T-periodic and, hence, bounded for all $t \geq 0$. On the other hand, the integration of the term $\bar{a}(\varepsilon)$ on $[0,t]$ results in $t\bar{a}(\varepsilon)$. Thus,

$$x(t,\varepsilon) - x_0(t) = \{\exp[\varepsilon t \bar{a}(\varepsilon)] \exp[\varepsilon \Delta(t,\varepsilon)] - 1\} \, \eta$$

Except for the case $\bar{a}(\varepsilon) = 0$, the approximation error will be $O(\varepsilon)$ only on $O(1)$ time intervals. A careful examination of the approximation error suggests that a better approximation of $x(t,\varepsilon)$ is $\exp[\varepsilon t \bar{a}(\varepsilon)]\eta$ or even $\exp[\varepsilon t \bar{a}(0)]\eta$, since $\bar{a}(\varepsilon) - \bar{a}(0) = O(\varepsilon)$. Let us try $\bar{x}(\varepsilon t) = \exp[\varepsilon t \bar{a}(0)]\eta$ as an alternative approximation. The approximation error is given by

$$\begin{aligned} x(t,\varepsilon) - \bar{x}(\varepsilon t) &= \{\exp[\varepsilon t \bar{a}(\varepsilon)] \exp[\varepsilon \Delta(t,\varepsilon)] - \exp[\varepsilon t \bar{a}(0)]\} \, \eta \\ &= \exp[\varepsilon t \bar{a}(0)] \{\exp[\varepsilon t (\bar{a}(\varepsilon) - \bar{a}(0))] \exp[\varepsilon \Delta(t,\varepsilon)] - 1\} \, \eta \end{aligned}$$

Noting that

$$\begin{aligned} \exp[\varepsilon \Delta(t,\varepsilon)] &= 1 + O(\varepsilon), \quad \forall\, t \geq 0 \\ \exp[\varepsilon t (\bar{a}(\varepsilon) - \bar{a}(0))] &= \exp[t O(\varepsilon^2)] \; = \; 1 + O(\varepsilon), \quad \forall\, t \in [0, b/\varepsilon] \\ \exp[\varepsilon t \bar{a}(0)] &= O(1), \quad \forall\, t \in [0, b/\varepsilon] \end{aligned}$$

for any finite $b > 0$, we conclude that $x(t,\varepsilon) - \bar{x}(\varepsilon t) = O(\varepsilon)$ on time intervals of order $O(1/\varepsilon)$, which confirms the conjecture that the approximation $\bar{x}(\varepsilon t) = \exp[\varepsilon t \bar{a}(0)]\eta$ is better than the approximation $x_0(t) = \eta$. Note that $\bar{x}(\varepsilon t)$ is the solution of the average system

$$\dot{x} = \varepsilon \bar{a}(0) x, \quad x(0) = \eta \tag{10.22}$$

whose right-hand side is the average of the right-hand side of (10.21) at $\varepsilon = 0$. △

In this example, we have arrived at the average system (10.22) through our knowledge of the closed-form expression of the exact solution of (10.21). Such closed-form expressions are available only in very special cases. However, the plausibility of averaging is not dependent on the special features of the example. Let us reason the idea of averaging in a different way. The right-hand side of (10.21) is multiplied by a positive constant ε. When ε is small, the solution x will vary "slowly" with t relative to the periodic fluctuation of $a(t,\varepsilon)$. It is intuitively clear that if the response of a system is much slower than the excitation, then such response will be determined predominantly by the average of the excitation. This intuition has its roots in linear system theory, where we know that if the bandwidth of the system is much smaller than the bandwidth of the input, then the system will act as a low-pass filter that rejects the high-frequency component of the input.

If the solution of (10.21) is determined predominantly by the average of the fluctuation of $a(t,\varepsilon)$, then it is reasonable, in order to get an $O(\varepsilon)$ approximation, that the function $a(t,\varepsilon)$ be replaced by its average. This two-time-scale interpretation of averaging is not dependent on the special features of Example 10.7, nor is it dependent on the linearity of the system. It is a plausible idea that works in a more general setup, as we shall see in the rest of the chapter.

Consider the system

$$\dot{x} = \varepsilon f(t,x,\varepsilon) \tag{10.23}$$

where f and its partial derivatives with respect to (x,ε) up to the second order are continuous and bounded for $(t,x,\varepsilon) \in [0,\infty) \times D_0 \times [0,\varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain. Moreover, $f(t,x,\varepsilon)$ is T-periodic in t for some $T > 0$ and ε is positive. We associate with (10.23) an autonomous average system

$$\dot{x} = \varepsilon f_{\text{av}}(x) \tag{10.24}$$

where

$$f_{\text{av}}(x) = \frac{1}{T}\int_0^T f(\tau,x,0)\ d\tau \tag{10.25}$$

The basic problem in the averaging method is to determine in what sense the behavior of the autonomous system (10.24) approximates the behavior of the nonautonomous system (10.23). We will address this problem by showing, via a change of variables, that the nonautonomous system (10.23) can be represented as a perturbation of the autonomous system (10.24). Define

$$u(t,x) = \int_0^t h(\tau,x)\ d\tau \tag{10.26}$$

where

$$h(t,x) = f(t,x,0) - f_{\text{av}}(x) \tag{10.27}$$

Since $h(t,x)$ is T-periodic in t and has zero mean, the function $u(t,x)$ is T-periodic in t. Hence, $u(t,x)$ is bounded for all $(t,x) \in [0,\infty) \times D_0$. Moreover, $\partial u/\partial t$ and $\partial u/\partial x$, given by

$$\frac{\partial u}{\partial t} = h(t,x), \qquad \frac{\partial u}{\partial x} = \int_0^t \frac{\partial h}{\partial x}(\tau,x)\ d\tau$$

are T-periodic in t and bounded on $[0,\infty) \times D_0$. Here, we have used the fact that $\partial h/\partial x$ is T-periodic in t and has zero mean. Consider the change of variables

$$x = y + \varepsilon u(t,y) \tag{10.28}$$

Differentiating both sides with respect to t, we obtain

$$\dot{x} = \dot{y} + \varepsilon\frac{\partial u}{\partial t}(t,y) + \varepsilon\frac{\partial u}{\partial y}(t,y)\ \dot{y}$$

Substituting for $\dot{x}$ from (10.23), we find that the new state variable y satisfies the equation

$$\begin{aligned}\left[I + \varepsilon\frac{\partial u}{\partial y}\right]\dot{y} &= \varepsilon f(t, y + \varepsilon u, \varepsilon) - \varepsilon\frac{\partial u}{\partial t} \\ &= \varepsilon f(t, y + \varepsilon u, \varepsilon) - \varepsilon f(t, y, 0) + \varepsilon f_{\text{av}}(y) \\ &\overset{\text{def}}{=} \varepsilon f_{\text{av}}(y) + \varepsilon p(t, y, \varepsilon)\end{aligned}$$

where

$$p(t, y, \varepsilon) = [f(t, y + \varepsilon u, \varepsilon) - f(t, y, \varepsilon)] + [f(t, y, \varepsilon) - f(t, y, 0)]$$

The function $p(t, y, \varepsilon)$ is T-periodic in t and, using mean value theorem, can be expressed as

$$p(t, y, \varepsilon) = F_1(t, y, \varepsilon u, \varepsilon)\varepsilon u + F_2(t, y, \varepsilon)\varepsilon$$

Because $\partial u/\partial y$ is bounded on $[0, \infty) \times D_0$, the matrix $I + \varepsilon\partial u/\partial y$ is nonsingular for sufficiently small ε, and

$$\left[I + \varepsilon\frac{\partial u}{\partial y}\right]^{-1} = I + O(\varepsilon)$$

Therefore, the state equation for y is given by

$$\dot{y} = \varepsilon f_{\text{av}}(y) + \varepsilon^2 q(t, y, \varepsilon) \tag{10.29}$$

where $q(t, y, \varepsilon)$ is T-periodic in t and f_{av}, q, and their first partial derivatives with respect to (y, ε) are continuous and bounded on $[0, \infty) \times D_0$ for sufficiently small ε. This equation is a perturbation of the average system (10.24). By extending the arguments used in the previous three sections, we can determine the basis for approximating the solutions of (10.29) by the solutions of the average system (10.24).

The change of time variable $s = \varepsilon t$ transforms (10.29) into

$$\frac{dy}{ds} = f_{\text{av}}(y) + \varepsilon q(s/\varepsilon, y, \varepsilon) \tag{10.30}$$

where $q(s/\varepsilon, y, \varepsilon)$ is εT-periodic in s and bounded on $[0, \infty) \times D_0$ for sufficiently small ε. By applying Theorems 3.4 and 3.5 on continuity of solutions with respect to initial states and parameters, we see that if the average system

$$\frac{dy}{ds} = f_{\text{av}}(y)$$

has a unique solution $\bar{y}(s)$ defined on $[0, b]$, $\bar{y}(s) \in D$ for all $s \in [0, b]$, and $y(0, \varepsilon) - x_{\text{av}}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the perturbed system (10.30) will have a unique solution defined for all $s \in [0, b]$ and the two solutions will be $O(\varepsilon)$ close. Since $t = s/\varepsilon$ and $x - y = O(\varepsilon)$ by (10.28), the solution of the average system (10.24) provides an $O(\varepsilon)$ approximation for the solution of (10.23) over the time interval $[0, b/\varepsilon]$ in the t time scale.

Suppose the average system (10.24) has an exponentially stable equilibrium point at the origin and D is a domain that contains the origin. Let $V(y)$ be the Lyapunov function provided by (the converse Lyapunov) Theorem 4.17. Then, for any compact subset Ω of the region of attraction of the origin, there is a constant $c > 0$ such that Ω lies in the interior of the compact set $\{V(y) \leq c\}$. Suppose $y_{\rm av}(0) \in \Omega$ and $y(0, \varepsilon) - y_{\rm av}(0) = O(\varepsilon)$. Applying Theorem 9.1 shows that the $O(\varepsilon)$ approximation will be valid for all $s \geq 0$, that is, for all $t \geq 0$.

Finally, Theorem 10.3 shows that (10.30) has a unique, exponentially stable, (εT)-periodic solution $\bar{y}(s/\varepsilon, \varepsilon)$ in an $O(\varepsilon)$ neighborhood of the origin. The periodic solution has period εT in the s time scale, that is, period T in the t time scale. By (10.28), we see that (10.23) has a T-periodic solution

$$\bar{x}(t, \varepsilon) = \bar{y}(t, \varepsilon) + \varepsilon u(t, \bar{y}(t, \varepsilon))$$

Because u is bounded, the periodic solution $\bar{x}(t, \varepsilon)$ lies in an $O(\varepsilon)$ neighborhood of the origin. We summarize these conclusions in the next theorem.

Theorem 10.4 *Let $f(t, x, \varepsilon)$ and its partial derivatives with respect to (x, ε) up to the second order be continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain. Suppose f is T-periodic in t for some $T > 0$ and ε is a positive parameter. Let $x(t, \varepsilon)$ and $x_{\rm av}(\varepsilon t)$ denote the solutions of* (10.23) *and* (10.24), *respectively.*

- *If $x_{\rm av}(\varepsilon t) \in D \ \forall \ t \in [0, b/\varepsilon]$ and $x(0, \varepsilon) - x_{\rm av}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t, \varepsilon)$ is defined and*

$$x(t, \varepsilon) - x_{\rm av}(\varepsilon t) = O(\varepsilon) \quad \text{on } [0, b/\varepsilon]$$

- *If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system* (10.24), *$\Omega \subset D$ is a compact subset of its region of attraction, $x_{\rm av}(0) \in \Omega$, and $x(0, \varepsilon) - x_{\rm av}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t, \varepsilon)$ is defined and*

$$x(t, \varepsilon) - x_{\rm av}(\varepsilon t) = O(\varepsilon) \quad \text{for all } t \in [0, \infty)$$

- *If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system* (10.24), *then there exist positive constants ε^* and k such that, for all $0 < \varepsilon < \varepsilon^*$,* (10.23) *has a unique, exponentially stable, T-periodic solution $\bar{x}(t, \varepsilon)$ with the property $\|\bar{x}(t, \varepsilon)\| \leq k\varepsilon$.* ◇

If $f(t, 0, \varepsilon) = 0$ for all $(t, \varepsilon) \in [0, \infty) \times [0, \varepsilon_0]$, the origin will be an equilibrium point of (10.23). By the uniqueness of the T-periodic solution $\bar{x}(t, \varepsilon)$, it follows that $\bar{x}(t, \varepsilon)$ is the trivial solution $x = 0$. In this case, the theorem ensures that the origin is an exponentially stable equilibrium point of (10.23).

Example 10.8 Consider the linear system

$$\dot{x} = \varepsilon A(t)x$$

where $A(t+T) = A(t)$ and $\varepsilon > 0$. Let

$$\bar{A} = \frac{1}{T}\int_0^T A(\tau)\, d\tau$$

The average system is given by

$$\dot{x} = \varepsilon \bar{A} x$$

It has an equilibrium point at $x = 0$. Suppose the matrix $\bar{A}$ is Hurwitz. Then, it follows from Theorem 10.4 that, for sufficiently small ε, $\dot{x} = \varepsilon A(t)x$ has a unique T-periodic solution in an $O(\varepsilon)$ neighborhood of the origin $x = 0$. However, $x = 0$ is an equilibrium point for the system. Hence, the periodic solution is the trivial solution $x(t) = 0$. Consequently, we conclude that, for sufficiently small ε, $x = 0$ is an exponentially stable equilibrium point for the nonautonomous system $\dot{x} = \varepsilon A(t)x$. △

Example 10.9 Consider the scalar system

$$\dot{x} = \varepsilon(x \sin^2 t - 0.5x^2) = \varepsilon f(t, x)$$

The function $f(t,x)$ is π-periodic in t. The average function $f_{\text{av}}(x)$ is given by

$$f_{\text{av}}(x) = \frac{1}{\pi}\int_0^\pi (x\sin^2 t - 0.5x^2)\, dt = 0.5(x - x^2)$$

The average system

$$\dot{x} = 0.5\varepsilon(x - x^2)$$

has two equilibrium points at $x = 0$ and $x = 1$. The Jacobian df_{av}/dx evaluated at these equilibria is given by

$$\begin{aligned} \left.\frac{df_{\text{av}}}{dx}\right|_{x=0} &= \left.(0.5 - x)\right|_{x=0} &= 0.5 \\ \left.\frac{df_{\text{av}}}{dx}\right|_{x=1} &= \left.(0.5 - x)\right|_{x=1} &= -0.5 \end{aligned}$$

Thus, for sufficiently small ε, the system has an exponentially stable π-periodic solution in an $O(\varepsilon)$ neighborhood of $x = 1$. Moreover, by sketching the function $x - x^2$, it can be seen that the region of attraction of $x = 1$ is $(0, \infty)$. Hence, for initial states in the compact interval $[a, b] \subset (0, \infty)$, solving the average system with the same initial state as the original system yields the approximation

$$x(t, \varepsilon) - x_{\text{av}}(\varepsilon t) = O(\varepsilon), \quad \forall\, t \geq 0$$

Suppose we want to calculate a second-order approximation. We need to use the change of variables (10.28) to represent the problem as a standard perturbation problem and then proceed to approximate the solution, as we have done in Section 10.1. Using (10.26), we find that the function $u(t,x)$ is given by

$$u(t,x) = \int_0^t (x\sin^2\tau - 0.5x^2 - 0.5x + 0.5x^2)\, d\tau = -\tfrac{1}{4}x\sin 2t$$

The change of variables of (10.28) takes the form

$$x = y - \tfrac{1}{4}\varepsilon y \sin 2t = \left(1 - \tfrac{1}{4}\varepsilon\sin 2t\right) y$$

Differentiating both sides with respect to t, we obtain

$$\dot{x} = \left(1 - \tfrac{1}{4}\varepsilon\sin 2t\right)\dot{y} - \tfrac{1}{2}\varepsilon y\cos 2t$$

Hence,

$$\dot{y} = \frac{\varepsilon}{1 - (\varepsilon/4)\sin 2t}\left(x\sin^2 t - \tfrac{1}{2}x^2 + \tfrac{1}{2}y\cos 2t\right)$$

Substituting x in term of y, and expanding the term $1/[1-(\varepsilon/4)\sin 2t]$ in the power series

$$\frac{1}{1-(\varepsilon/4)\sin 2t} = 1 + \tfrac{1}{4}\varepsilon\sin 2t + O(\varepsilon^2)$$

we arrive at the equation

$$\dot{y} = \tfrac{1}{2}\varepsilon(y - y^2) + \tfrac{1}{16}\varepsilon^2(y\sin 4t + 2y^2\sin 2t) + O(\varepsilon^3)$$

where the system appears as a perturbation of the average system. In order to find a second-order approximation, we need to calculate y_0 and y_1 in the finite Taylor series

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 R_y$$

We know that $y_0 = x_{\text{av}}$, the solution of the average system. The equation for y_1 is given by

$$\dot{y}_1 = \varepsilon\left[\left(\tfrac{1}{2} - y_0(t)\right)y_1 + \tfrac{1}{16}y_0(t)\sin 4t + \tfrac{1}{8}y_0^2(t)\sin 2t\right], \quad y_1(0) = 0$$

where we have assumed that the initial state $x(0)$ is independent of ε. Using (10.28), we obtain a second-order approximation of x as

$$x = \left(1 - \tfrac{1}{4}\varepsilon\sin 2t\right)x_{\text{av}}(\varepsilon t) + \varepsilon y_1(t,\varepsilon) + O(\varepsilon^2)$$

Figure 10.5 shows the solution of the exact system, the average system, and the second-order approximation for $x(0) = 0.7$ and $\varepsilon = 0.3$. The figure illustrates clearly how the solution of the average system averages the exact solution. The second-order approximation is almost indistinguishable from the exact solution, but we can see the difference as the solution reaches steady state. △

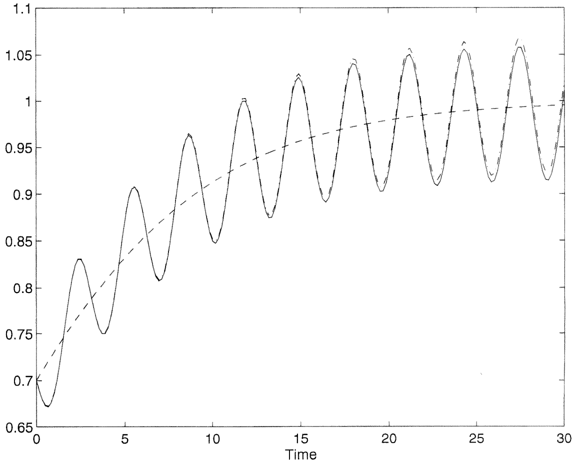

Figure 10.5: The exact (solid), average (dashed), and second-order (dash-dot) solutions of Example 10.9 with $\varepsilon = 0.3$.

Example 10.10 Consider the suspended pendulum of Section 1.2.1 and assume that the suspension point is subjected to vertical vibrations of small amplitude and high frequency. Suppose the motion of the suspension point is described by $a \sin \omega t$, where a is the amplitude and ω is the frequency. Writing Newton's law in the tangential direction (perpendicular to the rod), the equation of motion is[7]

$$m(l\ddot{\theta} - a\omega^2 \sin \omega t \sin \theta) = -mg \sin \theta - k(l\dot{\theta} + a\omega \cos \omega t \sin \theta)$$

Assume that $a/l \ll 1$ and $\omega_0/\omega \ll 1$, where $\omega_0 = \sqrt{g/l}$ is the frequency of free oscillations of the pendulum in the vicinity of the lower equilibrium position $\theta = 0$. Let $\varepsilon = a/l$ and write $\omega_0/\omega = \alpha\varepsilon$, where $\alpha = \omega_0 l/\omega a$. Let $\beta = k/m\omega_0$ and change the time scale from t to $\tau = \omega t$. In the new time scale, the equation of motion is

$$\frac{d^2\theta}{d\tau^2} + \alpha\beta\varepsilon\frac{d\theta}{d\tau} + (\alpha^2\varepsilon^2 - \varepsilon \sin \tau) \sin \theta + \alpha\beta\varepsilon^2 \cos \tau \sin \theta = 0$$

With

$$x_1 = \theta, \quad x_2 = \frac{1}{\varepsilon}\frac{d\theta}{d\tau} + \cos \tau \sin \theta$$

as state variables, the state equation is given by

$$\frac{dx}{d\tau} = \varepsilon f(\tau, x) \tag{10.31}$$

---

[7] To derive this equation, write expressions for the x- and y-coordinates of the bob as $x = l \sin \theta$ and $y = l \cos \theta - a \sin \omega t$. Then, show that the velocity and acceleration of the bob in the tangential direction are $(l\dot{\theta} + a\omega \cos \omega t \sin \theta)$ and $(l\ddot{\theta} - a\omega^2 \sin \omega t \sin \theta)$, respectively. The friction force is assumed to be viscous friction proportional to the velocity of the bob with a friction coefficient k.

where

$$\begin{aligned} f_1(\tau, x) &= x_2 - \sin x_1 \cos \tau \\ f_2(\tau, x) &= -\alpha\beta x_2 - \alpha^2 \sin x_1 + x_2 \cos x_1 \cos \tau - \sin x_1 \cos x_1 \cos^2 \tau \end{aligned}$$

The function $f(\tau, x)$ is 2π-periodic in τ. The average system is given by

$$\frac{dx}{d\tau} = \varepsilon f_{\mathrm{av}}(x) \tag{10.32}$$

where

$$\begin{aligned} f_{\mathrm{av}1}(x) &= \frac{1}{2\pi} \int_0^{2\pi} f_1(\tau, x)\, d\tau &= x_2 \\ f_{\mathrm{av}2}(x) &= \frac{1}{2\pi} \int_0^{2\pi} f_2(\tau, x)\, d\tau &= -\alpha\beta x_2 - \alpha^2 \sin x_1 - \tfrac{1}{4} \sin 2x_1 \end{aligned}$$

In arriving at these expressions, we have used the fact that the average of $\cos \tau$ is zero, while the average of $\cos^2 \tau$ is $1/2$. Both the original system (10.31) and the average system (10.32) have equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$, which correspond to the equilibrium positions $\theta = 0$ and $\theta = \pi$. With a fixed suspension point, the equilibrium position $\theta = 0$ is exponentially stable, while the equilibrium position $\theta = \pi$ is unstable. Let us see what a vibrating suspension point will do to the system. To apply Theorem 10.4, we analyze the stability properties of the equilibrium points of the average system (10.32) via linearization. The Jacobian of $f_{\mathrm{av}}(x)$ is given by

$$\frac{\partial f_{\mathrm{av}}}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 \cos x_1 - 0.5 \cos 2x_1 & -\alpha\beta \end{bmatrix}$$

At the equilibrium point $(x_1 = 0, x_2 = 0)$, the Jacobian

$$\begin{bmatrix} 0 & 1 \\ -\alpha^2 - 0.5 & -\alpha\beta \end{bmatrix}$$

is Hurwitz for all positive values of α and β. Therefore, Theorem 10.4 says that, for sufficiently small ε, the original system (10.31) has a unique exponentially stable 2π-periodic solution in an $O(\varepsilon)$ neighborhood of the origin. Because the origin is an equilibrium point for the original system, the periodic solution is the trivial solution $x = 0$. In this case, Theorem 10.4 confirms that, for sufficiently small ε, the origin is an exponentially stable equilibrium point for the original system (10.31). In other words, exponential stability of the lower equilibrium position of the pendulum is preserved under (small-amplitude, high-frequency) vibration of the suspension point. At the equilibrium point $(x_1 = \pi, x_2 = 0)$, the Jacobian

$$\begin{bmatrix} 0 & 1 \\ \alpha^2 - 0.5 & -\alpha\beta \end{bmatrix}$$

is Hurwitz for $0 < \alpha < 1/\sqrt{2}$ and $\beta > 0$. Noting, again, that $(x_1 = \pi,\ x_2 = 0)$ is an equilibrium point for the original system, and applying Theorem 10.4, we are led to the conclusion that if $\alpha < 1/\sqrt{2}$, then the upper equilibrium position $\theta = \pi$ is an exponentially stable equilibrium point for the original system (10.31) for sufficiently small ε. This is an intriguing finding, because it shows that the unstable upper equilibrium position of the pendulum can be stabilized by vibrating the suspension point vertically with small amplitude and high frequency.[8] △

10.5 Weakly Nonlinear Second-Order Oscillators

Consider the second-order system

$$\ddot{y} + \omega^2 y = \varepsilon g(y, \dot{y}) \tag{10.33}$$

where $g(\cdot,\cdot)$ is sufficiently smooth and $|g|$ is bounded by $k|y|$ or $k|\dot{y}|$ on compact sets of $(y, \dot{y})$; k is a positive constant. Choosing $x_1 = y$ and $x_2 = \dot{y}/\omega$ as state variables, we obtain the state equation

$$\begin{aligned} \dot{x}_1 &= \omega x_2 \\ \dot{x}_2 &= -\omega x_1 + \frac{\varepsilon}{\omega} g(x_1, \omega x_2) \end{aligned}$$

Representing the system in the polar coordinates

$$x_1 = r \sin\phi, \qquad x_2 = r\cos\phi$$

we have

$$\dot{r} = \frac{1}{r}(x_1\dot{x}_1 + x_2\dot{x}_2) = \frac{\varepsilon}{\omega} g(r\sin\phi, \omega r\cos\phi)\cos\phi \tag{10.34}$$

$$\dot{\phi} = \frac{1}{r^2}(x_2\dot{x}_1 - x_1\dot{x}_2) = \omega - \frac{\varepsilon}{\omega r} g(r\sin\phi, \omega r\cos\phi)\sin\phi \tag{10.35}$$

The second term on the right-hand side of (10.35) is $O(\varepsilon)$ on bounded sets of r, as a consequence of the assumption that $|g|$ is bounded by $k|y|$ or $k|\dot{y}|$. Hence, the right-hand side of (10.35) is positive for sufficiently small ε. Divide (10.34) by (10.35) to obtain

---

[8] The idea of introducing high-frequency, zero-mean vibrations in the parameters of a dynamic system in order to modify the properties of the system in a desired manner has been generalized into a *principle of vibrational control.* (See [22] and [127].)

$$\frac{dr}{d\phi} = \frac{\varepsilon g(r\sin\phi, \omega r\cos\phi)\cos\phi}{\omega^2 - (\varepsilon/r)g(r\sin\phi, \omega r\cos\phi)\sin\phi}$$

We rewrite this equation as

$$\frac{dr}{d\phi} = \varepsilon f(\phi, r, \varepsilon) \tag{10.36}$$

where

$$f(\phi, r, \varepsilon) = \frac{g(r\sin\phi, \omega r\cos\phi)\cos\phi}{\omega^2 - (\varepsilon/r)g(r\sin\phi, \omega r\cos\phi)\sin\phi}$$

If we view ϕ as the independent variable, then (10.36) takes the form (10.23), where $f(\phi, r, \varepsilon)$ is 2π-periodic in ϕ. The function $f_{\text{av}}(r)$ is given by

$$f_{\text{av}}(r) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi, r, 0)\ d\phi = \frac{1}{2\pi\omega^2}\int_0^{2\pi} g(r\sin\phi, \omega r\cos\phi)\cos\phi\ d\phi$$

Suppose the average system

$$\frac{dr}{d\phi} = \varepsilon f_{\text{av}}(r) \tag{10.37}$$

has an equilibrium point r^*, where $[\partial f_{\text{av}}/\partial r](r^*) < 0$; then, there is $\varepsilon^* > 0$ such that $\forall\ 0 < \varepsilon < \varepsilon^*$, (10.36) has a unique exponentially stable 2π-periodic solution $r = R(\phi, \varepsilon)$ in an $O(\varepsilon)$ neighborhood of r^*. This, by itself, does not say that (10.33) has a periodic solution with respect to t. More work is needed to reach that conclusion. Substituting $r = R(\phi, \varepsilon)$ into (10.35) yields

$$\dot{\phi} = \omega - \frac{\varepsilon}{\omega R(\phi, \varepsilon)} g(R(\phi, \varepsilon)\sin\phi, \omega R(\phi, \varepsilon)\cos\phi)\sin\phi$$

Let $\phi^*(t, \varepsilon)$ be the solution of this equation starting at $\phi^*(0, \varepsilon) = 0$. To show that (10.33) has a periodic solution, we need to show that there exists $T = T(\varepsilon) > 0$, generally dependent on ε, such that

$$\phi^*(t + T, \varepsilon) = 2\pi + \phi^*(t, \varepsilon),\ \forall\ t \geq 0 \tag{10.38}$$

For then,

$$R(\phi^*(t + T, \varepsilon), \varepsilon) = R(2\pi + \phi^*(t, \varepsilon), \varepsilon) = R(\phi^*(t, \varepsilon), \varepsilon)$$

which implies that $R(\phi^*(t, \varepsilon), \varepsilon)$ is T-periodic in t. Because

$$\phi^*(t + \tau, \varepsilon) = \phi^*(t, \varepsilon) + \omega\tau + O(\varepsilon)$$

for bounded $\tau \geq 0$, it can be easily seen that, for sufficiently small ε, (10.38) has a unique solution $T(\varepsilon) = 2\pi/\omega + O(\varepsilon)$.

The image of the solution $r = R(\phi^*(t, \varepsilon), \varepsilon)$ in the state plane x_1–x_2 is a closed orbit in the neighborhood of the circle $r = r^*$. Since the periodic solution $r = R(\phi, \varepsilon)$ is exponentially stable, the closed orbit will attract all solutions in its neighborhood; that is, the closed orbit is a stable limit cycle.

Example 10.11 The Van der Pol equation

$$\ddot{y} + y = \varepsilon \dot{y}(1 - y^2)$$

is a special case of (10.33) with $\omega = 1$ and $g(y, \dot{y}) = \dot{y}(1 - y^2)$. The function $f_{\rm av}(r)$ is given by

$$\begin{aligned} f_{\rm av}(r) &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 - r^2 \sin^2 \phi\right) r \cos^2 \phi \, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} r \cos^2 \phi \, d\phi - \frac{1}{2\pi} \int_0^{2\pi} r^3 \sin^2 \phi \cos^2 \phi \, d\phi \\ &= \frac{1}{2} r - \frac{1}{8} r^3 \end{aligned}$$

The average system

$$\frac{dr}{d\phi} = \varepsilon \left(\tfrac{1}{2} r - \tfrac{1}{8} r^3\right)$$

has three equilibrium points at $r = 0$, $r = 2$, and $r = -2$. Since by definition $r \geq 0$, the negative root is rejected. We check stability of the equilibria via linearization. The Jacobian matrix is given by

$$\frac{df_{\rm av}}{dr} = \tfrac{1}{2} - \tfrac{3}{8} r^2$$

and

$$\left.\frac{df_{\rm av}}{dr}\right|_{r=0} = \tfrac{1}{2} > 0; \qquad \left.\frac{df_{\rm av}}{dr}\right|_{r=2} = -1 < 0$$

Thus, the equilibrium point $r = 2$ is exponentially stable. Therefore, for sufficiently small ε, the Van der Pol equation has a stable limit cycle in an $O(\varepsilon)$ neighborhood of $r = 2$. The period of oscillation is $O(\varepsilon)$ close to 2π. This stable limit cycle was observed in Example 2.6 via simulation. △

Let us conclude by noting that the foregoing procedure may be used to show the existence of an unstable limit cycle. This can be done by reversing time in (10.33), that is, replacing t by $\tau = -t$. If the system has a stable limit cycle in reverse time, it will have an unstable limit cycle in forward time.

10.6 General Averaging

Consider the system

$$\dot{x} = \varepsilon f(t, x, \varepsilon) \tag{10.39}$$

where f and its partial derivatives with respect to (x, ε) up to the second order are continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [0, \varepsilon_0]$, for every compact set

$D_0 \subset D$. The parameter ε is positive, and $D \subset R^n$ is a domain. The averaging method applies to the system (10.39) in cases more general than the case when $f(t, x, \varepsilon)$ is periodic in t. In particular, it applies when the function $f(t, x, 0)$ has a well-defined average $f_{\rm av}(x)$ according to the next definition.

Definition 10.2 *A continuous, bounded function* $g : [0, \infty) \times D \to R^n$ *is said to have an average* $g_{\rm av}(x)$ *if the limit*

$$g_{\rm av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} g(\tau, x)\ d\tau$$

exists and

$$\left\| \frac{1}{T} \int_t^{t+T} g(\tau, x)\ d\tau - g_{\rm av}(x) \right\| \le k\sigma(T), \quad \forall\ (t, x) \in [0, \infty) \times D_0$$

for every compact set $D_0 \subset D$*, where* k *is a positive constant (possibly dependent on* D_0*) and* $\sigma : [0, \infty) \to [0, \infty)$ *is a strictly decreasing, continuous, bounded function such that* $\sigma(T) \to 0$ *as* $T \to \infty$*. The function* σ *is called the convergence function.*

Example 10.12

- Let $g(t, x) = \sum_{k=1}^{N} g_k(t, x)$, where $g_k(t, x)$ is periodic in t of period T_k, with $T_i \ne T_j$ when $i \ne j$. The function g is not periodic[9] in t, but it has the average

  $$g_{\rm av}(x) = \sum_{k=1}^{N} g_{k_{\rm av}}(x)$$

  where $g_{k_{\rm av}}$ is the average of the periodic function $g_k(t, x)$, as defined in Section 10.4. The convergence function σ is of order $O(1/T)$ as $T \to \infty$. We can take it as $\sigma(T) = 1/(T + 1)$.

- The average of

  $$g(t, x) = \frac{1}{1 + t} h(x)$$

  is zero, and the convergence function σ can be taken as $\sigma(T) = (1/T) \ln(1+T)$.

△

Suppose now that $f(t, x, 0)$ has the average function $f_{\rm av}(x)$ with convergence function σ. Let

$$h(t, x) = f(t, x, 0) - f_{\rm av}(x) \tag{10.40}$$

---

[9] This function is called almost periodic. An introduction to the theory of almost periodic functions can be found in [59] or [75].

The function $h(t,x)$ has zero average with σ as its convergence function. Suppose the Jacobian matrix $\partial h/\partial x$ has zero average with the same convergence function σ. Define

$$w(t,x,\eta) = \int_0^t h(\tau,x)\exp[-\eta(t-\tau)]\, d\tau \tag{10.41}$$

for some positive constant η. At $\eta = 0$, the function $w(t,x,0)$ satisfies

$$\begin{aligned} \|w(t+\delta,x,0) - w(t,x,0)\| &= \left\| \int_0^{t+\delta} h(\tau,x)\, d\tau - \int_0^t h(\tau,x)\, d\tau \right\| \\ &= \left\| \int_t^{t+\delta} h(\tau,x)\, d\tau \right\| \le k\delta\sigma(\delta) \end{aligned} \tag{10.42}$$

This implies, in particular, that

$$\|w(t,x,0)\| \le kt\sigma(t), \quad \forall\ (t,x) \in [0,\infty) \times D_0$$

since $w(0,x,0) = 0$. Integrating the right-hand side of (10.41) by parts, we obtain

$$\begin{aligned} w(t,x,\eta) &= w(t,x,0) - \eta \int_0^t \exp[-\eta(t-\tau)] w(\tau,x,0)\, d\tau \\ &= \exp(-\eta t) w(t,x,0) - \eta \int_0^t \exp[-\eta(t-\tau)]\ [w(\tau,x,0) - w(t,x,0)]\, d\tau \end{aligned}$$

where the second equality is obtained by adding and subtracting

$$\eta \int_0^t \exp[-\eta(t-\tau)]\, d\tau\ w(t,x,0)$$

to the right-hand side. Using (10.42), we obtain

$$\|w(t,x,\eta)\| \le kt\exp(-\eta t)\sigma(t) + k\eta \int_0^t \exp[-\eta(t-\tau)](t-\tau)\sigma(t-\tau)\, d\tau \tag{10.43}$$

This inequality can be used to show that $\eta\|w(t,x,\eta)\|$ is uniformly bounded by $k\alpha(\eta)$ for some class $\mathcal{K}$ function α. For example, if $\sigma(t) = 1/(t+1)$, then

$$\eta\|w(t,x,\eta)\| \le k\eta\exp(-\eta t) + k\eta^2 \int_0^t \exp[-\eta(t-\tau)]\, d\tau = k\eta$$

Defining $\alpha(\eta) = \eta$, we have $\eta\|w(t,x,\eta)\| \le k\alpha(\eta)$. If $\sigma(t) = 1/(t^r + 1)$ with $0 < r < 1$, then

$$\begin{aligned} \eta\|w(t,x,\eta)\| &\le k\eta t^{(1-r)} e^{-\eta t} + k\eta^2 \int_0^t e^{-\eta(t-\tau)} (t-\tau)^{(1-r)}\, d\tau \\ &\le k\eta \left(\frac{1-r}{\eta}\right)^{1-r} e^{-(1-r)} + k\eta^2 \int_0^\infty e^{-\eta s} s^{(1-r)}\, ds \\ &\le k\eta \left(\frac{1-r}{\eta}\right)^{1-r} e^{-(1-r)} + k\eta^2 \frac{\Gamma(2-r)}{\eta^{(2-r)}} \le kk_1\eta^r \end{aligned}$$

where $\Gamma(\cdot)$ denotes the standard gamma function. Defining $\alpha(\eta) = k_1\eta^r$, we have $\eta\|w(t,x,\eta)\| \leq k\alpha(\eta)$. In general, it can be shown (Exercise 10.19) that there is a class $\mathcal{K}$ function α such that

$$\eta\|w(t,x,\eta)\| \leq k\alpha(\eta), \quad \forall\ (t,x) \in [0,\infty) \times D_0 \tag{10.44}$$

Without loss of generality, we can choose $\alpha(\eta)$ such that $\alpha(\eta) \geq c\eta$ for $\eta \in [0,1]$, where c is a positive constant. The partial derivatives $[\partial w/\partial t]$ and $[\partial w/\partial x]$ are given by

$$\begin{aligned}\frac{\partial w}{\partial t} &= h(t,x) - \eta w(t,x,\eta)\\ \frac{\partial w}{\partial x} &= \int_0^t \frac{\partial h}{\partial x}(\tau,x)\exp[-\eta(t-\tau)]\ d\tau\end{aligned}$$

Because $[\partial h/\partial x]$ possesses the same properties of h that have been used to arrive at (10.44), it is clear that we can repeat the previous derivations to show that

$$\eta\left\|\frac{\partial w}{\partial x}\right\| \leq k\alpha(\eta), \quad \forall\ (t,x) \in [0,\infty) \times D_0 \tag{10.45}$$

There is no loss of generality in using the same class $\mathcal{K}$ function in both (10.44) and (10.45), since the calculated estimates will differ only in the positive constant that multiplies the η dependent term, so we can define α by using the larger of the two constants.

The function $w(t,x,\eta)$ that we have just defined possesses all the key properties of the function $u(t,x)$ of Section 10.4. The only difference is that the function w is parameterized in a parameter η in such a way that the bounds on w and $[\partial w/\partial x]$ are of the form $k\alpha(\eta)/\eta$ for some class $\mathcal{K}$ function α. We did not need to parameterize u in terms of any parameter. In fact, $u(t,x)$ is nothing more than the function $w(t,x,\eta)$ evaluated at $\eta = 0$. This should come as no surprise because in the periodic case, the convergence function $\sigma(t) = 1/(t+1)$; hence, $\alpha(\eta)/\eta = 1$.

From this point on, the analysis will be very similar to that of Section 10.4. We define the change of variables

$$x = y + \varepsilon w(t,y,\varepsilon) \tag{10.46}$$

The term $\varepsilon w(t,y,\varepsilon)$ is of order $O(\alpha(\varepsilon))$; thus, for sufficiently small ε, the change of variables of (10.46) is well defined, since the matrix $[I + \varepsilon\partial w/\partial y]$ is nonsingular. In particular,

$$\left[I + \varepsilon\frac{\partial w}{\partial y}\right]^{-1} = I + O(\alpha(\varepsilon))$$

Proceeding as in Section 10.4, we can show that the state equation for y is given by

$$\dot{y} = \varepsilon f_{\text{av}}(y) + \varepsilon\alpha(\varepsilon)q(t,y,\varepsilon) \tag{10.47}$$

where $q(t, y, \varepsilon)$ is bounded on $[0, \infty) \times D_0$ for sufficiently small ε. In arriving at (10.47), we have used the fact that $\alpha(\varepsilon) \geq c\varepsilon$. Equation (10.47) is a perturbation of the average system

$$\dot{x} = \varepsilon f_{\text{av}}(x) \tag{10.48}$$

It is similar to (10.29), except that the coefficient of the q term is $\varepsilon\alpha(\varepsilon)$, instead of ε^2. This observation leads to the following theorem, which is similar to Theorem 10.4, except that the estimates $O(\varepsilon)$ are replaced by the estimates $O(\alpha(\varepsilon))$.

Theorem 10.5 *Let $f(t, x, \varepsilon)$ and its partial derivatives with respect to (x, ε) up to the second order be continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $\varepsilon > 0$ and $D \subset R^n$ is a domain. Suppose $f(t, x, 0)$ has the average function $f_{\text{av}}(x)$ on $[0, \infty) \times D$ and the Jacobian of $h(t, x) = f(t, x, 0) - f_{\text{av}}(x)$ has zero average with the same convergence function as f. Let $x(t, \varepsilon)$ and $x_{\text{av}}(\varepsilon t)$ denote the solutions of* (10.39) *and* (10.48), *respectively, and α be the class $\mathcal{K}$ function appearing in the estimates of* (10.44) *and* (10.45).

- *If $x_{\text{av}}(\varepsilon t) \in D \ \forall \ t \in [0, b/\varepsilon]$ and $x(0, \varepsilon) - x_{\text{av}}(0) = O(\alpha(\varepsilon))$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t, \varepsilon)$ is defined and*

$$x(t, \varepsilon) - x_{\text{av}}(\varepsilon t) = O(\alpha(\varepsilon)) \ \text{ on } [0, b/\varepsilon]$$

- *If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system* (10.48), *$\Omega \subset D$ is a compact subset of its region of attraction, $x_{\text{av}}(0) \in \Omega$, and $x(0, \varepsilon) - x_{\text{av}}(0, \varepsilon) = O(\alpha(\varepsilon))$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t, \varepsilon)$ is defined and*

$$x(t, \varepsilon) - x_{\text{av}}(\varepsilon t) = O(\alpha(\varepsilon)) \ \text{ for all } t \in [0, \infty)$$

- *If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system* (10.48) *and $f(t, 0, \varepsilon) = 0$ for all $(t, \varepsilon) \in [0, \infty) \times [0, \varepsilon_0]$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin is an exponentially stable equilibrium point of the original system* (10.39). ◇

Proof: By expressing (10.47) in the $s = \varepsilon t$ time scale, applying Theorems 3.4 and 3.5, and using the change of variables of (10.46), we can conclude the first part of the theorem. For the second part, we apply Theorem 9.1 on continuity of solutions on the infinite interval. Finally, by using $h(t, 0) = 0$, $w(t, 0, \eta) = 0$, and the bound $\|\partial w/\partial x\| \leq k\alpha(\eta)/\eta$, we see that the estimate on w can be revised to

$$\eta\|w(t, x, \eta)\| \leq k\alpha(\eta)\|x\|$$

The assumption $f(t, 0, \varepsilon) = 0$ and differentiability of f with respect to ε imply that $f(t, x, \varepsilon)$ is Lipschitz in ε, linearly in x; that is,

$$\|f(t, x, \varepsilon) - f(t, x, 0)\| \leq L_1\varepsilon\|x\|$$

Using these estimates, it can be verified that the function $q(t, y, \varepsilon)$ in (10.47) satisfies the inequality $\|q(t, y, \varepsilon)\| \leq L\|y\|$ with some positive constant L for $(t, y, \varepsilon) \in [0, \infty) \times D_1 \times [0, \varepsilon_1]$, where $D_1 = \{\|y\| < r_1\}$ and r_1 and ε_1 are chosen small enough. By (the converse Lyapunov) Theorem 4.14 and Lemma 9.1, we conclude that, for sufficiently small ε, the origin is an exponentially stable equilibrium point for the original system (10.39). □

Example 10.13 Consider the linear system

$$\dot{x} = \varepsilon A(t)x$$

where $\varepsilon > 0$. Suppose $A(t)$ and its derivatives up to the second order are continuous and bounded. Moreover, suppose $A(t)$ has an average

$$A_{\text{av}} = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} A(\tau)\, d\tau$$

in the sense of Definition 10.2. The average system is given by

$$\dot{x} = \varepsilon A_{\text{av}} x$$

Suppose A_{av} is Hurwitz. By Theorem 10.5, we conclude that the origin of the original time-varying system is exponentially stable for sufficiently small ε. Suppose further that the matrix $A(t) = A_{\text{tr}}(t) + A_{\text{ss}}(t)$ is the sum of a transient component $A_{\text{tr}}(t)$ and a steady-state component $A_{\text{ss}}(t)$. The transient component decays to zero exponentially fast; that is,

$$\|A_{\text{tr}}(t)\| \leq k_1 \exp(-\gamma t), \quad k_1 > 0, \ \gamma > 0$$

while the elements of the steady-state component are formed of a finite sum of sinusoids with distinct frequencies. The average of the transient component is zero, since

$$\frac{1}{T} \int_t^{t+T} \|A_{\text{tr}}(\tau)\|\, d\tau \leq \frac{1}{T} \int_t^{t+T} k_1 e^{-\gamma\tau}\, d\tau = \frac{k_1 e^{-\gamma t}}{\gamma T} \left[1 - e^{-\gamma T}\right] \leq \frac{k_2}{T+1}$$

Recalling the first case of Example 10.12, we see that $A(t)$ has an average with convergence function $\sigma(T) = 1/(T+1)$. Hence, the class $\mathcal{K}$ function of Theorem 10.5 is $\alpha(\eta) = \eta$. Let $x(t, \varepsilon)$ and $x_{\text{av}}(\varepsilon t)$ denote solutions of the original and average systems, which start from the same initial state. By Theorem 10.5,

$$x(t, \varepsilon) - x_{\text{av}}(\varepsilon t) = O(\varepsilon), \quad \forall\, t \geq 0$$

△

10.7 Exercises

10.1 If $\delta(\varepsilon) = O(\varepsilon)$, is it $O(\varepsilon^{1/2})$? Is it $O(\varepsilon^{3/2})$?

10.2 If $\delta(\varepsilon) = \varepsilon^{1/n}$, where $n > 1$ is a positive integer, is there a positive integer N such that $\delta(\varepsilon) = O(\varepsilon^N)$?

10.3 Consider the initial value problem

$$\begin{aligned} \dot{x}_1 &= -(0.2+\varepsilon)x_1 + \frac{\pi}{4} - \tan^{-1} x_1 + \varepsilon \tan^{-1} x_2, \quad x_1(0) = \eta_1 \\ \dot{x}_2 &= -(0.2+\varepsilon)x_2 + \frac{\pi}{4} - \tan^{-1} x_2 + \varepsilon \tan^{-1} x_1, \quad x_2(0) = \eta_2 \end{aligned}$$

(a) Find an $O(\varepsilon)$ approximation.

(b) Find an $O(\varepsilon^2)$ approximation.

(c) Investigate the validity of the approximation on the infinite interval.

(d) Calculate, using a computer program, the exact solution, the $O(\varepsilon)$ approximation, and the $O(\varepsilon^2)$ approximation for $\varepsilon = 0.1$, $\eta_1 = 0.5$, and $\eta_2 = 1.5$ on the time interval $[0, 3]$. Comment on the accuracy of the approximation.

Hint: In parts (a) and (b), it is sufficient to give the equations defining the approximation. You are not required to find an analytic closed-form expression for the approximation.

10.4 Repeat Exercise 10.3 for the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + \varepsilon x_1^3$$

In part (d), let $\varepsilon = 0.1$, $\eta_1 = 1.0$, $\eta_2 = 0.0$, and the time interval be $[0, 5]$.

10.5 Repeat Exercise 10.3 for the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = \varepsilon x_1 - x_2 - \tfrac{1}{3}x_2^3$$

In part (d), let $\varepsilon = 0.2$, $\eta_1 = 1.0$, $\eta_2 = 0.0$, and the time interval be $[0, 4]$.

10.6 ([166]) Repeat Exercise 10.3 for the system

$$\dot{x}_1 = x_1 - x_1^2 + \varepsilon x_1 x_2, \qquad \dot{x}_2 = 2x_2 - x_2^2 - \varepsilon x_1 x_2$$

In part (d), let $\varepsilon = 0.2$, $\eta_1 = 0.5$, $\eta_2 = 1.0$, and the time interval be $[0, 4]$.

10.7 Repeat Exercise 10.3 for the system

$$\dot{x}_1 = -x_1 + x_2(1 + x_1) + \varepsilon(1 + x_1)^2, \qquad \dot{x}_2 = -x_1(x_1 + 1)$$

In part (d), let $\varepsilon = -0.1$, $\eta_1 = -1$, and $\eta_2 = 2$. Repeat the calculation for $\varepsilon = -0.05$ and $\varepsilon = -0.2$ and comment on the accuracy of the approximation.

10.8 Consider the initial value problem

$$\begin{aligned}\dot{x}_1 &= -x_1 + \varepsilon x_2, \quad x_1(0) = \eta \\ \dot{x}_2 &= -x_2 - \varepsilon x_1, \quad x_2(0) = \eta\end{aligned}$$

Find an $O(\varepsilon)$ approximation. Calculate the exact and approximate solutions at $\varepsilon = 0.1$ for two different sets of initial conditions: (1) $\eta = 1$, (2) $\eta = 10$. Comment on the approximation accuracy. Explain any discrepancy with Theorem 10.1.

10.9 ([70]) Study, using the averaging method, each of the following scalar systems.

(1) $\dot{x} = \varepsilon(x - x^2)\sin^2 t$ **(2)** $\dot{x} = \varepsilon(x\cos^2 t - \frac{1}{2}x^2)$

(3) $\dot{x} = \varepsilon(-x + \cos^2 t)$ **(4)** $\dot{x} = -\varepsilon x \cos t$

10.10 For each of the following systems, show that, for sufficiently small $\varepsilon > 0$, the origin is exponentially stable:

(1)

$$\begin{aligned}\dot{x}_1 &= \varepsilon x_2 \\ \dot{x}_2 &= -\varepsilon(1 + 2\sin t)x_2 - \varepsilon(1 + \cos t)\sin x_1\end{aligned}$$

(2)

$$\begin{aligned}\dot{x}_1 &= \varepsilon[(-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2] \\ \dot{x}_2 &= \varepsilon[(-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2]\end{aligned}$$

(3)

$$\dot{x} = \varepsilon\;(-x\sin^2 t + x^2 \sin t + xe^{-t}), \quad \varepsilon > 0$$

10.11 Consider the system

$$\begin{aligned}\dot{x}_1 &= \varepsilon[(-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2] \\ \dot{x}_2 &= \varepsilon[(-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2] + e^{-t}\end{aligned}$$

Show that there is $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$ and all $x(0) \in R^2$, $x(t) \to 0$ as $t \to \infty$.

10.12 Consider the system $\dot{y} = Ay + \varepsilon g(t, y, \varepsilon)$, $\varepsilon > 0$, where the $n \times n$ matrix A has only simple eigenvalues on the imaginary axis.

(a) Show that $\exp(At)$ and $\exp(-At)$ are bounded for all $t \geq 0$.

(b) Show that the change of variables $y = \exp(At)x$ transforms the system into the form $\dot{x} = \varepsilon f(t, x, \varepsilon)$, where $f = \exp(-At)g(t, \exp(At)x, \varepsilon)$.

10.13 ([166]) Study Mathieu's equation $\ddot{y} + (1 + 2\varepsilon\cos 2t)y = 0$, $\varepsilon > 0$, using the averaging method.
Hint: Use Exercise 10.12.

10.14 (**[166]**) Study the equation $\ddot{y}+y = 8\varepsilon(\dot{y})^2 \cos t$ using the averaging method. Hint: Use Exercise 10.12.

10.15 Apply the averaging method to study the existence of limit cycles for each of the second-order systems that follow. If there is a limit cycle, estimate its location in the state plane and the period of oscillation, and determine whether it is stable or unstable.

(1) $\ddot{y} + y = -\varepsilon\dot{y}(1 - y^2)$ **(2)** $\ddot{y} + y = \varepsilon\dot{y}(1 - y^2) - \varepsilon y^3$

(3) $\ddot{y} + y = -\varepsilon\left(1 - \frac{3\pi}{4}|y|\right)\dot{y}$ **(4)** $\ddot{y} + y = -\varepsilon\left(1 - \frac{3\pi}{4}|\dot{y}|\right)\dot{y}$

(5) $\ddot{y} + y = -\varepsilon(\dot{y} - y^3)$ **(6)** $\ddot{y} + y = \varepsilon\dot{y}(1 - y^2 - \dot{y}^2)$

10.16 Consider the second-order system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon[x_1 + x_2(1 - x_1^2 - x_2^2)], \quad \varepsilon > 0$$

(a) Show that, for sufficiently small ε, the system has a stable limit cycle.

(b) Show that the system has no periodic orbits when $\varepsilon > 1$.

10.17 Consider Rayleigh's equation

$$m\frac{d^2u}{dt^2} + ku = \lambda\left[1 - \alpha\left(\frac{du}{dt}\right)^2\right]\frac{du}{dt}$$

where m, k, λ, and α are positive constants.

(a) Using the dimensionless variables $y = u/u^*$, $\tau = t/t^*$, and $\varepsilon = \lambda/\lambda^*$, where $(u^*)^2\alpha k = m/3$, $t^* = \sqrt{m/k}$, and $\lambda^* = \sqrt{km}$, show that the equation can be normalized to

$$\ddot{y} + y = \varepsilon\left(\dot{y} - \tfrac{1}{3}\dot{y}^3\right)$$

in which $\dot{y}$ denotes the derivative of y with respect to τ.

(b) Apply the averaging method to show that the normalized Rayleigh equation has a stable limit cycle. Estimate the location of the limit cycle in the plane $(y, \dot{y})$.

(c) Using a numerical algorithm, obtain the phase portrait of the normalized Rayleigh equation in the plane $(y, \dot{y})$ for

(i) $\varepsilon = 1$, (ii) $\varepsilon = 0.1$, and (iii) $\varepsilon = 0.01$,

Compare with the results of part (b).

10.18 Consider Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

where A, a, c, k, m, and ω are positive constants.

(a) Taking $x_1 = y$, $x_2 = \dot{y}$, $\tau = \omega t$, and $\varepsilon = 1/\omega$, show that the equation can be represented as $dx/d\tau = \varepsilon f(\tau, x, \varepsilon)$.

(b) Show that the system has an exponentially stable periodic solution for sufficiently large ω. Estimate the frequency of oscillation and the location of the periodic orbit in the phase plane.

10.19 Verify (10.44).
Hint: Start from (10.43) and use the fact that $\sigma(t)$ is bounded for $t \leq 1/\sqrt{\eta}$, while for $t \geq 1/\sqrt{\eta}$, $\sigma(t) \leq \sigma(1/\sqrt{\eta})$.

10.20 Study, using general averaging, the scalar system

$$\dot{x} = \varepsilon\left(\sin^2 t + \sin 1.5t + e^{-t}\right)x$$

10.21 ([**168**]) The output of an nth-order linear time-invariant single-input–single-output system can be represented by $y(t) = \theta^T w(t)$, where θ is a $(2n+1)$-dimensional vector of constant parameters and $w(t)$ is an auxiliary signal that can be synthesized from the system's input and output without knowing θ. Suppose that the vector θ is unknown and denote its value by θ^*. In identification experiments, the parameter $\theta(t)$ is updated by using an adaptation law of the form $\dot{\theta} = -\varepsilon e(t)w(t)$, where $e(t) = [\theta(t) - \theta^*]^T w(t)$ is the error between the actual system's output and the estimated output obtained by using $\theta(t)$. Let $\phi(t) = \theta(t) - \theta^*$ denote the parameter error.

(a) Show that $\dot{\phi} = \varepsilon A(t)\phi$, where $A(t) = -w(t)w^T(t)$.

(b) Using (general) averaging, derive a condition on $w(t)$, which ensures that, for sufficiently small ε, $\theta(t) \rightarrow \theta^*$ as $t \rightarrow \infty$.

Chapter 11

Singular Perturbations

While the perturbation method of Section 10.1 applies to state equations that depend smoothly on a small parameter ε, in this chapter we face a more difficult perturbation problem characterized by discontinuous dependence of system properties on the perturbation parameter ε. We will study the so-called *standard singular perturbation model*

$$\begin{aligned} \dot{x} &= f(t, x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(t, x, z, \varepsilon) \end{aligned}$$

where setting $\varepsilon = 0$ causes a fundamental and abrupt change in the dynamic properties of the system, as the differential equation $\varepsilon \dot{z} = g$ degenerates into the algebraic or transcendental equation

$$0 = g(t, x, z, 0)$$

The essence of the theory developed in this chapter is that the discontinuity of solutions caused by singular perturbations can be avoided if analyzed in separate time scales. This multitime-scale approach is a fundamental characteristic of the singular perturbation method.

In Section 11.1, we define the standard singular perturbation model and illustrate, via examples, some of its physical sources. In Section 11.2, we study the two-time-scale properties of the standard model and give a trajectory approximation result, based on the decomposition of the model into reduced (slow) and boundary-layer (fast) models. The approximation result is extended in Section 11.3 to the infinite-time interval. The intuition behind the time-scale decomposition becomes more transparent with a geometric viewpoint, which we present in Section 11.4. The time-scale decomposition of Section 11.2 is used in Section 11.5 to analyze the stability of equilibrium points via Lyapunov's method.

11.1 The Standard Singular Perturbation Model

The singular perturbation model of a dynamical system is a state model where the derivatives of some of the states are multiplied by a small positive parameter ε; that is,

$$\begin{aligned} \dot{x} &= f(t, x, z, \varepsilon) \qquad (11.1) \\ \varepsilon \dot{z} &= g(t, x, z, \varepsilon) \qquad (11.2) \end{aligned}$$

We assume that the functions f and g are continuously differentiable in their arguments for $(t, x, z, \varepsilon) \in [0, t_1] \times D_x \times D_z \times [0, \varepsilon_0]$, where $D_x \subset R^n$ and $D_z \subset R^m$ are open connected sets. When we set $\varepsilon = 0$ in (11.1) and (11.2), the dimension of the state equation reduces from $n + m$ to n because the differential equation (11.2) degenerates into the equation

$$0 = g(t, x, z, 0) \qquad (11.3)$$

We say that the model (11.1)–(11.2) is in *standard form* if (11.3) has $k \geq 1$ isolated real roots

$$z = h_i(t, x), \quad i = 1, 2, \ldots, k \qquad (11.4)$$

for each $(t, x) \in [0, t_1] \times D_x$. This assumption ensures that a well-defined n-dimensional reduced model will correspond to each root of (11.3). To obtain the ith reduced model, we substitute (11.4) into (11.1), at $\varepsilon = 0$, to obtain

$$\dot{x} = f(t, x, h(t, x), 0) \qquad (11.5)$$

where we have dropped the subscript i from h. It will be clear from the context which root of (11.3) we are using. This model is sometimes called a *quasi-steady-state model*, because z, whose velocity $\dot{z} = g/\varepsilon$ can be large when ε is small and $g \neq 0$, may rapidly converge to a root of (11.3), which is the equilibrium of (11.2). We will discuss this two-time-scale property of (11.1) and (11.2) in the next section. The model (11.5) is also known as the *slow model*.

Modeling a physical system in the singularly perturbed form may not be easy. It is not always clear how to pick the parameters to be considered as small. Fortunately, in many applications, our knowledge of physical processes and components of the system sets us on the right track.[1] The following four examples illustrate four different "typical" ways of choosing the parameter ε. In the first example, ε is chosen as a small time constant. This is the most popular source of singularly perturbed models and, historically, the case that motivated interest in singular perturbations. Small time constants, masses, capacitances, and similar "parasitic" parameters that increase the order of a model are quite common in physical systems. In the interest of model simplification, we usually neglect these parasitic parameters to reduce the order of the model. Singular perturbations legitimize this ad hoc model simplification and provide tools for improving oversimplified models. In the second example, the parameter ε is the reciprocal of a high-gain parameter in a feedback system. The example represents an important source of singularly perturbed models. The use of high-gain parameters, or more precisely, parameters that are driven asymptotically toward infinity, in the design of feedback control systems is quite common. A typical approach to the analysis and design of high-gain feedback systems is to model them in the singularly perturbed form. In the third example, the parameter ε is a parasitic resistor in an electric circuit. Although neglecting the parasitic resistor reduces the order of the model, it does it in a way that is quite distinct from neglecting a parasitic time constant. Modeling the system in the standard singularly perturbed form involves a careful choice of the state variables. In the fourth example, the parameter ε is the ratio of the natural frequency of the car body to the natural frequency of the tire in an automotive suspension model. The special feature of this example is that it cannot be modeled in the standard singularly perturbed form without ε-dependent scaling of the state variables.

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[1] More about modeling physical systems in the singularly perturbed form can be found in [38], [105, Chapter 1], and [104, Chapter 4].

Example 11.1 An armature-controlled DC motor can be modeled by the second-order state equation

$$\begin{aligned} J\frac{d\omega}{dt} &= ki \\ L\frac{di}{dt} &= -k\omega - Ri + u \end{aligned}$$

where i, u, R, and L are the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, and ki and $k\omega$ are, respectively, the torque and the back electromotive force (e.m.f.) developed with constant excitation flux. The first state equation is a mechanical torque equation, and the second one is an equation for the electric transient in the armature circuit. Typically, L is "small" and can play the role of our parameter ε. This means that, with $\omega = x$ and $i = z$, the motor's model is in the standard form of (11.1)–(11.2) whenever $R \neq 0$. Neglecting L, we solve

$$0 = -k\omega - Ri + u$$

to obtain (the unique root)

$$i = \frac{u - k\omega}{R}$$

and substitute it into the torque equation. The resulting model

$$J\dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$

is the commonly used first-order model of the DC motor. As we discussed in Chapter 10, it is preferable to choose the perturbation parameter ε as a dimensionless

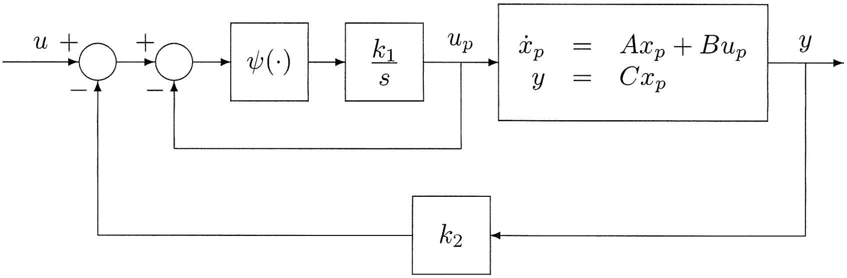

Figure 11.1: Actuator control with high-gain feedback.

ratio of two physical parameters. To that end, let us define the dimensionless variables

$$\omega_r = \frac{\omega}{\Omega}; \quad i_r = \frac{iR}{k\Omega}; \quad u_r = \frac{u}{k\Omega}$$

and rewrite the state equation as

$$\begin{aligned} T_m \frac{d\omega_r}{dt} &= i_r \\ T_e \frac{di_r}{dt} &= -\omega_r - i_r + u_r \end{aligned}$$

where $T_m = JR/k^2$ is the mechanical time constant and $T_e = L/R$ is the electrical time constant. Since $T_m \gg T_e$, we let T_m be the time unit; that is, we introduce the dimensionless time variable $t_r = t/T_m$ and rewrite the state equation as

$$\begin{aligned} \frac{d\omega_r}{dt_r} &= i_r \\ \frac{T_e}{T_m} \frac{di_r}{dt_r} &= -\omega_r - i_r + u_r \end{aligned}$$

This scaling has brought the model into the standard form with a physically meaningful dimensionless parameter

$$\varepsilon = \frac{T_e}{T_m} = \frac{Lk^2}{JR^2}$$

△

Example 11.2 Consider the feedback control system of Figure 11.1. The inner loop represents actuator control with high-gain feedback. The high-gain parameter is the integrator constant k_1. The plant is a single-input–single-output nth-order system represented by the state model $\{A, B, C\}$. The nonlinearity $\psi(\cdot) \in (0, \infty]$; that is,

$$\psi(0) = 0 \text{ and } y\psi(y) > 0, \ \forall \ y \neq 0$$

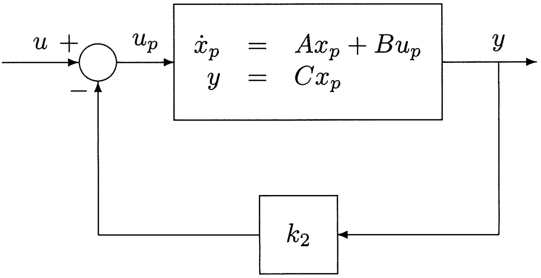

Figure 11.2: Simplified block diagram of Figure 11.1.

The state equation for the closed-loop system is

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu_p \\ \frac{1}{k_1}\dot{u}_p &= \psi(u - u_p - k_2 Cx_p) \end{aligned}$$

With $\varepsilon = 1/k_1$, $x_p = x$, and $u_p = z$, the model takes the form of (11.1)–(11.2). Setting $\varepsilon = 0$, or equivalently $k_1 = \infty$, we solve

$$\psi(u - u_p - k_2 Cx_p) = 0$$

to obtain

$$u_p = u - k_2 Cx_p$$

which is the unique root since $\psi(\cdot)$ vanishes only at its origin. The resulting reduced model

$$\dot{x}_p = (A - Bk_2C)x_p + Bu$$

is the model of the simplified block diagram of Figure 11.2, where the whole inner loop in Figure 11.1 is replaced by a direct connection. △

Example 11.3 Consider again the electric circuit of Example 10.4, shown in Figure 10.2. The differential equations for the voltages across the capacitors are

$$\begin{aligned} C\dot{v}_1 &= \frac{1}{R}(E - v_1) - \psi(v_1) - \frac{1}{R_c}(v_1 - v_2) \\ C\dot{v}_2 &= \frac{1}{R}(E - v_2) - \psi(v_2) - \frac{1}{R_c}(v_2 - v_1) \end{aligned}$$

In Example 10.4, we analyzed the circuit for a "large" resistor R_c, which was idealized to be open circuit when $1/R_c$ was set to zero. This time, let us study the circuit for a "small" R_c. Setting $R_c = 0$ replaces the resistor with a short-circuit connection that puts the two capacitors in parallel. In a well-defined model for this simplified circuit, the two capacitors in parallel should be replaced by one equivalent capacitor, which means that the model of the simplified circuit will be of order one. To represent the model order reduction as a singular perturbation, let us start with the seeming choice $\varepsilon = R_c$ and rewrite the state equation as

$$\begin{aligned} \varepsilon \dot{v}_1 &= \frac{\varepsilon}{CR}(E - v_1) - \frac{\varepsilon}{C}\psi(v_1) - \frac{1}{C}(v_1 - v_2) \\ \varepsilon \dot{v}_2 &= \frac{\varepsilon}{CR}(E - v_2) - \frac{\varepsilon}{C}\psi(v_2) - \frac{1}{C}(v_2 - v_1) \end{aligned}$$

If the preceding model were in the form of (11.1)–(11.2), both v_1 and v_2 would be considered as z variables, and (11.3) would be

$$v_1 - v_2 = 0$$

However, the roots of this equation are not isolated, which violates the basic assumption that the roots of (11.3) should be isolated. Therefore, with v_1 and v_2 as z variables, the model is not in the standard form. Let us now try another choice of the state variables. Take[2]

$$x = \tfrac{1}{2}(v_1 + v_2); \qquad z = \tfrac{1}{2}(v_1 - v_2)$$

The state equation for the new variables is

$$\begin{aligned} \dot{x} &= \frac{1}{CR}(E - x) - \frac{1}{2C}[\psi(x+z) + \psi(x-z)] \\ \varepsilon \dot{z} &= -\left(\frac{\varepsilon}{CR} + \frac{2}{C}\right) z - \frac{\varepsilon}{2C}[\psi(x+z) - \psi(x-z)] \end{aligned}$$

Now the unique root of (11.3) is $z = 0$, which results in the reduced model

$$\dot{x} = -\frac{1}{CR}(E - x) - \frac{1}{C}\psi(x)$$

This model represents the simplified circuit of Figure 11.3, where each pair of similar parallel branches is replaced by an equivalent single branch. To obtain ε as a dimensionless parameter, we normalize x, z, and ψ as

$$x_r = \frac{x}{E}; \qquad z_r = \frac{z}{E}; \qquad \psi_r(v) = \frac{R}{E}\psi(Ev)$$

and normalize the time variable as $t_r = t/CR$ to obtain the singularly perturbed model

$$\begin{aligned} \frac{dx_r}{dt_r} &= 1 - x_r - \frac{1}{2}[\psi_r(x_r + z_r) + \psi_r(x_r - z_r)] \\ \varepsilon \frac{dz_r}{dt_r} &= -(\varepsilon + 2) z_r - \frac{\varepsilon}{2}[\psi_r(x_r + z_r) - \psi_r(x_r - z_r)] \end{aligned}$$

where $\varepsilon = R_c/R$ is dimensionless. △

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[2] This choice of state variables follows from a systematic procedure described in [38].

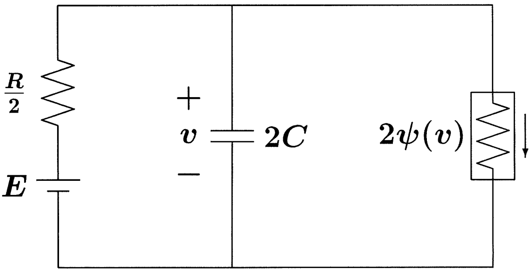

Figure 11.3: Simplified circuit when $R_c = 0$.

Example 11.4 A quarter-car model of automotive suspension is shown in Figure 11.4, where m_s and m_u are the car body and tire masses, k_s and k_t are the spring constants of the strut and tire, b_s is the damper (shock absorber) constant, and F is a force generated by a force actuator that may be used in active and semi-active suspension. When $F = 0$, we have the traditional passive suspension. The distances d_s, d_u, and d_r are the elevations of the car, tire, and road surface, respectively, from a reference point. From Newton's law, the balance of forces acting on m_s and m_u results in the equations

$$m_s\ddot{d}_s + b_s(\dot{d}_s - \dot{d}_u) + k_s(d_s - d_u) = F$$

$$m_u\ddot{d}_u + b_s(\dot{d}_u - \dot{d}_s) + k_s(d_u - d_s) + k_t(d_u - d_r) = -F$$

In a typical car, the natural frequency $\sqrt{k_t/m_u}$ of the tire is about 10 times the natural frequency $\sqrt{k_s/m_s}$ of the car body and strut. We therefore define the parameter

$$\varepsilon = \sqrt{\frac{k_s/m_s}{k_t/m_u}} = \sqrt{\frac{k_s m_u}{k_t m_s}}$$

This mass-spring system is of interest because it cannot be transformed into a standard singularly perturbed model without an ε-dependent scaling. The tire stiffness $k_t = O(1/\varepsilon^2)$ tends to infinity as $\varepsilon \to 0$. For the tire potential energy $k_t(d_u - d_r)^2/2$ to remain bounded, the displacement $d_u - d_r$ must be $O(\varepsilon)$; that is, the scaled displacement $(d_u - d_r)/\varepsilon$ must remain finite. In addition to this scaling, we normalize all variables to be dimensionless. Distances are divided by some distance ℓ, velocities by $\ell\sqrt{k_s/m_s}$, forces by ℓk_s, and time by $\sqrt{m_s/k_s}$. Thus, to express the system in the standard singularly perturbed form, we introduce the slow and fast variables as

$$x = \begin{bmatrix} (d_s - d_u)/\ell \\ (\dot{d}_s/\ell)\sqrt{m_s/k_s} \end{bmatrix}, \quad z = \begin{bmatrix} (d_u - d_r)/(\varepsilon\ell) \\ (\dot{d}_u/\ell)\sqrt{m_s/k_s} \end{bmatrix}$$

and take $u = F/(k_s \ell)$ as the control input, $w = (\dot{d}_r/\ell)\sqrt{m_s/k_s}$ as the disturbance input, and $t_r = t\sqrt{k_s/m_s}$ as the dismensionless time. The resulting singularly perturbed model is

$$\begin{aligned} \frac{dx_1}{dt_r} &= x_2 - z_2 \\ \frac{dx_2}{dt_r} &= -x_1 - \beta(x_2 - z_2) + u \\ \varepsilon\frac{dz_1}{dt_r} &= z_2 - w \\ \varepsilon\frac{dz_2}{dt_r} &= \alpha x_1 - \alpha\beta(z_2 - x_2) - z_1 - \alpha u \end{aligned}$$

where

$$\alpha = \sqrt{\frac{k_s m_s}{k_t m_u}}, \quad \beta = \frac{b_s}{\sqrt{k_s m_s}}$$

For typical cars with passive suspension, the parameters α, β, and ε take values in the ranges $[0.6, 1.2]$, $[0.5, 0.8]$, and $[0.08, 0.135]$, respectively. In active/semiactive suspension, the damping constant may be reduced as the force actuator provides additional damping. Setting $\varepsilon = 0$ results in the reduced model

$$\begin{aligned} \frac{dx_1}{dt_r} &= x_2 - w \\ \frac{dx_2}{dt_r} &= -x_1 - \beta(x_2 - w) + u \end{aligned}$$

which corresponds to the simplified one-degree-of-freedom model shown in Figure 11.4. △

11.2 Time-Scale Properties of the Standard Model

Singular perturbations cause a multitime-scale behavior of dynamical systems characterized by the presence of slow and fast transients in the system's response to external stimuli. Loosely speaking, the slow response is approximated by the reduced model (11.5), while the discrepancy between the response of the reduced model and that of the full model (11.1)–(11.2) is the fast transient. To see this point, let us consider the problem of solving the state equation

$$\dot{x} = f(t, x, z, \varepsilon), \quad x(t_0) = \xi(\varepsilon) \tag{11.6}$$

$$\varepsilon\dot{z} = g(t, x, z, \varepsilon), \quad z(t_0) = \eta(\varepsilon) \tag{11.7}$$

where $\xi(\varepsilon)$ and $\eta(\varepsilon)$ depend smoothly on ε and $t_0 \in [0, t_1)$. Let $x(t, \varepsilon)$ and $z(t, \varepsilon)$ denote the solution of the full problem of (11.6) and (11.7). When we define the

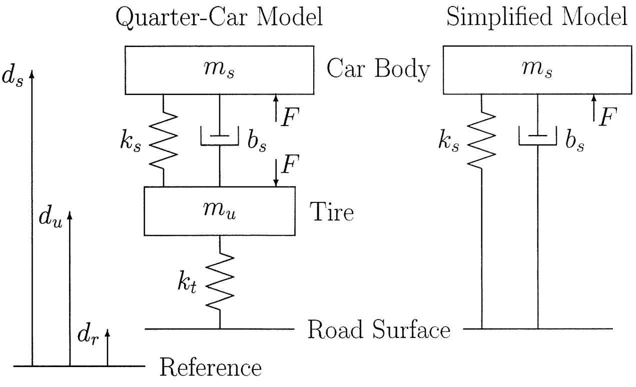

Figure 11.4: Quarter-Car Model of Automotive suspension.

corresponding problem for the reduced model (11.5), we can only specify n initial conditions, since the model is nth order. Naturally, we retain the initial state for x to obtain the reduced problem

$$\dot{x} = f(t, x, h(t, x), 0), \quad x(t_0) = \xi_0 \overset{\text{def}}{=} \xi(0) \tag{11.8}$$

Denote the solution of (11.8) by $\bar{x}(t)$. Because the variable z has been excluded from the reduced model and substituted by its "quasi-steady-state" $h(t, x)$, the only information we can obtain about z by solving (11.8) is to compute

$$\bar{z}(t) \overset{\text{def}}{=} h(t, \bar{x}(t))$$

which describes the quasi-steady-state behavior of z when $x = \bar{x}$. By contrast to the original variable z starting at t_0 from a prescribed $\eta(\varepsilon)$, the quasi-steady-state $\bar{z}$ is not free to start from a prescribed value, and there may be a large discrepancy between its initial value $\bar{z}(t_0) = h(t_0, \xi_0)$ and the prescribed initial state $\eta(\varepsilon)$. Thus, $\bar{z}(t)$ cannot be a uniform approximation of $z(t, \varepsilon)$. The best we can expect is that the estimate

$$z(t, \varepsilon) - \bar{z}(t) = O(\varepsilon)$$

will hold on an interval excluding t_0, that is, for $t \in [t_b, t_1]$, where $t_b > t_0$. On the other hand, it is reasonable to expect the estimate

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon)$$

to hold uniformly for all $t \in [t_0, t_1]$, since

$$x(t_0, \varepsilon) - \bar{x}(t_0) = \xi(\varepsilon) - \xi(0) = O(\varepsilon)$$

If the error $z(t, \varepsilon) - \bar{z}(t)$ is indeed $O(\varepsilon)$ over $[t_b, t_1]$, then it must be true that during the initial ("boundary-layer") interval $[t_0, t_b]$, the variable z approaches $\bar{z}$. Let us remember that the speed of z can be high, because $\dot{z} = g/\varepsilon$. In fact, having set $\varepsilon = 0$ in (11.2), we have made the transient of z instantaneous whenever $g \neq 0$. From our previous study of the stability of equilibrium points, it should be clear that we cannot expect z to converge to its quasi-steady-state $\bar{z}$, unless certain stability conditions are satisfied. Such conditions will result from the forthcoming analysis.

It is more convenient in the analysis to perform the change of variables

$$y = z - h(t, x) \tag{11.9}$$

that shifts the quasi-steady-state of z to the origin. In the new variables (x, y), the full problem is

$$\begin{aligned} \dot{x} &= f(t, x, y + h(t, x), \varepsilon), \quad x(t_0) = \xi(\varepsilon) && (11.10) \\ \varepsilon \dot{y} &= g(t, x, y + h(t, x), \varepsilon) - \varepsilon \frac{\partial h}{\partial t} \\ &\quad - \varepsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \varepsilon), \quad y(t_0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)) && (11.11) \end{aligned}$$

The quasi-steady-state of (11.11) is now $y = 0$, which when substituted into (11.10) results in the reduced model (11.8). To analyze (11.11), let us note that $\varepsilon \dot{y}$ may remain finite even when ε tends to zero and $\dot{y}$ tends to infinity. We set

$$\varepsilon \frac{dy}{dt} = \frac{dy}{d\tau}; \text{ hence, } \frac{d\tau}{dt} = \frac{1}{\varepsilon}$$

and use $\tau = 0$ as the initial value at $t = t_0$. The new time variable $\tau = (t - t_0)/\varepsilon$ is "stretched"; that is, if ε tends to zero, τ tends to infinity even for finite t only slightly larger than t_0 by a fixed (independent of ε) difference. In the τ time scale, (11.11) is represented by

$$\begin{aligned} \frac{dy}{d\tau} &= g(t, x, y + h(t, x), \varepsilon) - \varepsilon \frac{\partial h}{\partial t} \\ &\quad - \varepsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \varepsilon), \quad y(0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)) && (11.12) \end{aligned}$$

The variables t and x in the foregoing equation will be slowly varying since, in the τ time scale, they are given by

$$t = t_0 + \varepsilon \tau, \qquad x = x(t_0 + \varepsilon \tau, \varepsilon)$$

Setting $\varepsilon = 0$ freezes these variables at $t = t_0$ and $x = \xi_0$, and reduces (11.12) to the autonomous system

$$\frac{dy}{d\tau} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0), \quad y(0) = \eta(0) - h(t_0, \xi_0) \stackrel{\text{def}}{=} \eta_0 - h(t_0, \xi_0) \tag{11.13}$$

which has equilibrium at $y = 0$. If this equilibrium point is asymptotically stable and $y(0)$ belongs to its region of attraction, it is reasonable to expect that the solution of (11.13) will reach an $O(\varepsilon)$ neighborhood of the origin during the boundary-layer interval. Beyond this interval, we need a stability property that guarantees that $y(\tau)$ will remain close to zero, while the slowly varying parameters (t, x) move away from their initial values (t_0, ξ_0). To analyze this situation, we allow the frozen parameters to take values in the region of the slowly varying parameters (t, x).[3] Assume that the solution $\bar{x}(t)$ of the reduced problem is defined for $t \in [0, t_1]$ and $\bar{x}(t) \in D_x \subset R^n$, for some domain D_x. We rewrite (11.13) as

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0) \tag{11.14}$$

where $(t, x) \in [0, t_1] \times D_x$ are treated as fixed parameters. We will refer to (11.14) as the boundary-layer model or boundary-layer system. Sometimes, we will also refer to (11.13) as the boundary-layer model. This should cause no confusion, because (11.13) is an evaluation of (11.14) for a given initial time and initial state. The crucial stability property we need for (11.14) is exponential stability of its origin, uniformly in the frozen parameters, as stated in the next definition.

Definition 11.1 *The equilibrium point* $y = 0$ *of the boundary-layer system* (11.14) *is exponentially stable, uniformly in* $(t, x) \in [0, t_1] \times D_x$, *if there exist positive constants* k, γ, *and* ρ_0 *such that the solutions of* (11.14) *satisfy*

$$\|y(\tau)\| \leq k\|y(0)\| \exp(-\gamma\tau), \; \forall \; \|y(0)\| < \rho_0, \; \forall \; (t, x) \in [0, t_1] \times D_x, \; \forall \; \tau \geq 0 \tag{11.15}$$

Aside from trivial cases where the solution of the boundary layer model may be known in closed form, verification of exponential stability of the origin will have to be done either by linearization or via Lyapunov analysis. It can be shown (Exercise 11.5) that if the Jacobian matrix $[\partial g/\partial y]$ satisfies the eigenvalue condition

$$\mathrm{Re}\left[\lambda\left\{\frac{\partial g}{\partial y}(t, x, h(t, x), 0)\right\}\right] \leq -c < 0, \quad \forall \; (t, x) \in [0, t_1] \times D_x \tag{11.16}$$

then there exist constants k, γ, and ρ_0 for which (11.15) is satisfied. This, of course, is a local result; that is, the constant ρ_0 could be very small. Alternatively, it can be shown (Exercise 11.6) that if there is a Lyapunov function $V(t, x, y)$ that satisfies

$$c_1\|y\|^2 \leq V(t, x, y) \leq c_2\|y\|^2 \tag{11.17}$$

$$\frac{\partial V}{\partial y} g(t, x, y + h(t, x), 0) \leq -c_3\|y\|^2 \tag{11.18}$$

for $(t, x, y) \in [0, t_1] \times D_x \times D_y$, where $D_y \subset R^m$ is a domain that contains the origin, then (11.15) is satisfied with the estimates

$$\rho_0 = \rho\sqrt{c_1/c_2}, \; k = \sqrt{c_2/c_1}, \; \gamma = c_3/2c_2 \tag{11.19}$$

in which $B_\rho \subset D_y$.

---

[3] Recall from Section 9.6 that if the origin of (11.13) is exponentially stable, uniformly in the frozen parameters (t_0, ξ_0), then it will remain exponentially stable when these parameters are replaced by the slowly varying variables (t, x).

Theorem 11.1 *Consider the singular perturbation problem of* (11.6) *and* (11.7) *and let* $z = h(t, x)$ *be an isolated root of* (11.3). *Assume that the following conditions are satisfied for all*

$$[t, x, z - h(t, x), \varepsilon] \in [0, t_1] \times D_x \times D_y \times [0, \varepsilon_0]$$

for some domains $D_x \subset R^n$ *and* $D_y \subset R^m$, *in which* D_x *is convex and* D_y *contains the origin:*

- *The functions* f, g, *their first partial derivatives with respect to* (x, z, ε), *and the first partial derivative of* g *with respect to* t *are continuous; the function* $h(t, x)$ *and the Jacobian* $[\partial g(t, x, z, 0)/\partial z]$ *have continuous first partial derivatives with respect to their arguments; the initial data* $\xi(\varepsilon)$ *and* $\eta(\varepsilon)$ *are smooth functions of* ε.

- *The reduced problem* (11.8) *has a unique solution* $\bar{x}(t) \in S$, *for* $t \in [t_0, t_1]$, *where* S *is a compact subset of* D_x.

- *The origin is an exponentially stable equilibrium point of the boundary-layer model* (11.14), *uniformly in* (t, x); *let* $\mathcal{R}_y \subset D_y$ *be the region of attraction of* (11.13) *and* Ω_y *be a compact subset of* $\mathcal{R}_y$.

Then, there exists a positive constant ε^* *such that for all* $\eta_0 - h(t_0, \xi_0) \in \Omega_y$ *and* $0 < \varepsilon < \varepsilon^*$, *the singular perturbation problem of* (11.6) *and* (11.7) *has a unique solution* $x(t, \varepsilon)$, $z(t, \varepsilon)$ *on* $[t_0, t_1]$, *and*

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon) \tag{11.20}$$

$$z(t, \varepsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\varepsilon) = O(\varepsilon) \tag{11.21}$$

hold uniformly for $t \in [t_0, t_1]$, *where* $\hat{y}(\tau)$ *is the solution of the boundary-layer model* (11.13). *Moreover, given any* $t_b > t_0$, *there is* $\varepsilon^{**} \leq \varepsilon^*$ *such that*

$$z(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon) \tag{11.22}$$

holds uniformly for $t \in [t_b, t_1]$ *whenever* $\varepsilon < \varepsilon^{**}$. ◇

Proof: See Appendix C.17.

This theorem is known as Tikhonov's theorem.[4] Its proof uses the stability properties of the boundary-layer model to show that

$$\|y(t, \varepsilon)\| \leq k_1 \exp\left[\frac{-\alpha(t - t_0)}{\varepsilon}\right] + \varepsilon\delta$$

The preceding bound is used in (11.10) to prove (11.20), which is plausible, since $\int_0^t \exp(-\alpha s/\varepsilon)\, ds$ is $O(\varepsilon)$. The proof ends with error analysis of (11.11) in the τ time scale to prove (11.21) and (11.22).

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[4] There are other versions of Tikhonov's theorem which use slightly different technical assumptions. (See, for example, [105, Chapter 1, Theorem 3.1].)

Example 11.5 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= z, & x(0) &= \xi_0 \\ \varepsilon\dot{z} &= -x - z + u(t), & z(0) &= \eta_0 \end{aligned}$$

for the DC motor of Example 11.1. Suppose $u(t) = t$ for $t \geq 0$ and we want to solve the state equation over the interval $[0, 1]$. The unique root of (11.3) is $h(t, x) = -x + t$ and the boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -y$$

Clearly, the origin of the boundary-layer system is globally exponentially stable. The reduced problem

$$\dot{x} = -x + t, \quad x(0) = \xi_0$$

has the unique solution

$$\bar{x}(t) = t - 1 + (1 + \xi_0)\exp(-t)$$

The boundary-layer problem

$$\frac{dy}{d\tau} = -y, \quad y(0) = \eta_0 + \xi_0$$

has the unique solution

$$\hat{y}(\tau) = (\eta_0 + \xi_0)\exp(-\tau)$$

From Theorem 11.1, we have

$$x - [t - 1 + (1 + \xi_0)\exp(-t)] = O(\varepsilon)$$

$$z - \left[(\eta_0 + \xi_0)\exp\left(\frac{-t}{\varepsilon}\right) + 1 - (1 + \xi_0)\exp(-t)\right] = O(\varepsilon)$$

for all $t \in [0, 1]$. The $O(\varepsilon)$ approximation of z clearly exhibits a two-time-scale behavior. It starts with a fast transient $(\eta_0 + \xi_0)\exp(-t/\varepsilon)$, which is the so-called boundary-layer part of the solution. After the decay of this transient, z remains close to $[1 - (1 + \xi_0)\exp(-t)]$, which is the slow (quasi-steady-state) part of the solution. The two-time-scale behavior is significant only in z, while x is predominantly slow. In fact, x has a fast (boundary-layer) transient, but it is $O(\varepsilon)$. Since this system is linear, we can characterize its two-time-scale behavior via modal analysis. It can be easily seen that the system has one slow eigenvalue λ_1, which is $O(\varepsilon)$ close to the eigenvalue of the reduced model, that is, $\lambda_1 = -1 + O(\varepsilon)$, and one fast eigenvalue $\lambda_2 = \lambda/\varepsilon$, where λ is $O(\varepsilon)$ close to the eigenvalue of the boundary-layer model, that is, $\lambda_2 = [-1 + O(\varepsilon)]/\varepsilon$. The exact solutions of x and z will be linear combinations of the slow mode $\exp(\lambda_1 t)$, the fast mode $\exp(\lambda t/\varepsilon)$, and a steady-state component due to the input $u(t) = t$. By actually calculating the modal decomposition, it can be verified that the coefficient of the fast mode in x is $O(\varepsilon)$. This can be done for linear systems in general. (See Exercise 11.14.) △

Example 11.6 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= Ax + Bz, & x(0) &= \xi_0 \\ \varepsilon \dot{z} &= \psi(u(t) - z - k_2 Cx), & z(0) &= \eta_0 \end{aligned}$$

for the high-gain feedback system of Example 11.2. Suppose $u(t) = 1$ for $t \geq 0$ and $\psi(\cdot) = \tan^{-1}(\cdot)$. The unique root of (11.3) is $h(t,x) = 1 - k_2 Cx$ and the boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = \tan^{-1}(-y) = -\tan^{-1}(y)$$

The Jacobian

$$\left.\frac{\partial g}{\partial y}\right|_{y=0} = -\left.\frac{1}{1+y^2}\right|_{y=0} = -1$$

is Hurwitz; hence, the origin of the boundary-layer model is exponentially stable. It is also clear that the origin is globally asymptotically stable. Since the reduced problem

$$\dot{x} = (A - Bk_2C)x + B, \quad x(0) = \xi_0$$

is linear, it is clear that all the assumptions of Theorem 11.1 are satisfied, and we can proceed to approximate x and z in terms of the solutions of the reduced and boundary-layer problems. △

Example 11.7 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x^2(1+t)/z, & x(0) &= 1 \\ \varepsilon \dot{z} &= -[z + (1+t)x]\ z\ [z - (1+t)], & z(0) &= \eta_0 \end{aligned}$$

Equation (11.3), which takes the form

$$0 = -[z + (1+t)x]\ z\ [z - (1+t)]$$

has three isolated roots

$$z = -(1+t)x, \quad z = 0, \quad \text{and} \quad z = 1+t$$

in the region $\{t \geq 0 \text{ and } x > k\}$, where $0 < k < 1$. Consider first the root $z = -(1+t)x$. The boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -y[y - (1+t)x][y - (1+t)x - (1+t)]$$

A sketch of the right-hand side function, Figure 11.5(a), shows that the origin is asymptotically stable with $y < (1+t)x$ as its region of attraction. Taking $V(y) = y^2$,

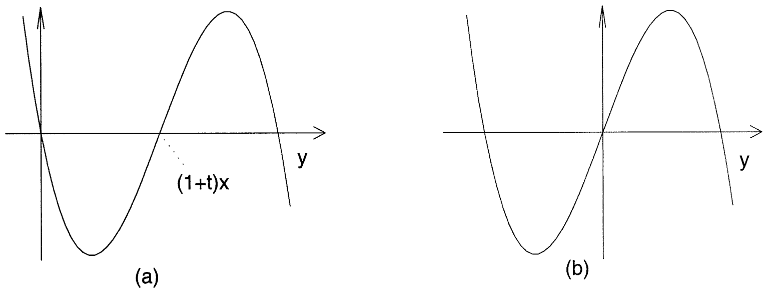

Figure 11.5: RHS of boundary-layer model: (a) $z = -(1+t)x$, (b) $z = 0$.

it can be easily verified that V satisfies (11.17) and (11.18) for $y \le \rho < (1+t)x$. The reduced problem

$$\dot{x} = -x, \quad x(0) = 1$$

has the unique solution $\bar{x}(t) = \exp(-t)$ for all $t \ge 0$. The boundary-layer problem with $t = 0$ and $x = 1$,

$$\frac{dy}{d\tau} = -y(y-1)(y-2), \quad y(0) = \eta_0 + 1$$

has a unique decaying solution $\hat{y}(\tau)$ for $\eta_0 < 0$. Consider next the root $z = 0$. The boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -[y + (1+t)x] \; y \; [y - (1+t)]$$

A sketch of the right-hand side function, Figure 11.5(b), shows that the origin is unstable. Consequently, Theorem 11.1 does not apply to this case. Finally, the boundary-layer model for the root $z = 1 + t$ is

$$\frac{dy}{d\tau} = -[y + (1+t) + (1+t)x][y + (1+t)]y$$

Similar to the first case, it can be shown that the origin is exponentially stable uniformly in (t, x). The reduced problem

$$\dot{x} = x^2, \quad x(0) = 1$$

has the unique solution $\bar{x}(t) = 1/(1-t)$ for all $t \in [0, 1)$. Notice that $\bar{x}(t)$ has a finite escape time at $t = 1$. However, Theorem 11.1 still holds for $t \in [0, t_1]$ with $t_1 < 1$. The boundary-layer problem with $t = 0$ and $x = 1$,

$$\frac{dy}{d\tau} = -(y+2)(y+1)y, \quad y(0) = \eta_0 - 1$$

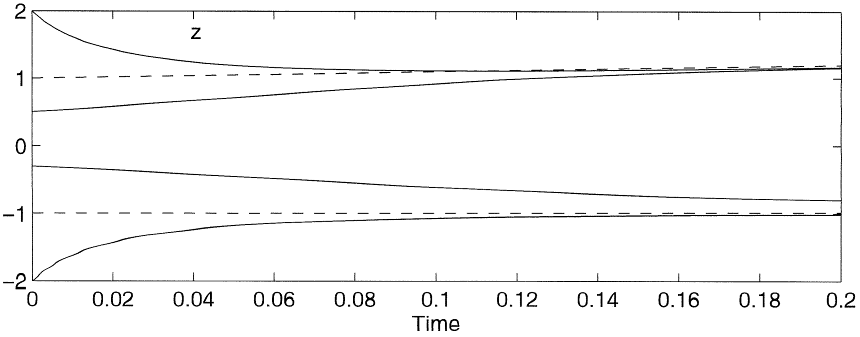

Figure 11.6: Simulation results for z of Example 11.7 at $\varepsilon = 0.1$: reduced solution (dashed); exact solution (solid).

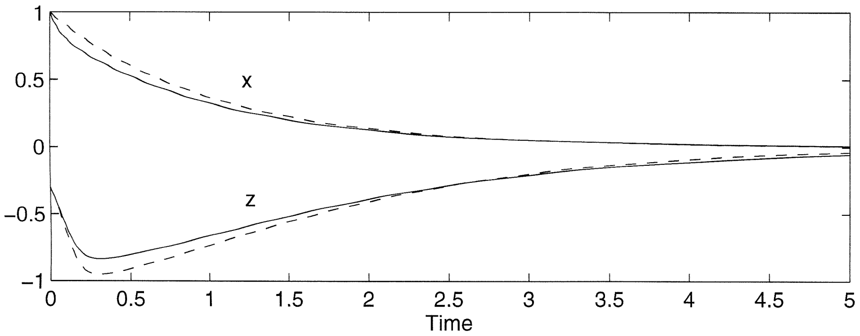

Figure 11.7: Exact (solid) and approximate (dashed) solutions for Example 11.7 at $\varepsilon = 0.1$.

has a unique decaying solution $\hat{y}(\tau)$ for $\eta_0 > 0$. Among the three roots of (11.3), only two roots, $h = -(1+t)x$ and $h = 1+t$, give rise to valid reduced models. Theorem 11.1 applies to the root $h = -(1+t)x$ if $\eta_0 < 0$ and to the root $h = 1+t$ if $\eta_0 > 0$. Figures 11.6 and 11.7 show simulation results at $\varepsilon = 0.1$. Figure 11.6 shows z for four different values of η_0, two for each reduced model. Figure 11.7 shows the exact and approximate solutions of x and z for $\eta_0 = -0.3$. The trajectories of Figure 11.6 clearly exhibit a two-time-scale behavior. They start with a fast transient of $z(t,\varepsilon)$ from η_0 to $\bar{z}(t)$. After the decay of this transient, they remain close to $\bar{z}(t)$. In the case $\eta_0 = -0.3$, the convergence to $\bar{z}(t)$ does not take place within the time interval $[0, 0.2]$. The same case is shown in Figure 11.7 on a longer time interval, where we can see $z(t,\varepsilon)$ approaching $\bar{z}(t)$. Figure 11.7 illustrates the $O(\varepsilon)$ asymptotic approximation result of Tikhonov's theorem. △

11.3 Singular Perturbation on the Infinite Interval

Theorem 11.1 is valid only on $O(1)$ time intervals. This fact can be easily seen from the proof of the theorem. In particular, it is established in (C.81) that

$$\|x(t,\varepsilon) - \bar{x}(t)\| \le \varepsilon k_3[1 + t_1 - t_0]\exp[L_6(t_1 - t_0)]$$

For any finite t_1, the foregoing estimate is $O(\varepsilon)$, but it is not $O(\varepsilon)$ uniformly in t for all $t \ge t_0$. For the latter statement to hold, we need to show that

$$\|x(t,\varepsilon) - \bar{x}(t)\| \le \varepsilon k, \quad \forall\ t \in [t_0, \infty)$$

This can be done under some additional stability conditions. In the next theorem, we require the reduced system (11.5) to have an exponentially stable equilibrium point at the origin and use a Lyapunov function to estimate its region of attraction.

Theorem 11.2 *Consider the singular perturbation problem of* (11.6) *and* (11.7) *and let* $z = h(t,x)$ *be an isolated root of* (11.3). *Assume that the following conditions are satisfied for all*

$$[t, x, z - h(t,x), \varepsilon] \in [0,\infty) \times D_x \times D_y \times [0, \varepsilon_0]$$

for some domains $D_x \subset R^n$ *and* $D_y \subset R^m$*, which contain their respective origins:*

- *On any compact subset of* $D_x \times D_y$*, the functions* f*,* g*, their first partial derivatives with respect to* (x, z, ε)*, and the first partial derivative of* g *with respect to* t *are continuous and bounded,* $h(t,x)$ *and* $[\partial g(t,x,z,0)/\partial z]$ *have bounded first partial derivatives with respect to their arguments, and* $[\partial f(t,x,h(t,x),0)/\partial x]$ *is Lipschitz in* x*, uniformly in* t*; the initial data* $\xi(\varepsilon)$ *and* $\eta(\varepsilon)$ *are smooth functions of* ε*;*

- *the origin is an exponentially stable equilibrium point of the reduced system* (11.5)*; there is a Lyapunov function* $V(t,x)$ *that satisfies the conditions of Theorem* 4.9 *for* (11.5) *for* $(t,x) \in [0,\infty) \times D_x$ *and* $\{W_1(x) \le c\}$ *is a compact subset of* D_x*;*

- *the origin is an exponentially stable equilibrium point of the boundary-layer system* (11.14)*, uniformly in* (t,x)*; let* $\mathcal{R}_y \subset D_y$ *be the region of attraction of* (11.13) *and* Ω_y *be a compact subset of* $\mathcal{R}_y$*.*

Then, for each compact set $\Omega_x \subset \{W_2(x) \le \rho c,\ 0 < \rho < 1\}$ *there is a positive constant* ε^* *such that for all* $t_0 \ge 0$*,* $\xi_0 \in \Omega_x$*,* $\eta_0 - h(t_0, \xi_0) \in \Omega_y$*, and* $0 < \varepsilon < \varepsilon^*$*, the singular perturbation problem of* (11.6) *and* (11.7) *has a unique solution* $x(t,\varepsilon)$*,* $z(t,\varepsilon)$ *on* $[t_0,\infty)$*, and*

$$x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon) \tag{11.23}$$

$$z(t,\varepsilon) - h(t,\bar{x}(t)) - \hat{y}(t/\varepsilon) = O(\varepsilon) \tag{11.24}$$

hold uniformly for $t \in [t_0, \infty)$, *where* $\bar{x}(t)$ *and* $\hat{y}(\tau)$ *are the solutions of the reduced and boundary-layer problems* (11.8) *and* (11.13). *Moreover, given any* $t_b > t_0$, *there is* $\varepsilon^{**} \le \varepsilon^*$ *such that*

$$z(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon) \tag{11.25}$$

holds uniformly for $t \in [t_b, \infty)$ *whenever* $\varepsilon < \varepsilon^{**}$. ◇

Proof: See Appendix C.18.

If the reduced system (11.5) is autonomous, the set Ω_x in Theorem 11.2 can be any compact subset of its region of attraction. This is a consequence of (the converse Lyapunov) Theorem 4.17, which provides a Lyapunov function $V(x)$ such that any compact subset of the region of attraction is in the interior of a compact set of the form $\{V(x) \le c\}$.

Example 11.8 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= 1 - x - \frac{1}{2}[\psi(x+z) + \psi(x-z)], \qquad x(0) = \xi_0 \\ \varepsilon \dot{z} &= -(\varepsilon + 2)z - \frac{\varepsilon}{2}[\psi(x+z) - \psi(x-z)], \quad z(0) = \eta_0 \end{aligned}$$

for the electric circuit of Example 11.3, and assume that

$$\psi(v) = a\left[\exp\left(\frac{v}{b}\right) - 1\right], \quad a > 0, \ b > 0$$

We have dropped the subscript r as we copied these equations from Example 11.3. The differentiability and Lipschitz conditions of Theorem 11.2 are satisfied on any compact set of (x, z). The reduced model

$$\dot{x} = 1 - x - a\left[\exp\left(\frac{x}{b}\right) - 1\right] \stackrel{\text{def}}{=} f_o(x)$$

has a unique equilibrium point at $x = p^*$, where p^* is the unique root of $f_o(p^*) = 0$. It can be easily seen that $0 < p^* < 1$. The Jacobian

$$\left.\frac{df_o}{dx}\right|_{x=p^*} = -1 - \frac{a}{b}\exp\left(\frac{p^*}{b}\right) < -1$$

is negative; hence, the equilibrium point $x = p^*$ is exponentially stable. Moreover, by sketching the function $f_o(x)$, it can be seen that $x = p^*$ is globally asymptotically stable. The change of variables $\tilde{x} = x - p^*$ shifts the equilibrium point to the origin. The boundary-layer model

$$\frac{dz}{d\tau} = -2z$$

is independent of x, and its origin is globally exponentially stable. Thus, all the conditions of Theorem 11.2 are satisfied globally and the estimates of (11.23) through (11.25), with $h = 0$, hold for all $t \ge 0$ and for any bounded initial state (ξ_0, η_0). △

Example 11.9 Consider the adaptive control of a plant represented by the second-order transfer function

$$\tilde{P}(s) = \frac{k_p}{(s - a_p)(\varepsilon s + 1)}$$

where a_p, $k_p > 0$, and $\varepsilon > 0$ are unknown parameters. The parameter ε represents a small "parasitic" time constant. Suppose we have neglected ε and simplified the transfer function to

$$P(s) = \frac{k_p}{s - a_p}$$

We may now proceed to design the adaptive controller for this first-order transfer function. In Section 1.2.6, a model reference adaptive controller is given by

$$\begin{aligned} u &= \theta_1 r + \theta_2 y_p \\ \dot{\theta}_1 &= -\gamma(y_p - y_m)r \\ \dot{\theta}_2 &= -\gamma(y_p - y_m)y_p \end{aligned}$$

where y_p, u, r, and y_m are the plant output, the control input, the reference input, and the reference model output, respectively. With (the first-order model of) the plant and the reference model represented by

$$\dot{y}_p = a_p y_p + k_p u$$

and

$$\dot{y}_m = a_m y_m + k_m r, \qquad k_m > 0$$

it is shown in Section 1.2.6 that the closed-loop adaptive control system is represented by the third-order state equation

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p\phi_1 r + k_p\phi_2(e_o + y_m) \\ \dot{\phi}_1 &= -\gamma e_o r \\ \dot{\phi}_2 &= -\gamma e_o(e_o + y_m) \end{aligned}$$

where $e_o = y_p - y_m$, $\phi_1 = \theta_1 - \theta_1^*$, $\phi_2 = \theta_2 - \theta_2^*$, $\theta_1^* = k_m/k_p$, and $\theta_2^* = (a_m - a_p)/k_p$. Define

$$x = \begin{bmatrix} e_o & \phi_1 & \phi_2 \end{bmatrix}^T$$

as the state vector and rewrite the state equation as

$$\dot{x} = f_0(t, x)$$

where $f_0(t, 0) = 0$. We will refer to this third-order state equation as the nominal adaptive control system, which is the model we use in the stability analysis. We assume that the origin of the model is exponentially stable.[5] When the adaptive controller is applied to the actual system, the closed-loop system will be different from this nominal model. Let us represent the situation as a singular perturbation problem. The actual second-order model of the plant can be represented by the singularly perturbed model

$$\begin{aligned} \dot{y}_p &= a_p y_p + k_p z \\ \varepsilon \dot{z} &= -z + u \end{aligned}$$

By repeating the derivations of Section 1.2.6, it can be seen that the actual adaptive control system is represented by the singularly perturbed model

$$\begin{aligned} \dot{x} &= f_0(t,x) + K[z - h(t,x)] \\ \varepsilon \dot{z} &= -z + h(t,x) \end{aligned}$$

where

$$h(t,x) = u = (\theta_1^* + \phi_1)r(t) + (\theta_2^* + \phi_2)(e_o + y_m(t)), \quad K = [k_p, 0, 0]^T$$

The signal $y_m(t)$ is the output of a Hurwitz transfer function driven by $r(t)$. Therefore, it has the same smoothness and boundedness properties of $r(t)$. In particular, if $r(t)$ has continuous and bounded derivatives up to order N, the same will be true for $y_m(t)$. Let us analyze this singularly perturbed system. At $\varepsilon = 0$, we have $z = h(t,x)$ and the reduced model is

$$\dot{x} = f_0(t,x)$$

which is the closed-loop model of the nominal adaptive control system. We have assumed that the origin of the model is exponentially stable. The boundary-layer model

$$\frac{dy}{d\tau} = -y$$

is independent of (t,x) and its origin is globally exponentially stable. If the reference input $r(t)$ and its derivative $\dot{r}(t)$ are bounded, all the assumptions of Theorem 11.2 will be satisfied on any compact set of (x,z). Let $\bar{x}$ denote the solution of the nominal adaptive control system and $x(t,\varepsilon)$ denote the solution of the actual adaptive control system, both starting from the same initial state. By Theorem 11.2, we conclude that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$,

---

[5] It is shown in Example 8.12 that this will be the case under a persistence of excitation condition. In particular, the origin will be exponentially stable if $r(t) = a \sin \omega t$. A word of caution at this point: Note that our analysis in this example assumes that $r(t)$ is fixed and studies the asymptotic behavior of the system for small ε. As we fix the value of ε at some small numerical value, our underlying assumption puts a constraint on $r(t)$–in particular, on the input frequency ω. If we start to increase ω, we may reach a point where the conclusions of the example are no longer valid because a high-frequency input may violate the slowly varying nature of the slow variable x. For example, the signal $\dot{r}(t)$, which is of order $O(\omega)$, may violate our assumption that $\dot{r}$ is of order $O(1)$ with respect to ε.

$$x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$$

where $O(\varepsilon)$ holds uniformly in t for all $t \geq t_0$. This result shows robustness to unmodeled fast dynamics. △

11.4 Slow and Fast Manifolds

In this section, we give a geometric view of the two-time-scale behavior of the solutions of (11.1)–(11.2) as trajectories in R^{n+m}. In order to use the concept of invariant manifolds,[6] we restrict our discussion to autonomous systems. Furthermore, to simplify the notation, we take f and g to be independent of ε. Thus, we consider the following simpler form of the singularly perturbed system (11.1)–(11.2):

$$\dot{x} = f(x,z) \tag{11.26}$$
$$\varepsilon\dot{z} = g(x,z) \tag{11.27}$$

Let $z = h(x)$ be an isolated root of $0 = g(x,z)$ and suppose the assumptions of Theorem 11.1 are satisfied for this root. The equation $z = h(x)$ describes an n-dimensional manifold in the $(n+m)$-dimensional state space of (x,z). It is an invariant manifold for the system

$$\dot{x} = f(x,z) \tag{11.28}$$
$$0 = g(x,z) \tag{11.29}$$

since a trajectory of (11.28)–(11.29) that starts in the manifold $z = h(x)$ will remain in the manifold for all future time (for which the solution is defined). The motion in this manifold is described by the reduced model

$$\dot{x} = f(x,h(x))$$

Theorem 11.1 shows that trajectories of (11.26)–(11.27), which start in an $O(\varepsilon)$ neighborhood of $z = h(x)$, will remain within an $O(\varepsilon)$ neighborhood of $z = h(x)$. This motivates the following question: Is there an analog of the invariant manifold $z = h(x)$ for $\varepsilon > 0$? It turns out that, under the assumptions of Theorem 11.1, there is a nearby invariant manifold for (11.26)–(11.27) that lies within an $O(\varepsilon)$ neighborhood of $z = h(x)$. We seek the invariant manifold for (11.26)–(11.27) in the form

$$z = H(x,\varepsilon) \tag{11.30}$$

where H is a sufficiently smooth (that is, sufficiently many times continuously differentiable) function of x and ε. The expression (11.30) defines an n-dimensional manifold, dependent on ε, in the $(n+m)$-dimensional state space of (x,z). For $z = H(x,\varepsilon)$ to be an invariant manifold of (11.26)–(11.27), it must be true that

---

[6] Invariant manifolds have been introduced in Section 8.1.

$$z(0,\varepsilon) - H(x(0,\varepsilon),\varepsilon) = 0 \Rightarrow z(t,\varepsilon) - H(x(t,\varepsilon),\varepsilon) \equiv 0, \quad \forall\, t \in J \subset [0,\infty)$$

where J is any time interval over which the solution $[x(t,\varepsilon), z(t,\varepsilon)]$ exists. Differentiating both sides of (11.30) with respect to t, multiplying through by ε, and substituting for $\dot{x}$, $\varepsilon\dot{z}$, and z from (11.26), (11.27), and (11.30), respectively, we obtain the *manifold condition*

$$0 = g(x, H(x,\varepsilon)) - \varepsilon \frac{\partial H}{\partial x} f(x, H(x,\varepsilon)) \tag{11.31}$$

which $H(x,\varepsilon)$ must satisfy for all x in the region of interest and all $\varepsilon \in [0,\varepsilon_0]$. At $\varepsilon = 0$, the partial differential equation (11.31) degenerates into

$$0 = g(x, H(x,0))$$

which shows that $H(x,0) = h(x)$. Since $0 = g(x,z)$ may have more than one isolated root $z = h(x)$, we may seek an invariant manifold for (11.26)–(11.27) in the neighborhood of each root. It can be shown[7] that there exist $\varepsilon^* > 0$ and a function $H(x,\varepsilon)$ satisfying the manifold condition (11.31) for all $\varepsilon \in [0,\varepsilon^*]$ and

$$H(x,\varepsilon) - h(x) = O(\varepsilon)$$

for bounded x. The invariant manifold $z = H(x,\varepsilon)$ is called a *slow manifold* for (11.26)–(11.27). For each slow manifold, there corresponds a slow model

$$\dot{x} = f(x, H(x,\varepsilon)) \tag{11.32}$$

which describes *exactly* the motion on that manifold.

In most cases, we cannot solve the manifold condition (11.31) exactly, but we can approximate $H(x,\varepsilon)$ arbitrarily closely as a Taylor series at $\varepsilon = 0$. The approximation procedure starts by substituting into (11.31) a Taylor series for $H(x,\varepsilon)$, namely,

$$H(x,\varepsilon) = H_0(x) + \varepsilon H_1(x) + \varepsilon^2 H_2(x) + \cdots$$

and by calculating $H_0(x)$, $H_1(x)$, and so on, by equating terms of like powers of ε. This requires the functions f and g to be continuously differentiable in their arguments a sufficient number of times. It is clear that $H_0(x) = H(x,0) = h(x)$. The equation for $H_1(x)$ is

$$\frac{\partial g}{\partial z}(x, h(x)) H_1(x) = \frac{\partial h}{\partial x} f(x, h(x))$$

---

[7] We will not prove the existence of the invariant manifold here. A proof can be done by a variation of the proof of (the center manifold) Theorem 8.1, given in Appendix C.15. (See [34, Section 2.7].) A proof under the basic assumptions of Theorem 11.1 can be found in [102].

and has a unique solution if the Jacobian $[\partial g/\partial z]$ at $z = h(x)$ is nonsingular. The nonsingularity of the Jacobian is implied by the eigenvalue condition (11.16). Similar to H_1, the equations for higher order terms will be linear and solvable if the Jacobian $[\partial g/\partial z]$ is nonsingular.

To introduce the notion of a fast manifold, we examine (11.26)–(11.27) in the $\tau = t/\varepsilon$ time scale. At $\varepsilon = 0$, $x(\tau) \equiv x(0)$, while $z(\tau)$ evolves according to

$$\frac{dz}{d\tau} = g(x(0), z)$$

approaching the equilibrium point $z = h(x(0))$. This motion describes trajectories (x, z) in R^{n+m}, which, for every given $x(0)$, lie in a fast manifold F_x defined by $x = x(0) = \text{constant}$ and rapidly descend to the manifold $z = h(x)$. For ε larger than zero, but small, the fast manifolds are "foliations" of solutions rapidly approaching the slow manifold. Let us illustrate this picture by two second-order examples.

Example 11.10 Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon \dot{z} &= \tan^{-1}(1 - z - x) \end{aligned}$$

At $\varepsilon = 0$, the slow manifold is $z = h(x) = 1 - x$. The corresponding slow model

$$\dot{x} = -2x + 1$$

has an asymptotically stable equilibrium at $x = 0.5$. Therefore trajectories on the manifold $z = 1 - x$ will be heading toward the point $P = (0.5, 0.5)$, as indicated by the arrow heads in Figure 11.8. Notice that $(0.5, 0.5)$ is an equilibrium point of the full system. The fast manifolds at $\varepsilon = 0$ are parallel to the z-axis, with the trajectories heading toward the slow manifold $z = 1 - x$. With this information, we can construct an approximate phase portrait of the system. For example, a trajectory starting at point A will move down vertically until it hits the manifold $z = 1 - x$ at point B. From B, the trajectory moves along the manifold toward the equilibrium point P. Similarly, a trajectory starting at point C will move up vertically to point D and then along the manifold to the equilibrium point P. For $\varepsilon > 0$, but small, the phase portrait of the system will be close to the approximate picture we have drawn at $\varepsilon = 0$. Figure 11.9 shows the phase portrait for $\varepsilon = 0.1$. The proximity of the two portraits is noticeable. △

Example 11.11 Consider the Van der Pol equation

$$\frac{d^2 v}{ds^2} - \mu(1 - v^2)\frac{dv}{ds} + v = 0$$

when $\mu \gg 1$. With

$$x = -\frac{1}{\mu}\frac{dv}{ds} + v - \frac{1}{3}v^3; \quad z = v$$

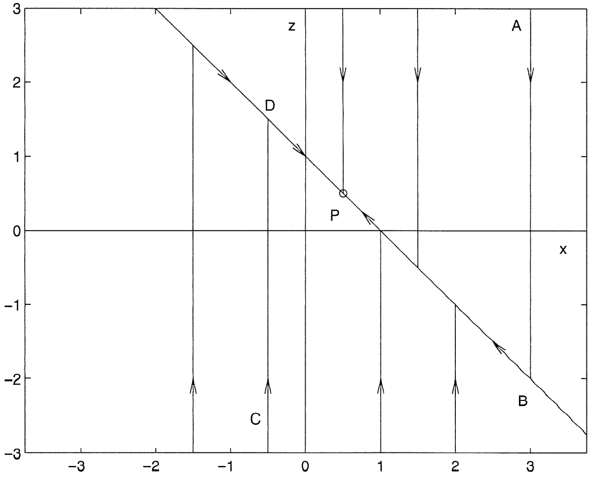

Figure 11.8: Approximate phase portrait of Example 11.10.

as state variables, $t = s/\mu$ as the time variable, and $\varepsilon = 1/\mu^2$, the system is represented by the standard singularly perturbed model

$$\begin{aligned} \dot{x} &= z \\ \varepsilon\dot{z} &= -x + z - \tfrac{1}{3}z^3 \end{aligned}$$

We already know by the Poincaré–Bendixson theorem (Example 2.9) that the Van der Pol equation has a stable limit cycle. What we would like to do here is to use singular perturbations to have a better estimate of the location of the limit cycle. At $\varepsilon = 0$, we need to solve for the roots $z = h(x)$ of

$$0 = -x + z - \tfrac{1}{3}z^3$$

The curve $-x+z-z^3/3 = 0$, the slow manifold at $\varepsilon = 0$, is sketched in Figure 11.10. For $x < -2/3$, there is only one root on the branch AB. For $-2/3 < x < 2/3$, there are three roots, one on each of the branches AB, BC, and CD. For $x > 2/3$, there is one root on the branch CD. For roots on the branch AB, the Jacobian

$$\frac{\partial g}{\partial z} = 1 - z^2 < 0, \quad \text{for } z^2 > 1$$

Thus, roots on the branch AB (excluding a neighborhood of point B) are exponentially stable. The same is true for roots on the branch CD (excluding a neighborhood of point C). On the other hand, roots on the branch BC are unstable because they lie in the region $z^2 < 1$. Let us construct an approximate phase portrait by using singular perturbations. We divide the state plane into three regions, depending

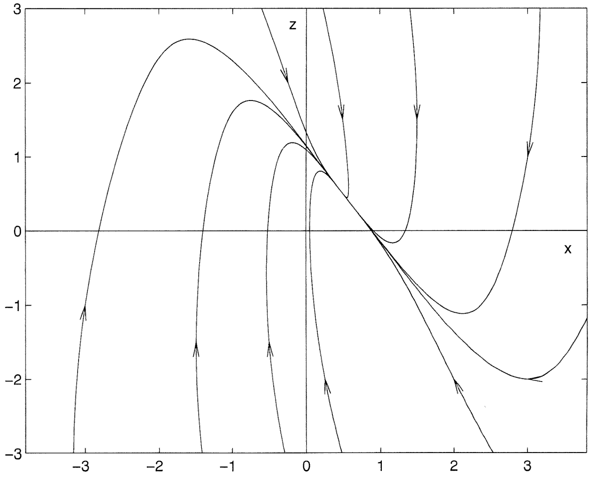

Figure 11.9: Phase portrait of Example 11.10 for $\varepsilon = 0.1$.

on the value of x. Trajectories starting in the region $x < -2/3$ will move parallel to the z-axis approaching the branch AB of the slow manifold. Trajectories starting in the region $-2/3 < x < 2/3$ will again be parallel to the z-axis, approaching either the branch AB or the branch CD, depending on the initial value of z. If the initial point is over the branch BC, the trajectory will approach AB; otherwise, it will approach CD. Finally, trajectories starting in the region $x > 2/3$ will approach the branch CD. For trajectories on the slow manifold itself, they will move along the manifold. The direction of motion can be determined by inspection of the vector field sign and is indicated in Figure 11.10. In particular, since $\dot{x} = z$, trajectories on the branch AB will be sliding down, while those on the branch CD will be climbing up. There is no point to talk about motion on the branch BC since there are no reduced models corresponding to the unstable roots on that branch. So far, we have formed an approximate phase portrait everywhere, except the branch BC and the neighborhoods of points B and C. We cannot use singular perturbation theory to predict the phase portrait in these regions. Let us investigate what happens in the neighborhood of B when ε is positive, but small. Trajectories sliding along the branch AB toward B are actually sliding along the exact slow manifold $z = H(x, \varepsilon)$. Since the trajectory is moving toward B, we must have $g < 0$. Consequently, the exact slow manifold must lie above the branch AB. Inspection of the vector field diagram in the neighborhood of B shows that the trajectory crosses the vertical line through B (that is, $x = 2/3$) at a point above B. Once the trajectory crosses this line, it belongs to the region of attraction of a stable root on the branch CD; therefore, the trajectory moves rapidly in a vertical line toward the branch CD. By a similar argument, it can be shown that a trajectory moving along the branch CD will cross the vertical line through C at a point below C and then will move

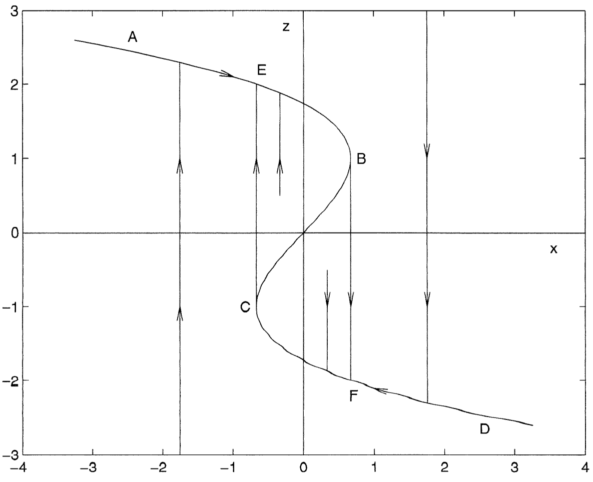

Figure 11.10: Approximate phase portrait of the Van der Pol oscillator.

vertically toward the branch AB. This completes the picture of the approximate portrait. Trajectories starting at any point are attracted to one of the two branches AB or CD, which they approach vertically. Once on the slow manifold, the trajectory will move toward the closed curve $E - B - F - C - E$, if not already on it, and will cycle through it. The exact limit cycle of the Van der Pol oscillator will lie within an $O(\varepsilon)$ neighborhood of this closed curve. The phase portrait for $\varepsilon = 0.1$, shown in Figure 11.11, confirms this prediction.

We can also estimate the period of oscillation of the periodic solution. The closed curve $E - B - F - C - E$ has two slow sides and two fast ones. Neglecting the time of the fast transients from B to F and from C to E, we estimate the period of oscillation by $t_{EB} + t_{FC}$. The time t_{EB} can be estimated from the reduced model

$$\begin{aligned} \dot{x} &= z \\ 0 &= -x + z - \tfrac{1}{3}z^3 \end{aligned}$$

Differentiating the second equation with respect to t and equating the expressions for $\dot{x}$ from the two equations, we obtain the equation

$$\dot{z} = \frac{z}{1 - z^2}$$

which, when integrated from E to B, yields $t_{EB} = (3/2) - \ln 2$. The time t_{FC} can be estimated similarly and, due to symmetry, $t_{EB} = t_{FC}$. Thus, the period of oscillation is approximated for small ε by $3 - 2\ln 2$. △

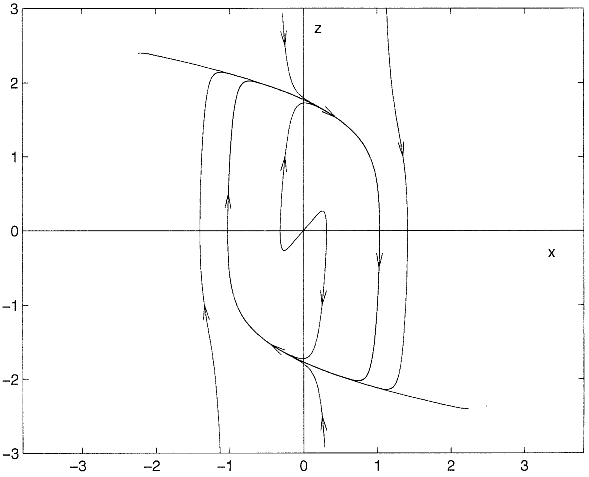

Figure 11.11: Phase portrait of the Van der Pol oscillator for $\varepsilon = 0.1$.

11.5 Stability Analysis

We consider the autonomous singularly perturbed system

$$\dot{x} = f(x,z) \tag{11.33}$$

$$\varepsilon\dot{z} = g(x,z) \tag{11.34}$$

and assume that the origin $(x = 0, z = 0)$ is an isolated equilibrium point and the functions f and g are locally Lipschitz in a domain that contains the origin. Consequently,

$$f(0,0) = 0, \quad g(0,0) = 0$$

We want to analyze the stability of the origin by examining the reduced and boundary-layer models. Let $z = h(x)$ be an isolated root of

$$0 = g(x,z)$$

defined for all $x \in D_x \subset R^n$, where D_x is a domain that contains $x = 0$. Suppose $h(0) = 0$. If $z = h(x)$ is the only root of $0 = g$, then it must vanish at the origin, since $g(0,0) = 0$. If there are two or more isolated roots, then one of them must vanish at $x = 0$, and that is the one we must work with. It is more convenient to work in the (x,y)-coordinates, where

$$y = z - h(x)$$

because this change of variables shifts the equilibrium of the boundary-layer model to the origin. In the new coordinates, the singularly perturbed system is

$$\dot{x} = f(x, y + h(x)) \tag{11.35}$$

$$\varepsilon\dot{y} \;=\; g(x, y + h(x)) - \varepsilon\,\frac{\partial h}{\partial x} f(x, y + h(x)) \tag{11.36}$$

Assuming that $\|h(x)\| \le \zeta(\|x\|)$ for all $x \in D_x$, where ζ is a class $\mathcal{K}$ function, the map $y = z - h(x)$ is stability preserving; that is, the origin of (11.33)–(11.34) is asymptotically stable if and only if the origin of (11.35)–(11.36) is asymptotically stable. The reduced system

$$\dot{x} = f(x, h(x)) \tag{11.37}$$

has equilibrium at $x = 0$ and the boundary-layer system

$$\frac{dy}{d\tau} = g(x, y + h(x)) \tag{11.38}$$

where $\tau = t/\varepsilon$ and x is treated as a fixed parameter, has equilibrium at $y = 0$. The main theme of our analysis is to assume that, for each of the two systems, the origin is asymptotically stable and that we have a Lyapunov function that satisfies the conditions of Lyapunov's theorem. In the case of the boundary-layer system, we require asymptotic stability of the origin to hold uniformly in the frozen parameter x. We have already defined what this means in the case of an exponentially stable origin (Definition 11.1). More generally, we say that the origin of (11.38) is asymptotically stable uniformly in x if the solutions of (11.38) satisfy

$$\|y(\tau)\| \le \beta(y(0), \tau), \;\; \forall\, \tau \ge 0, \; \forall\, x \in D_x$$

where β is a class $\mathcal{KL}$ function. This conditions will be implied by the conditions we will impose on the Lyapunov function for (11.38). Viewing the full singularly perturbed system (11.35)–(11.36) as an interconnection of the reduced and boundary-layer systems, we form a composite Lyapunov function candidate for the full system as a linear combination of the Lyapunov functions for the reduced and boundary-layer systems. We then proceed to calculate the derivative of the composite Lyapunov function along the trajectories of the full system and verify, under reasonable growth conditions on f and g, that the composite Lyapunov function will satisfy the conditions of Lyapunov's theorem for sufficiently small ε.

Let $V(x)$ be a Lyapunov function for the reduced system (11.37) such that

$$\frac{\partial V}{\partial x} f(x, h(x)) \le -\alpha_1 \psi_1^2(x) \tag{11.39}$$

for all $x \in D_x$, where $\psi_1 : R^n \to R$ is a positive definite function; that is, $\psi_1(0) = 0$ and $\psi_1(x) > 0$ for all $x \in D_x - \{0\}$. Let $W(x, y)$ be a Lyapunov function for the boundary-layer system (11.38) such that

$$\frac{\partial W}{\partial y} g(x, y + h(x)) \le -\alpha_2 \psi_2^2(y) \tag{11.40}$$

for all $(x, y) \in D_x \times D_y$, where $D_y \subset R^m$ is a domain that contains $y = 0$, and $\psi_2 : R^m \to R$ is a positive definite function; that is, $\psi_2(0) = 0$ and $\psi_2(y) > 0$ for all $y \in D_y - \{0\}$. We allow the Lyapunov function W to depend on x, since x is a parameter of the system and Lyapunov functions may, in general, depend on the system's parameters. Because x is not a true constant parameter, we have to keep track of the effect of the dependence of W on x. To ensure that the origin of (11.38) is asymptotically stable uniformly in x, we assume that $W(x, y)$ satisfies

$$W_1(y) \le W(x,y) \le W_2(y), \quad \forall\ (x,y) \in D_x \times D_y \tag{11.41}$$

for some positive definite continuous functions W_1 and W_2. Now consider the composite Lyapunov function candidate

$$\nu(x,y) = (1-d)V(x) + dW(x,y), \quad 0 < d < 1 \tag{11.42}$$

where the constant d is to be chosen. Calculating the derivative of ν along the trajectories of the full system (11.35)–(11.36), we obtain

$$\begin{aligned} \dot{\nu} &= (1-d)\frac{\partial V}{\partial x} f(x, y+h(x)) + \frac{d}{\varepsilon}\frac{\partial W}{\partial y} g(x, y+h(x)) \\ &\quad - d\frac{\partial W}{\partial y}\frac{\partial h}{\partial x} f(x, y+h(x)) + d\frac{\partial W}{\partial x} f(x, y+h(x)) \\ &= (1-d)\frac{\partial V}{\partial x} f(x, h(x)) + \frac{d}{\varepsilon}\frac{\partial W}{\partial y} g(x, y+h(x)) \\ &\quad + (1-d)\frac{\partial V}{\partial x}[f(x, y+h(x)) - f(x, h(x))] \\ &\quad + d\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y+h(x)) \end{aligned}$$

We have represented the derivative $\dot{\nu}$ as the sum of four terms. The first two terms are the derivatives of V and W along the trajectories of the reduced and boundary-layer systems. These two terms are negative definite in x and y, respectively, by inequalities (11.39) and (11.40). The other two terms represent the effect of the interconnection between the slow and fast dynamics, which is neglected at $\varepsilon = 0$. These terms are, in general, indefinite. The first of these two terms

$$\frac{\partial V}{\partial x}[f(x, y+h(x)) - f(x, h(x))]$$

represents the effect of the deviation of (11.35) from the reduced system (11.37). The other term

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y+h(x))$$

represents the deviation of (11.36) from the boundary-layer system (11.38), as well as the effect of freezing x during the boundary-layer analysis. Suppose that these perturbation terms satisfy

$$\frac{\partial V}{\partial x}[f(x, y+h(x)) - f(x, h(x))] \le \beta_1 \psi_1(x) \psi_2(y) \tag{11.43}$$

and

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y + h(x)) \le \beta_2 \psi_1(x)\psi_2(y) + \gamma\psi_2^2(y) \tag{11.44}$$

for some nonnegative constants β_1, β_2, and γ. Using inequalities (11.39), (11.40), (11.43), and (11.44), we obtain

$$\begin{aligned} \dot{\nu} \;&\le\; -(1-d)\alpha_1\psi_1^2(x) - \frac{d}{\varepsilon}\alpha_2\psi_2^2(y) + (1-d)\beta_1\psi_1(x)\psi_2(y) \\ &\quad\; + d\beta_2\psi_1(x)\psi_2(y) + d\gamma\psi_2^2(y) \\ &=\; -\psi^T(x,y)\Lambda\psi(x,y) \end{aligned}$$

where

$$\psi(x,y) = \begin{bmatrix} \psi_1(x) \\ \psi_2(y) \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} (1-d)\alpha_1 & -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 \\ -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 & d((\alpha_2/\varepsilon) - \gamma) \end{bmatrix}$$

The right-hand side of the last inequality is a quadratic form in ψ. The quadratic form is negative definite when

$$d(1-d)\alpha_1\left(\frac{\alpha_2}{\varepsilon} - \gamma\right) > \tfrac{1}{4}[(1-d)\beta_1 + d\beta_2]^2$$

which is equivalent to

$$\varepsilon < \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2]^2} \stackrel{\text{def}}{=} \varepsilon_d \tag{11.45}$$

The dependence of ε_d on d is sketched in Figure 11.12. It can be easily seen that the maximum value of ε_d occurs at $d^* = \beta_1/(\beta_1 + \beta_2)$ and is given by

$$\varepsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2} \tag{11.46}$$

It follows that the origin of (11.35)–(11.36) is asymptotically stable for all $\varepsilon < \varepsilon^*$. Theorem 11.3 summarizes our findings.

Theorem 11.3 *Consider the singularly perturbed system* (11.35) and (11.36). *Assume there are Lyapunov functions $V(x)$ and $W(x,y)$ that satisfy* (11.39) *through* (11.41), (11.43), *and* (11.44). *Let ε_d and ε^* be defined by* (11.45) *and* (11.46). *Then, the origin of* (11.35) and (11.36) *is asymptotically stable for all* $0 < \varepsilon < \varepsilon^*$. *Moreover, $\nu(x,y)$, defined by* (11.42), *is a Lyapunov function for* $\varepsilon \in (0, \varepsilon_d)$. ◇

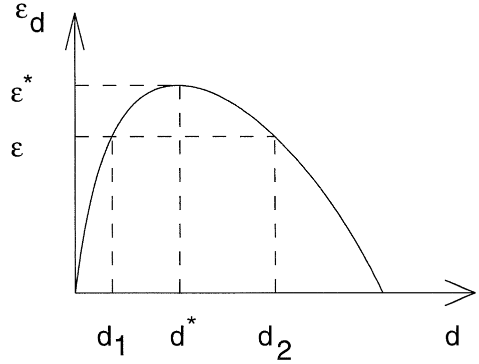

Figure 11.12: Upper bound on ε.

The stability analysis that led to Theorem 11.3 delineates a procedure for constructing Lyapunov functions for the singularly perturbed system (11.35)–(11.36). We start by studying the reduced and boundary-layer systems, searching for Lyapunov functions $V(x)$ and $W(x,y)$ that satisfy (11.39) through (11.41). Then inequalities (11.43) and (11.44), which we will refer to as the interconnection conditions, are checked. Several choices of V and W may be tried before one finds the desired Lyapunov functions. As a guideline in that search, notice that the interconnection conditions will be satisfied if

$$\left\|\frac{\partial V}{\partial x}\right\| \le k_1\psi_1(x); \quad \|f(x,h(x))\| \le k_2\psi_1(x)$$

$$\|f(x,y+h(x)) - f(x,h(x))\| \le k_3\psi_2(y)$$

$$\left\|\frac{\partial W}{\partial y}\right\| \le k_4\psi_2(y); \quad \left\|\frac{\partial W}{\partial x}\right\| \le k_5\psi_2(y)$$

A Lyapunov function $V(x)$ that satisfies (11.39) and $\|\partial V/\partial x\| \le k_1\psi_1(x)$ is known as a *quadratic-type* Lyapunov function, and ψ_1 is called a *comparison function.* Thus, the search would be successful if we could find quadratic-type Lyapunov functions V and W with comparison functions ψ_1 and ψ_2 such that $\|f(x,h(x))\|$ could be bounded by $\psi_1(x)$ and $\|f(x,y+h(x)) - f(x,h(x))\|$ could be bounded by $\psi_2(y)$. If we succeed in finding V and W, we can conclude that the origin is asymptotically stable for $\varepsilon < \varepsilon^*$. For a given $\varepsilon < \varepsilon^*$, there is a range (d_1, d_2), illustrated in Figure 11.12, such that for any $d \in (d_1, d_2)$, the function $\nu(x,y) = (1-d)V(x) + dW(x,y)$ is a valid Lyapunov function. The freedom in choosing d can be used to achieve other objectives, like improving estimates of the region of attraction.

Example 11.12 The second-order system

$$\begin{aligned}\dot{x} &= f(x,z) = x - x^3 + z\\ \varepsilon\dot{z} &= g(x,z) = -x - z\end{aligned}$$

has a unique equilibrium point at the origin. Let $y = z - h(x) = z + x$ and rewrite the system as

$$\begin{aligned}\dot{x} &= -x^3 + y\\ \varepsilon\dot{y} &= -y + \varepsilon(-x^3 + y)\end{aligned}$$

For the reduced system

$$\dot{x} = -x^3$$

we take $V(x) = (1/4)x^4$, which satisfies (11.39) with $\psi_1(x) = |x|^3$ and $\alpha_1 = 1$. For the boundary-layer system

$$\frac{dy}{d\tau} = -y$$

we take $W(y) = (1/2)y^2$, which satisfies (11.41) with $\psi_2(y) = |y|$ and $\alpha_2 = 1$. As for the interconnection conditions of (11.43) and (11.44), we have

$$\frac{\partial V}{\partial x}[f(x, y + h(x)) - f(x, h(x))] = x^3 y \le \psi_1\psi_2$$

and

$$\frac{\partial W}{\partial y} f(x, y + h(x)) = y(-x^3 + y) \le \psi_1\psi_2 + \psi_2^2$$

Note that $\partial W/\partial x = 0$. Hence, (11.43) and (11.44) are satisfied with $\beta_1 = \beta_2 = \gamma = 1$. Therefore, the origin is asymptotically stable for $\varepsilon < \varepsilon^* = 0.5$. In fact, since all the conditions are satisfied globally and $\nu(x,y) = (1-d)V(x) + dW(y)$ is radially unbounded, the origin is globally asymptotically stable for $\varepsilon < 0.5$. To see how conservative this bound is, let us note that the characteristic equation of the linearization at the origin is

$$\lambda^2 + \left(\frac{1}{\varepsilon} - 1\right)\lambda = 0$$

which shows that the origin is unstable for $\varepsilon > 1$. Since our example is a simple second-order system, we may calculate the derivative of the Lyapunov function

$$\nu(x,y) = \frac{1-d}{4}x^4 + \frac{d}{2}y^2$$

along the trajectories of the full singularly perturbed system and see if we can get a less conservative upper bound on ε compared with the one provided by Theorem 11.3:

$$\begin{aligned}\dot{\nu} &= (1-d)x^3(-x^3 + y) - \frac{d}{\varepsilon}y^2 + dy(-x^3 + y)\\ &= -(1-d)x^6 + (1-2d)x^3 y - d\left(\frac{1}{\varepsilon} - 1\right)y^2\end{aligned}$$

It is apparent that the choice $d = 1/2$ cancels the cross-product terms and yields

$$\dot{\nu} = -\frac{1}{2}x^6 - \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right)y^2$$

which is negative definite for all $\varepsilon < 1$. This estimate is indeed less conservative than that of Theorem 11.3. In fact, it is the actual range of ε for which the origin is asymptotically stable. △

Example 11.13 The system

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon\dot{z} &= \tan^{-1}(1 - x - z) \end{aligned}$$

has an equilibrium point at $(0.5, 0.5)$. The change of variables

$$\tilde{x} = x - 0.5; \quad \tilde{z} = z - 0.5$$

shifts the equilibrium point to the origin. To simplify the notation, let us drop the tilde and write the state equation as

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon\dot{z} &= -\tan^{-1}(x + z) \end{aligned}$$

The equation

$$0 = -\tan^{-1}(x + z)$$

has a unique root $z = h(x) = -x$. We apply the change of variables $y = z + x$ to obtain

$$\begin{aligned} \dot{x} &= -2x + y \\ \varepsilon\dot{y} &= -\tan^{-1} y + \varepsilon(-2x + y) \end{aligned}$$

For the reduced system, we take $V(x) = (1/2)x^2$, which satisfies (11.39) with $\alpha_1 = 2$ and $\psi_1(x) = |x|$. For the boundary-layer system, we take $W(y) = (1/2)y^2$ and (11.40) takes the form

$$\frac{dW}{dy}[-\tan^{-1} y] = -y\tan^{-1} y \le -\frac{\tan^{-1}\rho}{\rho}y^2$$

for all $y \in D_y = \{y \mid |y| < \rho\}$. Thus, (11.41) is satisfied with $\alpha_2 = (\tan^{-1}\rho)/\rho$ and $\psi_2(y) = |y|$. The interconnection conditions (11.43) and (11.44) are satisfied globally with $\beta_1 = 1$, $\beta_2 = 2$, and $\gamma = 1$. Hence, the origin is asymptotically stable for all $\varepsilon < \varepsilon^* = (\tan^{-1}\rho)/2\rho$. In fact, the origin is exponentially stable, since both ν and the negative definite upper bound on $\dot{\nu}$ are quadratic in (x, y). △

The Lyapunov analysis we have just presented can be extended to nonautonomous systems. We will not give the details here;[8] instead, we consider the case of exponential stability and use converse Lyapunov theorems to prove a result of conceptual importance.

Theorem 11.4 *Consider the singularly perturbed system*

$$\dot{x} = f(t, x, z, \varepsilon) \tag{11.47}$$

$$\varepsilon \dot{z} = g(t, x, z, \varepsilon) \tag{11.48}$$

Assume that the following assumptions are satisfied for all

$$(t, x, \varepsilon) \in [0, \infty) \times B_r \times [0, \varepsilon_0]$$

- $f(t, 0, 0, \varepsilon) = 0$ *and* $g(t, 0, 0, \varepsilon) = 0$.
- *The equation*

  $$0 = g(t, x, z, 0)$$

  has an isolated root $z = h(t, x)$ *such that* $h(t, 0) = 0$.
- *The functions* f, g, h, *and their partial derivatives up to the second order are bounded for* $z - h(t, x) \in B_\rho$.
- *The origin of the reduced system*

  $$\dot{x} = f(t, x, h(t, x), 0)$$

  is exponentially stable.
- *The origin of the boundary-layer system*

  $$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0)$$

  is exponentially stable, uniformly in (t, x).

Then, there exists $\varepsilon^* > 0$ *such that for all* $\varepsilon < \varepsilon^*$, *the origin of* (11.47)–(11.48) *is exponentially stable.* ◇

Proof: By Theorem 4.14, there is a Lyapunov function $V(t, x)$ for the reduced system that satisfies

$$c_1 \|x\|^2 \le V(t, x) \le c_2 \|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, h(t, x), 0) \le -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4 \|x\|$$

---

[8] A detailed treatment of the nonautonomous case can be found in [105, Section 7.5].

for some positive constants c_i, $i = 1, \ldots, 4$, and for $x \in B_{r_0}$, where $r_0 \le r$. By Lemma 9.8, there is a Lyapunov function $W(t, x, y)$ for the boundary-layer system that satisfies

$$b_1\|y\|^2 \le W(t,x,y) \le b_2\|y\|^2$$

$$\frac{\partial W}{\partial y} g(t,x,y+h(t,x),0) \le -b_3\|y\|^2$$

$$\left\|\frac{\partial W}{\partial y}\right\| \le b_4\|y\|$$

$$\left\|\frac{\partial W}{\partial t}\right\| \le b_5\|y\|^2; \qquad \left\|\frac{\partial W}{\partial x}\right\| \le b_6\|y\|^2$$

for some positive constants b_i, $i = 1, \ldots, 6$, and for $y \in B_{\rho_0}$, where $\rho_0 \le \rho$. Apply the change of variables

$$y = z - h(t,x)$$

to transform (11.47)–(11.48) into

$$\dot{x} = f(t,x,y+h(t,x),\varepsilon) \tag{11.49}$$

$$\begin{aligned} \varepsilon\dot{y} = {} & g(t,x,y+h(t,x),\varepsilon) - \varepsilon\frac{\partial h}{\partial t} \\ & - \varepsilon\frac{\partial h}{\partial x} f(t,x,y+h(t,x),\varepsilon) \end{aligned} \tag{11.50}$$

We are going to use

$$\nu(t,x,y) = V(t,x) + W(t,x,y)$$

as a Lyapunov function candidate for the system (11.49)–(11.50). In preparation for that, let us note the following estimates in the neighborhood of the origin: Since f and g vanish at the origin for all $\varepsilon \in [0, \varepsilon_0]$, they are Lipschitz in ε linearly in the state (x, y). In particular,

$$\|f(t,x,y+h(t,x),\varepsilon) - f(t,x,y+h(t,x),0)\| \le \varepsilon L_1(\|x\| + \|y\|)$$

$$\|g(t,x,y+h(t,x),\varepsilon) - g(t,x,y+h(t,x),0)\| \le \varepsilon L_2(\|x\| + \|y\|)$$

Also,

$$\|f(t,x,y+h(t,x),0) - f(t,x,h(t,x),0)\| \le L_3\|y\|$$

$$\|f(t,x,h(t,x),0)\| \le L_4\|x\|$$

$$\left\|\frac{\partial h}{\partial t}\right\| \le k_1\|x\|; \qquad \left\|\frac{\partial h}{\partial x}\right\| \le k_2$$

where we have used the fact that $f(t,x,h(t,x),0)$ and $h(t,x)$ vanish at $x = 0$ for all t. Using these estimates and the properties of the functions V and W, it can be verified that the derivative of ν along the trajectories of (11.49)–(11.50) satisfies the inequality

$$\begin{aligned}\dot{\nu} \;\le\; & -a_1\|x\|^2 + \varepsilon a_2\|x\|^2 - \frac{a_3}{\varepsilon}\|y\|^2 + a_4\|y\|^2 \\ & + a_5\|x\|\;\|y\| + a_6\|x\|\;\|y\|^2 + a_7\|y\|^3\end{aligned}$$

with positive a_1 and a_3 and nonnegative a_2 and a_4 to a_7. For all $\|y\| \le \rho_0$, this inequality simplifies to

$$\begin{aligned}\dot{\nu} \;\le\; & -a_1\|x\|^2 + \varepsilon a_2\|x\|^2 - \frac{a_3}{\varepsilon}\|y\|^2 + a_8\|y\|^2 + 2a_9\|x\|\;\|y\| \\ = \; & -\begin{bmatrix}\|x\| \\ \|y\|\end{bmatrix}^T \begin{bmatrix} a_1 - \varepsilon a_2 & -a_9 \\ -a_9 & (a_3/\varepsilon) - a_8\end{bmatrix}\begin{bmatrix}\|x\| \\ \|y\|\end{bmatrix}\end{aligned}$$

Thus, there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, we have

$$\dot{\nu} \le -2\gamma\nu$$

for some $\gamma > 0$. It follows that

$$\nu(t, x(t), y(t)) \le \exp[-2\gamma(t - t_0)]\nu(t_0, x(t_0), y(t_0))$$

and, from the properties of V and W,

$$\left\| \begin{matrix} x(t) \\ y(t) \end{matrix} \right\| \le K_1 \exp[-\gamma(t - t_0)] \left\| \begin{matrix} x(t_0) \\ y(t_0) \end{matrix} \right\|$$

Since $y = z - h(t, x)$ and $\|h(t, x)\| \le k_2\|x\|$, we obtain

$$\left\| \begin{matrix} x(t) \\ z(t) \end{matrix} \right\| \le K_2 \exp[-\gamma(t - t_0)] \left\| \begin{matrix} x(t_0) \\ z(t_0) \end{matrix} \right\|$$

which completes the proof of the theorem. □

Theorem 11.4 is conceptually important because it establishes robustness of exponential stability to unmodeled fast (high-frequency) dynamics. Quite often in the analysis of dynamical systems, we use reduced-order models obtained by neglecting small "parasitic" parameters. This reduction in the order of the model can be represented as a singular perturbation problem, where the full singularly perturbed model represents the actual system with the parasitic parameters and the reduced model is the simplified model used in the analysis. It is quite reasonable to assume that the boundary-layer model has an exponentially stable origin. In fact, if the dynamics associated with the parasitic elements were unstable, we should not have neglected them in the first place. The technicalities of assuming exponential stability instead of only asymptotic stability, or assuming that exponential stability holds uniformly, are quite reasonable in most applications. It is enough to mention that all these technicalities will automatically hold when the fast dynamics are linear. When the origin of the reduced model is exponentially stable, Theorem 11.4 assures us that the origin of the actual system will be exponentially stable, provided the neglected fast dynamics are sufficiently fast. The next example illustrates how this robustness property arises in control design.

Example 11.14 Consider the feedback stabilization of the system

$$\begin{aligned} \dot{x} &= f(t, x, v) \\ \varepsilon \dot{z} &= Az + Bu \\ v &= Cz \end{aligned}$$

where $f(t, 0, 0) = 0$ and A is a Hurwitz matrix. The system has an open-loop equilibrium point at the origin, and the control task is to design a state feedback control law to stabilize the origin. The linear part of this model represents actuator dynamics, which are, typically, much faster than the plant dynamics represented by the nonlinear equation $\dot{x} = f$. To simplify the design problem, we may neglect the actuator dynamics by setting $\varepsilon = 0$ and substituting $v = -CA^{-1}Bu$ into the plant equation. To simplify the notation, let us assume that $-CA^{-1}B = I$ and write the reduced model as

$$\dot{x} = f(t, x, u)$$

We use this model to design a state feedback control law $u = \gamma(t, x)$ such that the origin of the closed-loop model

$$\dot{x} = f(t, x, \gamma(t, x))$$

is exponentially stable. We will refer to this model as the nominal closed-loop system. Will the control law stabilize the actual system with the actuator dynamics included? When the control is applied to the actual system, the closed-loop equation is

$$\begin{aligned} \dot{x} &= f(t, x, Cz) \\ \varepsilon \dot{z} &= Az + B\gamma(t, x) \end{aligned}$$

We have a singular perturbation problem, where the full singularly perturbed model is the actual closed-loop system and the reduced model is the nominal closed-loop system. By design, the origin of the reduced model is exponentially stable. The boundary-layer model

$$\frac{dy}{d\tau} = Ay$$

is independent of (t, x) and its origin is exponentially stable since A is a Hurwitz matrix. Assuming that f and γ are smooth enough to satisfy the conditions of Theorem 11.4, we conclude that the origin of the actual closed-loop system is exponentially stable for sufficiently small ε. This result legitimizes the ad hoc model simplification process of neglecting the actuator dynamics. △

11.6 Exercises

11.1 Consider the *RC* circuit of Figure 11.13 and suppose the capacitor C_2 is small relative to C_1, while $R_1 = R_2 = R$. Represent the system in the standard singularly perturbed form.

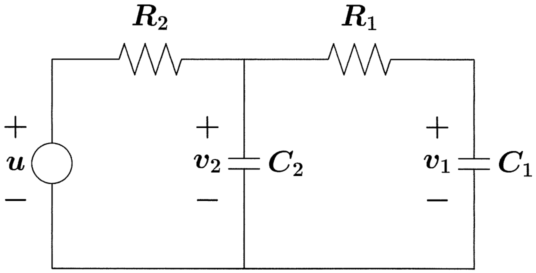

Figure 11.13: Exercises 11.1 and 11.2.

11.2 Consider the *RC* circuit of Figure 11.13 and suppose the resistor R_1 is small relative to R_2, while $C_1 = C_2 = C$. Represent the system in the standard singularly perturbed form.

11.3 Consider the tunnel diode circuit of Section 1.2.2 and suppose the inductance L is relatively small so that the time constant L/R is much smaller than the time constant CR. Represent the system as a standard singularly perturbed model with $\varepsilon = L/CR^2$.

11.4 (**[105]**) The feedback system of Figure 11.14 has a high-gain amplifier with gain k and a nonlinear element ψ. Represent the system as a standard singularly perturbed model with $\varepsilon = 1/k$.

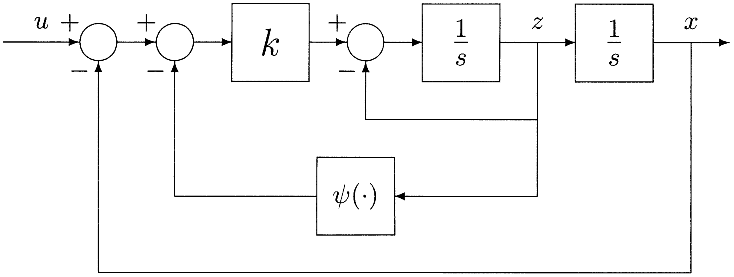

Figure 11.14: Exercise 11.4.

11.5 Show that if the Jacobian $[\partial g/\partial y]$ satisfies the eigenvalue condition (11.16), then there exist constants k, γ, and ρ_0 for which inequality (11.15) is satisfied.

11.6 Show that if there is a Lyapunov function satisfying (11.17) and (11.18), then inequality (11.15) is satisfied with the estimates of (11.19).

11.7 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x^2 + z, & x(0) &= \xi \\ \varepsilon \dot{z} &= x^2 - z + 1, & z(0) &= \eta \end{aligned}$$

(a) Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 0$. Simulate x and z for

$$(\mathbf{1})\ \varepsilon = 0.1 \quad \text{and} \quad (\mathbf{2})\ \varepsilon = 0.05$$

and compare with the approximation derived in part (a). In carrying out the computer simulation, note that the system has a finite escape time shortly after $t = 1$.

11.8 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x + z, \quad x(0) = \xi \\ \varepsilon \dot{z} &= -\frac{2}{\pi} \tan^{-1}\left(\frac{\pi}{2}(2x + z)\right), \quad z(0) = \eta \end{aligned}$$

(a) Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 1$. Simulate x and z for

$$(\mathbf{1})\ \varepsilon = 0.2 \quad \text{and} \quad (\mathbf{2})\ \varepsilon = 0.1$$

and compare with the approximation derived in part (a).

11.9 Consider the singularly perturbed system

$$\dot{x} = z, \qquad \varepsilon \dot{z} = -x - \varepsilon z - \exp(z) + 1 + u(t)$$

Find the reduced and boundary-layer models and analyze the stability properties of the boundary-layer model.

11.10 ([105]) Consider the singularly perturbed system

$$\dot{x} = \frac{x^2 t}{z}, \qquad \varepsilon \dot{z} = -(z + xt)(z - 2)(z - 4)$$

(a) How many reduced models can this system have?

(b) Investigate boundary-layer stability for each reduced model.

(c) Let $x(0) = 1$ and $z(0) = a$. Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$ for all values of a in the interval $[-2, 6]$.

11.11 Apply Theorem 11.2 to study the asymptotic behavior of the system

$$\dot{x} = -x + z - \sin t, \qquad \varepsilon\dot{z} = -z + \sin t$$

as $t \to \infty$.

11.12 (**[105]**) Find the exact slow manifold of the system

$$\dot{x} = xz^3, \qquad \varepsilon\dot{z} = -z - x^{4/3} + \tfrac{4}{3}\varepsilon x^{16/3}$$

11.13 (**[105]**) How many slow manifolds does the following system have? Which of these manifolds will attract trajectories of the system?

$$\dot{x} = -xz, \qquad \varepsilon\dot{z} = -(z - \sin^2 x)(z - e^{ax})(z - 2e^{2ax}), \quad a > 0$$

11.14 (**[105]**) Consider the linear autonomous singularly perturbed system

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z \end{aligned}$$

where $x \in R^n$, $z \in R^m$, and A_{22} is a Hurwitz matrix.

(a) Show that for sufficiently small ε, the system has an exact slow manifold $z = -L(\varepsilon)x$, where L satisfies the algebraic equation

$$-\varepsilon L(A_{11} - A_{12}L) = A_{21} - A_{22}L$$

(b) Show that the change of variables $\eta = z + L(\varepsilon)x$ transforms the system into a block triangular form.

(c) Show that the eigenvalues of the system cluster into a group of n slow eigenvalues of order $O(1)$ and m fast eigenvalues of order $O(1/\varepsilon)$.

(d) Let $H(\varepsilon)$ be the solution of the linear equation

$$\varepsilon(A_{11} - A_{12}L)H - H(A_{22} + \varepsilon LA_{12}) + A_{12} = 0$$

Show that the similarity transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I - \varepsilon HL & -\varepsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

transforms the system into the block modal form

$$\dot{\xi} = A_s(\varepsilon)\xi, \qquad \varepsilon\dot{\eta} = A_f(\varepsilon)\eta$$

where the eigenvalues of A_s and A_f/ε are, respectively, the slow and fast eigenvalues of the full singularly perturbed system.

(e) Show that the component of the fast mode in x is $O(\varepsilon)$.

(f) Give an independent proof of Tikhonov's theorem in the current case.

11.15 Consider the linear singularly perturbed system

$$\begin{aligned}\dot{x} &= A_{11}x + A_{12}z + B_1u(t), \qquad x(0) = \xi \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z + B_2u(t), \qquad z(0) = \eta\end{aligned}$$

where $x \in R^n$, $z \in R^m$, $u \in R^p$, A_{22} is Hurwitz, and $u(t)$ is uniformly bounded for all $t \geq 0$. Let $\bar{x}(t)$ be the solution of the reduced system

$$\dot{x} = A_0x + B_0u(t), \qquad x(0) = \xi$$

where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, and $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$.

(a) Show that $x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$ on any compact interval $[0, t_1]$.

(b) Show that if A_0 is Hurwitz, then $x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$ for all $t \geq 0$.

Hint: Use the transformation of the previous Exercise.

11.16 Consider the singularly perturbed system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 + z, \qquad \varepsilon\dot{z} = \tan^{-1}(1 - x_1 - z)$$

(a) Find the reduced and boundary-layer models.

(b) Analyze the stability properties of the boundary-layer model.

(c) Let $x_1(0) = x_2(0) = z(0) = 0$. Find an $O(\varepsilon)$ approximation of the solution. Using a numerical algorithm, calculate the exact and approximate solutions over the time interval $[0, 10]$ for $\varepsilon = 0.1$.

(d) Investigate the validity of the approximation on the infinite time interval.

(e) Show that the system has a unique equilibrium point and analyze its stability by using the singular perturbation approach. Is the equilibrium point asymptotically stable? Is it globally asymptotically stable? Is it exponentially stable? Calculate an upper bound ε^* on ε for which your stability analysis is valid.

11.17 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -2x + x^2 + z, \qquad \varepsilon\dot{z} = x - x^2 - z$$

In part (c), let $x(0) = z(0) = 1$ and the time interval be $[0, 5]$.

11.18 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = xz^3 \qquad \varepsilon\dot{z} = -2x^{4/3} - 2z$$

In part (c), let $x(0) = z(0) = 1$ and the time interval be $[0, 1]$.

11.19 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -x^3 + \tan^{-1}(z), \qquad \varepsilon\dot{z} = -x - z$$

In part (c), let $x(0) = -1$, $z(0) = 2$ and the time interval be $[0, 2]$.

11.20 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -x + z_1 + z_2 + z_1 z_2, \qquad \varepsilon\dot{z}_1 = -z_1, \qquad \varepsilon\dot{z}_2 = -z_2 - (x + z_1 + xz_1)$$

In part (c), let $x(0) = z_1(0) = z_2(0) = 1$ and the time interval be $[0, 2]$.

11.21 Consider the field-controlled DC motor of Exercise 1.17. Let $v_a = V_a =$ constant, and $v_f = U =$ constant.

(a) Show that the system has a unique equilibrium point at

$$I_f = \frac{U}{R_f}, \quad I_a = \frac{c_3 V_a}{c_3 R_a + c_1 c_2 U^2/R_f^2}, \quad \Omega = \frac{c_2 V_a U/R_f}{c_3 R_a + c_1 c_2 U^2/R_f^2}$$

We will use (I_f, I_a, Ω) as a nominal operating point.

(b) It is typical that the armature circuit time constant $T_a = L_a/R_a$ is much smaller than the field circuit time constant $T_f = L_f/R_f$ and the mechanical time constant. Therefore, the system can be modeled as a singularly perturbed system with i_f and ω as the slow variables and i_a as the fast variable. Taking $x_1 = i_f/I_f$, $x_2 = \omega/\Omega$, $z = i_a/I_a$, $u = v_f/U$, and $\varepsilon = T_a/T_f$, and using $t' = t/T_f$ as the time variable, show that the singularly perturbed model is given by

$$\dot{x}_1 = -x_1 + u, \qquad \dot{x}_2 = a(x_1 z - x_2), \qquad \varepsilon\dot{z} = -z - bx_1x_2 + c$$

where $a = L_f c_3/R_f J$, $b = c_1 c_2 U^2/c_3 R_a R_f^2$, $c = V_a/I_a R_a$, and $\dot{(\cdot)}$ denotes the derivative with respect to t'.

(c) Find the reduced and boundary-layer models.

(d) Analyze the stability properties of the boundary-layer model.

(e) Find an $O(\varepsilon)$ approximation of x and z.

(f) Investigate the validity of the approximation on the infinite time interval.

(g) Using a numerical algorithm, calculate the exact and approximate solutions for a unit step input at u and zero initial states over the time interval $[0, 10]$ for $\varepsilon = 0.2$ and $\varepsilon = 0.1$. Use the numerical data $c_1 = c_2 = \sqrt{2} \times 10^{-2}$ N-m/A, $c_3 = 6 \times 10^{-6}$ N-m-s/rad, $J = 10^{-6}$ N-m-s^2/rad, $R_a = R_f = 1\ \Omega$, $L_f = 0.2$ H, $V_a = 1$ V, and $U = 0.2$ V.

11.22 ([105]) Consider the singularly perturbed system

$$\dot{x} = -\eta(x) + az, \qquad \varepsilon\dot{z} = -\frac{x}{a} - z$$

where a is a positive constant and η is a smooth nonlinear function that satisfies

$$\eta(0) = 0 \ \text{ and } \ x\eta(x) > 0, \ \text{ for } \ x \in (-\infty, b) - \{0\}$$

for some $b > 0$. Investigate the stability of the origin for small ε by using the singular perturbation approach.

11.23 ([105]) The singularly perturbed system

$$\dot{x} = -2x^3 + z^2, \qquad \varepsilon\dot{z} = x^3 - \tan z$$

has an isolated equilibrium point at the origin.

(a) Show that asymptotic stability of the origin cannot be shown by linearization.

(b) Using the singular perturbation approach, show that the origin is asymptotically stable for $\varepsilon \in (0, \varepsilon^*)$. Estimate ε^* and the region of attraction.

11.24 ([105]) Let the assumptions of Theorem 11.3 hold with $\psi_1(x) = \|x\|$ and $\psi_2(y) = \|y\|$ and suppose, in addition, that $V(x)$ and $W(x, y)$ satisfy

$$k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2$$

$$k_3\|y\|^2 \leq W(x, y) \leq k_4\|y\|^2$$

$\forall\ (x, y) \in D_x \times D_y$, where k_1 to k_4 are positive constants. Show that the conclusions of Theorem 11.3 hold with exponential stability replacing asymptotic stability.

11.25 ([191]) Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \varepsilon\dot{y} &= Ay + \varepsilon g_1(x, y) \end{aligned}$$

where A is Hurwitz and f and g_1 are sufficiently smooth functions that vanish at the origin. Suppose there is a Lyapunov function $V(x)$ such that $[\partial V/\partial x]f(x, 0) \leq -\alpha_1\phi(x)$ in the domain of interest, where $\alpha_1 > 0$ and $\phi(x)$ is positive definite. Let P be the solution of the Lyapunov equation $PA + A^TP = -I$ and take $W(y) = y^TPy$.

(a) Suppose f and g_1 satisfy the inequalities

$$\|g_1(x,0)\|_2 \le k_1\phi^{1/2}(x), \quad k_1 \ge 0$$

$$\frac{\partial V}{\partial x}[f(x,y) - f(x,0)] \le k_2\phi^{1/2}(x)\|y\|_2, \quad k_2 \ge 0$$

in the domain of interest. Using the Lyapunov function candidate $\nu(x,y) = (1-d)V(x) + dW(y)$, $0 < d < 1$ and the analysis preceding Theorem 11.3, show that the origin is asymptotically stable for sufficiently small ε.

(b) As an alternative to Theorem 11.3, suppose f and g_1 satisfy the inequalities

$$\|g_1(x,0)\|_2 \le k_3\phi^{a}(x), \quad k_3 \ge 0, \quad 0 < a \le \tfrac{1}{2}$$

$$\frac{\partial V}{\partial x}[f(x,y) - f(x,0)] \le k_4\phi^{b}(x)\|y\|_2^c, \quad k_4 \ge 0, \quad 0 < b < 1, \quad c = \frac{1-b}{a}$$

in the domain of interest. Using the Lyapunov function candidate $\nu(x,y) = V(x) + (y^TPy)^\gamma$, where $\gamma = 1/2a$, show that the origin is asymptotically stable for sufficiently small ε.
Hint: Use Young's inequality

$$uw \le \frac{1}{\mu}u^p + \mu^{\frac{1}{p-1}}w^{\frac{p}{p-1}}, \forall\ u \ge 0, \ w \ge 0, \ \mu > 0, \ p > 1$$

to show that $\dot{\nu} \le -c_1\phi - c_2\|y\|_2^{2\gamma}$. Then show that the coefficients c_1 and c_2 can be made positive for sufficiently small ε.

(c) Give an example where the interconnection conditions of part (b) are satisfied, but not those of part (a).

11.26 ([99]) Consider the multiparameter singularly perturbed system

$$\begin{aligned}\dot{x} &= f(x, z_1, \ldots, z_m)\\ \varepsilon_i\dot{z}_i &= \eta_i(x) + \sum_{j=1}^{m} a_{ij}z_j, \qquad i = 1, \ldots, m\end{aligned}$$

where x is an n-dimensional vector, z_i's are scalar variables, and ε_i's are small positive parameters. Let $\varepsilon = \max_i \varepsilon_i$. This equation can be rewritten as

$$\begin{aligned}\dot{x} &= f(x,z)\\ \varepsilon D\dot{z} &= \eta(x) + Az\end{aligned}$$

where z and η are m-dimensional vectors whose components are z_i and η_i, respectively, A is an $m \times m$ matrix whose elements are a_{ij}, and D is an $m \times m$ diagonal matrix whose ith diagonal element is $\varepsilon_i/\varepsilon$. The diagonal elements of D are positive and bounded by one. Suppose the origin of the reduced system $\dot{x} = f(x, -A^{-1}\eta(x))$ is asymptotically stable and there is a Lyapunov function $V(x)$ that satisfies the conditions of Theorem 11.3. Suppose further that there is a diagonal matrix P with positive elements such that

$$PA + A^T P = -Q, \qquad Q > 0$$

Using

$$\nu(x, z) = (1-d)V(x) + d(z + A^{-1}\eta(x))^T PD(z + A^{-1}\eta(x)), \quad 0 < d < 1$$

as a Lyapunov function candidate, analyze the stability of the origin. State and prove a theorem similar to Theorem 11.3 for the multiparameter case. Your conclusion should allow the parameters ε_i's to be arbitrary, subject only to a requirement that they be sufficiently small.

11.27 ([105]) The singularly perturbed system

$$\dot{x}_1 = (a + x_2)x_1 + 2z, \qquad \dot{x}_2 = bx_1^2, \qquad \varepsilon\dot{z} = -x_1x_2 - z$$

where $a > 0$ and $b > 0$, has an equilibrium set $\{x_1 = 0,\ z = 0\}$. Study the asymptotic behavior of the solution, for small ε, using LaSalle's invariance principle. Hint: The asymptotic behavior of the reduced model has been studied in Example 4.10. Use a composite Lyapunov function and proceed as in Section 11.5. Notice, however, that Theorem 11.3 does not apply to the current problem.

11.28 Show that the origin of the system

$$\dot{x}_1 = x_2 + e^{-t}z, \qquad \dot{x}_2 = -x_2 + z, \qquad \varepsilon\dot{z} = -(x_1 + z) - (x_1 + z)^3$$

is globally exponentially stable for sufficiently small ε.

11.29 Consider the singularly perturbed system

$$\dot{x} = -x + \tan^{-1} z, \qquad \varepsilon\dot{z} = -x - z + u$$

(a) Find ε^* such that $\forall\ \varepsilon < \varepsilon^*$, the origin of the unforced system is globally asymptotically stable.

(b) Show that for each $\varepsilon < \varepsilon^*$, the system is input-to-state stable.

11.30 Consider the feedback connection of Figure 7.1, where the linear component is a singularly perturbed system represented by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2 + z \\ \varepsilon\dot{z} &= -z + u \\ y &= 2x_1 + x_2 \end{aligned}$$

and ψ is a smooth, memoryless, time-invariant nonlinearity that belongs to a sector $[0, k]$ for some $k > 0$.

(a) Represent the closed-loop system as a singularly perturbed system and find its reduced and boundary-layer models.

(b) Show that for every $k > 0$, there is $\varepsilon^* > 0$ such that the system is absolutely stable for all $0 < \varepsilon < \varepsilon^*$.

Chapter 12

Feedback Control

The last three chapters of the book deal with the design of feedback control. Various tools of nonlinear control design are introduced, including linearization, integral control, gain scheduling, feedback linearization, sliding mode control, Lyapunov redesign, backstepping, passivity-based control, and high-gain observers. Most of the nonlinear analysis tools we have learned so far come into play in these three chapters, solidifying our understanding of these tools. This chapter starts with a section on control problems that serves as an introduction to all three chapters. This is followed by four sections on classical tools, which proved to be useful in applications, namely, linearization, integral control, and gain scheduling. Feedback linearization is presented in Chapter 13 and various nonlinear design tools are presented in Chapter 14.

12.1 Control Problems

There are many control tasks that require the use of feedback. Depending on the design goals, there are several formulations of the control problem. The tasks of stabilization, tracking, and disturbance rejection or attenuation (and various combinations of them) lead to a number of control problems. In each problem, we may have a state feedback version where all state variables can be measured or an output feedback version where only an output vector, whose dimension is typically less than the dimension of the state, can be measured. In a typical control problem, there are additional goals for the design, like meeting certain requirements on the transient response or certain constraints on the control input. These requirements could be conflicting and the designer has to trade off various conflicting requirements. The desire to optimize this design tradeoff leads to various optimal control problems. When model uncertainty is taken into consideration, issues of sensitivity and robustness come into play. The attempt to design feedback control to cope with a wide range of model uncertainty leads to either robust or adaptive control problems. In robust control, the model uncertainty is characterized as perturbations of a nominal model. You may think of the nominal model as a point in a space and the perturbed models as points in a ball that contains the nominal model. A robust control design tries to meet the control objective for any model in the "ball of uncertainty." Adaptive control, on the other hand, parameterizes the uncertainty in terms of certain unknown parameters and tries to use feedback to learn these parameters on-line, that is, during the operation of the system. In a more elaborate adaptive scheme, the controller might be learning certain unknown nonlinear functions, rather than just learning some unknown parameters. There are also problem formulations that mix robust and adaptive control. In the current section, we describe the control problems we shall encounter in this chapter and the next two. We will limit our discussions to the basic tasks of stabilization, tracking, and disturbance rejection. We start with the stabilization problem, both state feedback and output feedback versions. Then we describe tracking and disturbance rejection problems. Some robust control problems will be described in Chapter 14, as needed.

The state feedback stabilization problem for the system

$$\dot{x} = f(t, x, u)$$

is the problem of designing a feedback control law

$$u = \gamma(t, x)$$

such that the origin $x = 0$ is a uniformly asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(t, x, \gamma(t, x))$$

The feedback control law $u = \gamma(t, x)$ is usually called "static feedback," because it is a memoryless function of x. Sometimes, we use a dynamic state feedback control

$$u = \gamma(t, x, z)$$

where z is the solution of a dynamical system driven by x; that is,

$$\dot{z} = g(t, x, z)$$

Common examples of dynamic state feedback control arise when we use integral control (Section 12.3) or adaptive control (Section 1.2.6).

The output feedback stabilization problem for the system

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned}$$

is the problem of designing a static output feedback control law

$$u = \gamma(t, y)$$

or a dynamic output feedback control law

$$\begin{aligned} u &= \gamma(t, y, z) \\ \dot{z} &= g(t, y, z) \end{aligned}$$

such that the origin is a uniformly asymptotically stable equilibrium point of the closed-loop system. In the case of dynamic feedback control, the origin to be stabilized is $(x = 0,\ z = 0)$. Dynamic feedback control is more common in output feedback schemes, since the lack of measurement of some of the state variables is usually compensated for by including "observers" or "observer-like" components in the feedback controller.

While the standard stabilization problem is defined as stabilization of an equilibrium point at the origin, we can use the same formulation to stabilize the system with respect to an arbitrary point $x_{\rm ss}$. For that we need the existence of a steady-state value of the input $u_{\rm ss}$ which can maintain equilibrium at $x_{\rm ss}$; namely,

$$0 = f(t, x_{\rm ss}, u_{\rm ss}), \qquad \forall\ t \geq 0$$

The change of variables

$$x_\delta = x - x_{\rm ss}, \qquad u_\delta = u - u_{\rm ss}$$

results in

$$\dot{x}_\delta = f(t, x_{\rm ss} + x_\delta, u_{\rm ss} + u_\delta) \stackrel{\rm def}{=} f_\delta(t, x_\delta, u_\delta)$$

where $f_\delta(t, 0, 0) \equiv 0$ for all $t \geq 0$. For output feedback problems, the output is redefined as

$$y_\delta = y - h(t, x_{\rm ss}, u_{\rm ss}) = h(t, x_{\rm ss} + x_\delta, u_{\rm ss} + u_\delta) - h(t, x_{\rm ss}, u_{\rm ss}) \stackrel{\rm def}{=} h_\delta(t, x_\delta, u_\delta)$$

in which $h_\delta(t, 0, 0) \equiv 0$ for all $t \geq 0$. We can now proceed to the solve the standard stabilization problem for the system

$$\begin{aligned} \dot{x}_\delta &= f_\delta(t, x_\delta, u_\delta) \\ y_\delta &= h_\delta(t, x_\delta, u_\delta) \end{aligned}$$

where u_δ is designed as feedback control of x_δ or y_δ. The overall control $u = u_\delta + u_{\rm ss}$ has a feedback component u_δ and a feedforward component $u_{\rm ss}$.

Naturally, the feedback stabilization problem is much simpler when the system is linear and time invariant:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

In this case, the state feedback control $u = -Kx$ preserves linearity of the open-loop system, and the origin of the closed-loop system

$$\dot{x} = (A - BK)x$$

is asymptotically stable if and only if the matrix $A - BK$ is Hurwitz. Thus, the state feedback stabilization problem reduces to a problem of designing a matrix K to assign the eigenvalues of $A - BK$ in the open left-half complex plane. Linear control theory[1] confirms that the eigenvalues of $A - BK$ can be arbitrarily assigned (subject only to the constraint that complex eigenvalues are in conjugate pairs) provided the pair (A, B) is controllable. Even if some eigenvalues of A are not controllable, stabilization is still possible, provided the uncontrollable eigenvalues have negative real parts. In this case, the pair (A, B) is called stabilizable, and the uncontrollable (open-loop) eigenvalues of A will be (closed-loop) eigenvalues of $A - BK$. If we can only measure the output y, we can use dynamic compensation, like the observer-based controller

$$\begin{aligned} u &= -K\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + H(y - C\hat{x} - Du) \end{aligned}$$

to stabilize the system. Here, the feedback gain K is designed as in state feedback, such that $A - BK$ is Hurwitz, while the observer gain H is designed such that $A - HC$ is Hurwitz. The closed-loop eigenvalues will consist of the eigenvalues of $A - BK$ and the eigenvalues of $A - HC$.[2] The stabilization of $A - HC$ is dual to the stabilization of $A - BK$ and requires observability (or at least detectability) of the pair (A, C).

For a general nonlinear system, the problem is more difficult and less understood. The most practical way to approach the stabilization problem for nonlinear systems is to appeal to the neat results available in the linear case, that is, via linearization. In Section 12.2, a feedback control law is designed by linearizing the system about the desired equilibrium point and designing a stabilizing linear feedback control for the linearization. The validity of this idea comes from Lyapunov's indirect method stated in Theorems 4.7 and 4.13. Clearly, this approach is local; that is, it can only guarantee asymptotic stability, but cannot, in general, prescribe a region of attraction nor achieve global asymptotic stability. In Section 12.5, we describe *gain scheduling*, a technique that aims at extending the region of validity of linearization by solving the stabilization problem at different operating points and allowing the controller to move from one design to another in a smooth or abrupt way. In Chapter 13, another linearization idea is presented. There, we deal with a special class of nonlinear systems that can be transformed into linear systems via feedback and (possibly) a change of variables. After this transformation, a stabilizing linear state feedback control is designed for the linear system. This linearization approach is different from the first one in that no approximation is used; it is exact. Exactness, however, assumes perfect knowledge of the state equation and uses that knowledge to cancel the nonlinearities of the system. Since perfect knowledge of the state equation and exact mathematical cancellation of terms are almost impossible, the implementation of this approach will almost always result in a closed-loop system, which is a perturbation of a nominal system whose origin is exponentially stable. The validity of the method draws upon Lyapunov theory for perturbed systems (Chapter 9), specifically regarding robustness of exponential stability.

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[1] See, for example, [9], [35], [110], or [158].

[2] This fact is usually referred to as the "separation principle," since the assignment of the closed-loop eigenvalues can be carried out in separate tasks for the state feedback and observer problems.

When a linear system is stabilized by feedback, the origin of the closed-loop system is globally asymptotically stable. This is not the case for nonlinear systems where different stabilization notions can be introduced. If the nonlinear system is stabilized via linearization, then the origin of the closed-loop system will be asymptotically stable. Without further analysis of the system, the region of attraction of the origin will be unknown. In this case, we say the feedback control achieves *local stabilization.* If the feedback control guarantees that a certain set is included in the region of attraction or if an estimate of the region of attraction is given, we say that the feedback control achieves *regional stabilization.* If the origin of the closed-loop system is globally asymptotically stable, we say that the control achieves *global stabilization.* If feedback control does not achieve global stabilization, but can be designed such that any given compact set (no matter how large) can be included in the region of attraction, we say that the feedback control achieves *semiglobal stabilization.* These four stabilization notions are illustrated by the next example.

Example 12.1 Suppose we want to stabilize the scalar system

$$\dot{x} = x^2 + u$$

by using state feedback. Linearization at the origin results in the linear system $\dot{x} = u$, which can be stabilized by $u = -kx$ with $k > 0$. When this control is applied to the nonlinear system, it results in

$$\dot{x} = -kx + x^2$$

whose linearization at the origin is $\dot{x} = -kx$. Thus, by Theorem 4.7, the origin is asymptotically stable, and we say that $u = -kx$ achieves local stabilization. In this example, it is not hard to see that the region of attraction is the set $\{x < k\}$. With this information, we say that $u = -kx$ achieves regional stabilization. By increasing k, we can expand the region of attraction. In fact, given any compact set $B_r = \{|x| \leq r\}$, we can include it in the region of attraction by choosing $k > r$. Hence, $u = -kx$ achieves semiglobal stabilization. It is important to notice that $u = -kx$ does not achieve global stabilization. In fact, for any finite k, there is a part of the state space (that is, $x \geq k$), which is not in the region of attraction. While semiglobal stabilization can include any compact set in the region of attraction, the control law is dependent on the given set and will not necessarily work with a bigger set. For a given r, we can choose $k > r$. Once k is fixed and the controller is implemented, if the initial state happens to be in the region $\{x > k\}$, the solution $x(t)$ will diverge to infinity. Global stabilization can be achieved by the nonlinear control law

$$u = -x^2 - kx$$

which cancels the open-loop nonlinearity and yields the linear closed-loop system $\dot{x} = -kx$. △

We turn now to the description of a more general control problem; namely, the tracking problem in the presence of disturbance. Here, we have a system modeled by

$$\begin{aligned} \dot{x} &= f(t, x, u, w) \\ y &= h(t, x, u, w) \\ y_m &= h_m(t, x, u, w) \end{aligned}$$

where x is the state, u is the control input, w is a disturbance input, y is the controlled output, and y_m is the measured output. The basic goal of the control problem is to design the control input so that the controlled output y tracks a reference signal r; that is,

$$e(t) = y(t) - r(t) \approx 0, \quad \forall\, t \geq t_0$$

where t_0 is the time at which control starts. Since the initial value of y depends on the initial state $x(t_0)$, meeting this requirement for all $t \geq t_0$ would require either presetting $x(t_0)$ or presetting the initial value of the reference signal by assuming knowledge of $x(t_0)$, which is not feasible in many applications. Therefore, we usually seek an asymptotic output tracking goal, where the tracking error e approaches zero as t tends to infinity; that is,

$$e(t) \to 0 \ \text{ as } \ t \to \infty$$

If asymptotic output tracking is achieved in the presence of input disturbance w, we say that we have achieved asymptotic disturbance rejection. When the exogenous signals r and w are generated by a known model, such as constant signals or sinusoidal signals of known frequencies, asymptotic output tracking and disturbance rejection can be achieved by including such model in the feedback controller.[3] This is so even when the system's model contain uncertain parameters. In the important special case of constant exogenous signals, where the goal is to asymptotically regulate y to a "set point" r, asymptotic regulation and disturbance rejection can be achieved by including "integral action" in the controller. This is the only way to achieve asymptotic regulation in the presence of parametric uncertainties, which explains the popularity of PI (proportional-integral) and PID (proportional-integral-derivative) controllers in industrial applications. The principle of using integral action is not tied in with linearity. This is shown in Section 12.3 where integral control is presented for a general nonlinear system; then in Section 12.4, we show how linearization can be used to design the stabilizing component of the integral controller. In Chapter 14, Sections 14.1.4 and 14.5.3, we show how PI and PID controllers can be designed as robust regulators of a class of nonlinear systems.

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[3] This is known as the "internal model principle." (See [32].)

For a general time-varying disturbance input $w(t)$, it might not be feasible to achieve asymptotic disturbance rejection. In such cases, we may attempt to achieve disturbance attenuation, which can take the form of a requirement to achieve ultimate boundedness of the tracking error with a prescribed tolerance; that is,

$$\|e(t)\| \leq \varepsilon, \quad \forall\, t \geq T$$

where ε is a prespecified (small) positive number. Alternatively, we may consider attenuating the closed-loop input–output map from the disturbance input w to the tracking error e. For example, if we consider w as an $\mathcal{L}_2$ signal, then our goal would be to minimize the $\mathcal{L}_2$ gain of the closed-loop input–output map from w to e, or at least, make this gain less than a prescribed tolerance.[4]

Feedback control laws for the tracking problem are classified in the same way we have seen in the stabilization problem. We speak of state feedback if x can be measured; that is, if $y_m = x$; otherwise, we speak of output feedback. Also, the feedback control law can be static or dynamic. The control law may achieve local, regional, semiglobal, or global tracking. The new element here is that these phrases refer not only to the size of the initial state, but to the size of the exogenous signals r and w as well. For example, in a typical problem, local tracking means tracking is achieved for sufficiently small initial states and sufficiently small exogenous signals, while global tracking means tracking is achieved for any initial state and any (r, w) in a prescribed class of exogenous signals.

12.2 Stabilization via Linearization

We illustrate the design-via-linearization approach by considering the stabilization problem. We start with state feedback control and then present output feedback.

For state feedback stabilization, consider the system

$$\dot{x} = f(x, u) \tag{12.1}$$

where $f(0,0) = 0$ and $f(x,u)$ is continuously differentiable in a domain $D_x \times D_u \subset R^n \times R^p$ that contains the origin $(x = 0,\ u = 0)$. We want to design a state feedback control law $u = \gamma(x)$ to stabilize the system. Linearization of (12.1) about $(x = 0,\ u = 0)$ results in the linear system

$$\dot{x} = Ax + Bu \tag{12.2}$$

where

$$A = \left.\frac{\partial f}{\partial x}(x,u)\right|_{x=0,u=0}; \qquad B = \left.\frac{\partial f}{\partial u}(x,u)\right|_{x=0,u=0}$$

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[4] This is the formulation of the H_∞ control problem. (See, for example, [20], [54], [61], [90], [199], and [219].)

Assume the pair (A, B) is controllable, or at least stabilizable. Design a matrix K to assign the eigenvalues of $A - BK$ to desired locations in the open left-half complex plane. Now apply the linear state feedback control $u = -Kx$ to the nonlinear system (12.1). The closed-loop system is

$$\dot{x} = f(x, -Kx) \tag{12.3}$$

Clearly, the origin is an equilibrium point of the closed-loop system. The linearization of (12.3) about the origin $x = 0$ is given by

$$\dot{x} = \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)\ (-K)\right]_{x=0} x = (A - BK)x$$

Since $A-BK$ is Hurwitz, it follows from Theorem 4.7 that the origin is an asymptotically stable equilibrium point of the closed-loop system (12.3). Actually, according to Theorem 4.13, the origin is exponentially stable. As a byproduct of the linearization approach, we can always find a Lyapunov function for the closed-loop system. Let Q be any positive-definite symmetric matrix and solve the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -Q$$

for P. Since $(A - BK)$ is Hurwitz, the Lyapunov equation has a unique positive definite solution (Theorem 4.6). The quadratic function $V(x) = x^T P x$ is a Lyapunov function for the closed-loop system in the neighborhood of the origin. We can use $V(x)$ to estimate the region of attraction.

Example 12.2 Consider the pendulum equation

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$$

where $a = g/l > 0$, $b = k/m \geq 0$, $c = 1/ml^2 > 0$, θ is the angle subtended by the rod and the vertical axis, and T is the torque applied to the pendulum. View the torque as the control input and suppose we want to stabilize the pendulum at an angle $\theta = \delta$. For the pendulum to maintain equilibrium at $\theta = \delta$, the torque must have a steady-state component T_{ss} that satisfies

$$0 = -a\sin\delta + cT_{\text{ss}}$$

Choose the state variables as $x_1 = \theta - \delta$, $x_2 = \dot{\theta}$ and the control variable as $u = T - T_{\text{ss}}$. The state equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

is in the standard form (12.1), where $f(0, 0) = 0$. Linearization of the system at the origin results in

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos(x_1 + \delta) & -b \end{bmatrix}_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

The pair (A, B) is controllable. Taking $K = [k_1 \;\; k_2]$, it can be easily verified that $A - BK$ is Hurwitz for

$$k_1 > -\frac{a\cos\delta}{c}, \quad k_2 > -\frac{b}{c}$$

The torque is given by

$$T = \frac{a\sin\delta}{c} - Kx = \frac{a\sin\delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}$$

We leave it to the reader (Exercise 12.1) to continue with the Lyapunov analysis of the closed-loop system. △

For output feedback stabilization, consider the system

$$\dot{x} = f(x, u) \tag{12.4}$$

$$y = h(x) \tag{12.5}$$

where $f(0,0) = 0$, $h(0) = 0$, and $f(x,u)$, $h(x)$ are continuously differentiable in a domain $D_x \times D_u \subset R^n \times R^p$ that contains the origin $(x = 0, \; u = 0)$. We want to design an output feedback control law (using only measurements of y) to stabilize the system. Linearization of (12.4)–(12.5) about $(x = 0, \; u = 0)$ results in the linear system

$$\dot{x} = Ax + Bu \tag{12.6}$$

$$y = Cx \tag{12.7}$$

where A and B are defined after (12.2) and

$$C = \left.\frac{\partial h}{\partial x}(x)\right|_{x=0}$$

Assume (A, B) is stabilizable and (A, C) is detectable, and design a linear dynamic output feedback controller

$$\dot{z} = Fz + Gy \tag{12.8}$$

$$u = Lz + My \tag{12.9}$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \tag{12.10}$$

is Hurwitz. An example of such design is the observer-based controller, where

$$z = \hat{x}, \quad F = A - BK - HC, \quad G = H, \quad L = -K, \quad M = 0$$

and K and H are designed such that $A-BK$ and $A-HC$ are Hurwitz. When the controller (12.8)–(12.9) is applied to the nonlinear system (12.4)–(12.5), it results in the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, Lz + Mh(x)) && (12.11) \\ \dot{z} &= Fz + Gh(x) && (12.12) \end{aligned}$$

It can be verified that the origin ($x = 0$, $z = 0$) is an equilibrium point of the closed-loop system (12.11)–(12.12) and linearization about the origin results in the Hurwitz matrix of (12.10). Thus, once again, we conclude that the origin is an exponentially stable equilibrium point of the closed-loop system (12.11)–(12.12). A Lyapunov function for the closed-loop system can be obtained by solving a Lyapunov equation for the Hurwitz matrix of (12.10).

Example 12.3 Reconsider the pendulum equation of Example 12.2, and suppose we measure the angle θ, but not the angular velocity $\dot{\theta}$. An output variable y can be taken as $y = x_1 = \theta - \delta$, and the state feedback controller of Example 12.2 can be implemented by using the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{x}_1)$$

Taking $H = [h_1 \; h_2]^T$, it can be verified that $A - HC$ will be Hurwitz if

$$h_1 + b > 0, \qquad h_1 b + h_2 + a\cos\delta > 0$$

The torque is given by

$$T = \frac{a\sin\delta}{c} - K\hat{x}$$

△

12.3 Integral Control

In Example 12.2, we considered the problem of regulating the pendulum's angle θ to a constant value δ. We reduced the problem to a stabilization problem by shifting the desired equilibrium point to the origin. While this approach is sound when the parameters of the system are known, it could be unacceptable under parameter perturbations. The control law

$$T = \frac{a\sin\delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}$$

comprises the steady-state component $T_{ss} = (a/c)\sin\delta$, which assigns the equilibrium value of θ, say θ_{ss}, at the desired angle δ, and the feedback component $-Kx$, which makes $A - BK$ Hurwitz. While the calculation of both components depends on the parameters of the system, the feedback part can be designed to be robust to a wide range of parameter perturbations. In particular, if we know an upper bound on the ratio a/c, that is, $a/c \leq \rho$, we can ensure that $A - BK$ will be Hurwitz by choosing k_1 and k_2 to satisfy

$$k_1 > \rho, \qquad k_2 > 0$$

The calculation of $T_{\rm ss}$, on the other hand, could be sensitive to parameter perturbations. Suppose $T_{\rm ss}$ is calculated by using nominal values a_0 and c_0 of a and c, respectively. The equilibrium point of the closed-loop system is given by

$$a \sin \theta_{\rm ss} = c \left[\frac{a_0}{c_0} \sin \delta - k_1(\theta_{\rm ss} - \delta) \right]$$

If $\delta = 0$ or $\delta = \pi$ (that is; the pendulum is stabilized at one of the open-loop equilibrium points), $T_{\rm ss} = 0$ and the foregoing equation yields $\theta_{\rm ss} = \delta$. In this case, the approach used in Example 12.2 will be robust to parameter perturbations. For other values of δ, the error in the steady-state angle could be unacceptable. For example, if $\delta = 45°$, $c = c_0/2$ (doubling the mass), $a = a_0$, and $k_1 = 3a_0/c_0$, we have $\theta_{\rm ss} \approx 36°$.

In this section, we present an integral control approach that ensures asymptotic regulation under all parameter perturbations that do not destroy the stability of the closed-loop system. The use of integral control is not tied in with linearity nor with the use of linearization to design the feedback controller. We present the approach for a general nonlinear system and then show in the next section how linearization can be used to design the feedback controller.

Consider the system

$$\begin{aligned} \dot{x} &= f(x, u, w) && (12.13) \\ y &= h(x, w) && (12.14) \\ y_m &= h_m(x, w) && (12.15) \end{aligned}$$

where $x \in R^n$ is the state, $u \in R^p$ is the control input, $y \in R^p$ is the controlled output, $y_m \in R^m$ is the measured output, and $w \in R^l$ is a vector of unknown constant parameters and disturbances. The function f, h, and h_m are continuously differentiable in (x, u) and continuous in w in a domain $D_x \times D_u \times D_w \subset R^n \times R^p \times R^l$. Let $r \in D_r \subset R^p$ be a constant reference that is available on line and set

$$v = \begin{bmatrix} r \\ w \end{bmatrix} \in D_v \stackrel{\rm def}{=} D_r \times D_w$$

We want to design feedback control such that

$$y(t) \to r \ \text{ as } \ t \to \infty$$

We assume that y can be measured; that is, y is a subset of y_m. The regulation task will be achieved by stabilizing the system at an equilibrium point where $y = r$.

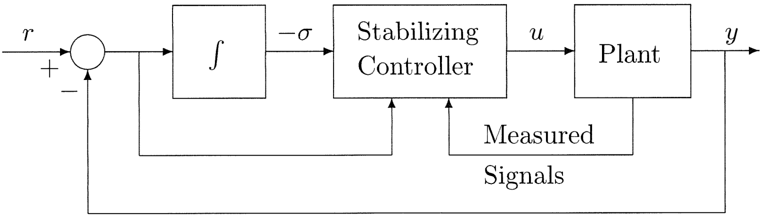

Figure 12.1: Integral control.

Towards that end, we assume that for each $v \in D_v$, there is a unique pair (x_{ss}, u_{ss}) that depends continuously on v and satisfies the equations

$$\begin{aligned} 0 &= f(x_{ss}, u_{ss}, w) && (12.16) \\ r &= h(x_{ss}, w) && (12.17) \end{aligned}$$

so that x_{ss} is the desired equilibrium point and u_{ss} is the steady-state control that is needed to maintain equilibrium at x_{ss}. To introduce integral action, we integrate the regulation error $e = y - r$:

$$\dot{\sigma} = e$$

Then we augment the integrator with the state equation (12.13) to obtain

$$\begin{aligned} \dot{x} &= f(x, u, w) && (12.18) \\ \dot{\sigma} &= h(x, w) - r && (12.19) \end{aligned}$$

For multioutput systems ($p > 1$), the integrator equation represents a stack of p integrators where each component of e is integrated. It is clear that integrating e requires both y and r to be available on line. The control task now is to design a stabilizing feedback controller that stabilizes the augmented state model (12.18)–(12.19) at an equilibrium point (x_{ss}, σ_{ss}) where σ_{ss} produces the desired u_{ss}. Figure 12.1 shows a block diagram representation of the integral control scheme.

The integral controller comprises two components: the integrator and the stabilizing controller. The integrator is sometimes called the internal model, since it duplicates the model of the equation $\dot{v} = 0$, which generates the exogenous constant signal v. The structure of the stabilizing controller depends on the measured signal. For example, in the case of state feedback; that is, when $y_m = x$, the stabilizing controller takes the form

$$u = \gamma(x, \sigma, e)$$

where γ is designed such that there is a unique σ_{ss} that satisfies the equation

$$\gamma(x_{ss}, \sigma_{ss}, 0) = u_{ss}$$

and the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, \gamma(x, \sigma, h(x, w) - r), w) \\ \dot{\sigma} &= h(x, w) - r \end{aligned}$$

has an asymptotically stable equilibrium point at $(x_{\rm ss}, \sigma_{\rm ss})$. At the equilibrium point, $y = r$, irrespective of the value of w. Hence, asymptotic regulation is achieved for all initial states in the region of attraction of $(x_{\rm ss}, \sigma_{\rm ss})$.

The fact that the integral controller of Figure 12.1 is robust to all parameter perturbations that do not destroy the stability of the closed-loop system can be intuitively explained as follows: The feedback controller creates an asymptotically stable equilibrium point. At this point, all signals must be constant. For the integrator $\dot{\sigma} = e$ to have a constant output σ, its input e must be zero. Thus, the inclusion of the integrator forces the regulation error to be zero at equilibrium. Parameter perturbations will change the equilibrium point, but the condition $e = 0$ at equilibrium will be maintained. Thus, as long as the perturbed equilibrium point remains asymptotically stable, regulation will be achieved.

The design of the stabilizing controller is not a trivial task because the closed-loop equation depends on the unknown vector w. In the next section, we will see a straightforward solution to this dilemma via linearization, but it will only guarantee local regulation. Nonlocal regulation can be achieved by using some of the nonlinear design tools of Chapter 14; an example is given in Section 14.1.4.

12.4 Integral Control via Linearization

We start by designing a state feedback integral controller; then we consider output feedback. We need to design $u = \gamma(x, \sigma, e)$ to stabilize the augmented state model (12.18)–(12.19) at $(x_{\rm ss}, \sigma_{\rm ss})$ where $u_{\rm ss} = \gamma(x_{\rm ss}, \sigma_{\rm ss}, 0)$. Since we are going to use linearization, it is reasonable to consider a linear feedback control law of the form

$$u = -K_1 x - K_2 \sigma - K_3 e \tag{12.20}$$

When the control (12.20) is applied to (12.18)–(12.19), it results in the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, -K_1 x - K_2 \sigma - K_3(h(x,w) - r), w) && (12.21) \\ \dot{\sigma} &= h(x, w) - r && (12.22) \end{aligned}$$

Equilibrium points $(\bar{x}, \bar{\sigma})$ of (12.21)–(12.22) satisfy the equations

$$\begin{aligned} 0 &= f(\bar{x}, \bar{u}, w) \\ 0 &= h(\bar{x}, w) - r \\ \bar{u} &= -K_1 \bar{x} - K_2 \bar{\sigma} \end{aligned}$$

By the assumption that the equilibrium equations (12.16) and (12.17) have a unique solution $(x_{\rm ss}, u_{\rm ss})$ in the domain of interest, we conclude that $\bar{x} = x_{\rm ss}$ and $\bar{u} = u_{\rm ss}$. By choosing K_2 to be nonsingular, we guarantee that there is a unique solution $\sigma_{\rm ss}$ of the equation

$$u_{\rm ss} = -K_1 x_{\rm ss} - K_2 \sigma_{\rm ss}$$

Our task now is to stabilize the equilibrium point (x_{ss}, σ_{ss}). Linearization of the closed-loop system (12.21)–(12.22) about (x_{ss}, σ_{ss}) yields

$$\dot{\xi}_\delta = (\mathcal{A} - \mathcal{B}\mathcal{K})\xi_\delta$$

where

$$\xi_\delta = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}(x,u,w)\right|_{x=x_{ss},u=u_{ss}}, \quad B = \left.\frac{\partial f}{\partial u}(x,u,w)\right|_{x=x_{ss},u=u_{ss}}, \quad C = \left.\frac{\partial h}{\partial x}(x,w)\right|_{x=x_{ss}}$$

The matrices A, B, and C are, in general, dependent on v. Suppose now that (A, B) is controllable (respectively, stabilizable) and[5]

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p \tag{12.23}$$

Then, $(\mathcal{A}, \mathcal{B})$ is controllable (respectively, stabilizable).[6] Design $\mathcal{K}$, independent of w, such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is Hurwitz for all $v \in D_v$.[7] For any such design, the matrix K_2 will be nonsingular.[8] Thus, (x_{ss}, σ_{ss}) is an exponentially stable equilibrium point of the closed-loop system (12.21)–(12.22), and all solutions starting in its region of attraction approach it as t tends to infinity. Consequently, $y(t) - r \to 0$ as $t \to \infty$. We note that the stabilization of (x_{ss}, σ_{ss}) is independent of K_3. Therefore, we can take $K_3 = 0$, or we may use it as an extra degree of freedom to improve performance.

In summary, assuming (A, B) is stabilizable and the rank condition (12.23) is satisfied, the state feedback control can be taken as

$$\begin{aligned} u &= -K_1 x - K_2 \sigma \\ \dot{\sigma} &= e = y - r \end{aligned}$$

where $\mathcal{K} = [K_1 \; K_2]$ is designed such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is Hurwitz.

Example 12.4 Consider the pendulum equation

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$$

---

[5] The rank condition (12.23) implies that the linear state model (A, B, C) has no transmission zeros at the origin.

[6] See Exercise 12.3.

[7] This is a robust stabilization problem that has been extensively studied in the linear control literature. (See, for example, [48] and [69].) Note that if $\mathcal{K}$ is designed to stabilize $\mathcal{A} - \mathcal{B}\mathcal{K}$ at some nominal parameters, then, due to continuous dependence of the eigenvalues of a matrix on its elements, $\mathcal{A} - \mathcal{B}\mathcal{K}$ will remain Hurwitz in some neighborhood of the nominal parameters.

[8] Had K_2 been singular, $\mathcal{A} - \mathcal{B}\mathcal{K}$ would have been singular as well, which contradicts the fact that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is Hurwitz.

where $a = g/l > 0$, $b = k/m \geq 0$, $c = 1/ml^2 > 0$, θ is the angle subtended by the rod and the vertical axis, and T is the torque applied to the pendulum. View T as the control input and suppose we want to regulate θ to δ. Taking $x_1 = \theta - \delta$, $x_2 = \dot{\theta}$, $u = T$, and $y = x_1$, we write the state equation as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1 + \delta) - bx_2 + cu \\ y &= x_1 \end{aligned}$$

It can be easily seen that the desired equilibrium point is

$$x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_{ss} = \frac{a}{c}\sin\delta$$

The matrices A, B, and C are given by

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Noting that $c > 0$, it can be easily verified that (A, B) is controllable and the rank condition (12.23) is satisfied. Taking $K_1 = [k_1 \; k_2]$ and $K_2 = k_3$, it can be verified, using the Routh–Hurwitz criterion, that $\mathcal{A} - \mathcal{B}\mathcal{K}$ will be Hurwitz if

$$b + k_2 c > 0, \quad (b + k_2 c)(a\cos\delta + k_1 c) - k_3 c > 0, \quad \text{and} \quad k_3 c > 0$$

Suppose we do not know the exact values of the parameters $a > 0$, $b \geq 0$, $c > 0$, but we know upper bounds ρ_1 on a/c and ρ_2 on $1/c$. Then, the choice

$$k_2 > 0, \quad k_3 > 0, \quad \text{and} \quad k_1 > \rho_1 + \frac{k_3}{k_2}\rho_2 \tag{12.24}$$

ensures that $\mathcal{A} - \mathcal{B}\mathcal{K}$ will be Hurwitz. The feedback control law is given by

$$\begin{aligned} u &= -k_1(\theta - \delta) - k_2\dot{\theta} - k_3\sigma \\ \dot{\sigma} &= \theta - \delta \end{aligned}$$

which is the classical PID controller. Comparing this feedback law with the one derived in Example 12.2 shows that we no longer calculate the steady-state torque needed to maintain the equilibrium position. Regulation will be achieved for all parameter perturbations that satisfy $(b + k_2 c)(a\cos\delta + k_1 c) - k_3 c > 0$. Figure 12.2 shows simulation results for regulating the pendulum to $\delta = \pi/4$ with integral action (Example 12.4) and without integral action (Example 12.2). In the first case, the feedback gains $k_1 = 8$, $k_2 = 2$, and $k_3 = 10$ assign the eigenvalues at -15.93, -2.93, and -2.14. In the second case, the feedback gains $k_1 = 3$, $k_2 = 0.7$ assign the eigenvalues at $-4 \pm j4.59$. In both cases, the nominal parameters are $a = c = 10$ and $b = 1$. In the perturbed case, b and c are reduced to 0.5 and 5, respectively, corresponding to doubling of the mass. The simulation shows the improvement in the steady-state response with integral action, which is achieved at the expense of a longer settling time and an increased torque during the transient period. △

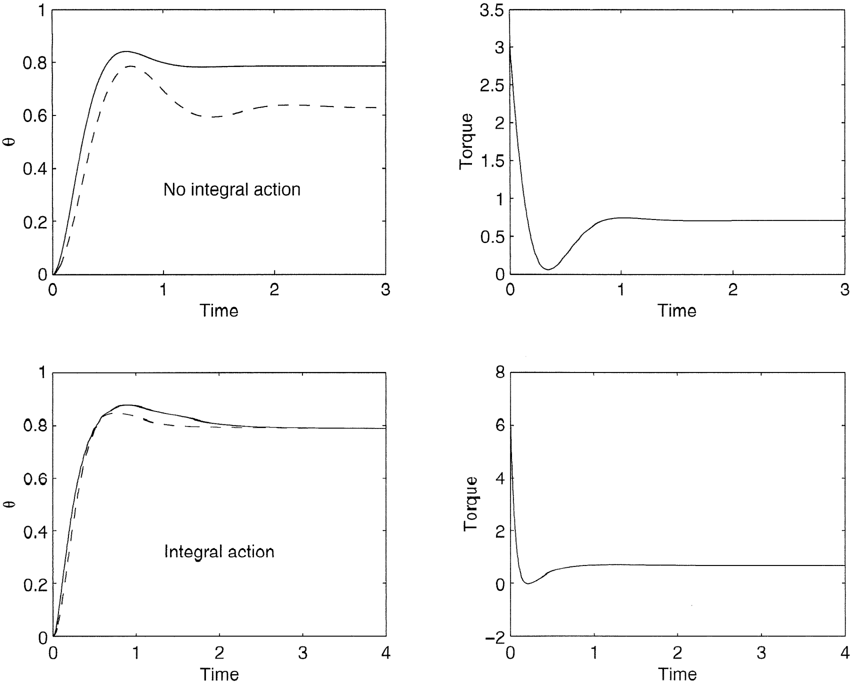

Figure 12.2: Simulation results for pendulum regulation under nominal (solid) and perturbed (dashed) parameters, with (Example 12.4) and without (Example 12.2) integral action.

In the more general case of output feeback, the integral controller can be taken as

$$\dot{\sigma} = e = y - r \tag{12.25}$$

$$\dot{z} = Fz + G_1\sigma + G_2 y_m \tag{12.26}$$

$$u = Lz + M_1\sigma + M_2 y_m + M_3 e \tag{12.27}$$

where F, G_1, G_2, L, M_1, M_2, and M_3 are designed, independent of w, such that

$$\mathcal{A}_c = \begin{bmatrix} A + BM_2C_m + BM_3C & BM_1 & BL \\ C & 0 & 0 \\ G_2C_m & G_1 & F \end{bmatrix}$$

is Hurwitz for all $v \in D_v$, where $C_m = [\partial h_m/\partial x](x_{\text{ss}}, w)$. This will ensure that

$$\begin{bmatrix} M_1 & L \\ G_1 & F \end{bmatrix}$$

is nonsingular and the equation

$$\begin{bmatrix} M_1 & L \\ G_1 & F \end{bmatrix} \begin{bmatrix} \sigma_{\rm ss} \\ z_{\rm ss} \end{bmatrix} = \begin{bmatrix} u_{\rm ss} - M_2 h_m(x_{\rm ss}, w) \\ -G_2 h_m(x_{\rm ss}, w) \end{bmatrix}$$

has a unique solution $(\sigma_{\rm ss}, z_{\rm ss})$. Thus, $(x_{\rm ss}, \sigma_{\rm ss}, z_{\rm ss})$ is the unique equilibrium point of the closed-loop system at which $u = u_{\rm ss}$ and $e = 0$. It can be verified that $\mathcal{A}_c$ is the linearization of the closed-loop system about $(x_{\rm ss}, \sigma_{\rm ss}, z_{\rm ss})$. Hence, the equilibrium point is exponentially stable and all solutions starting in its region of attraction approach it as t tends to infinity. Consequently, $y(t) - r \to 0$ as $t \to \infty$.

12.5 Gain Scheduling

The basic limitation of the design-via-linearization approach is the fact that the controller is guaranteed to work only in some neighborhood of a single operating (equilibrium) point. In this section, we introduce *gain scheduling*, a technique that can extend the validity of the linearization approach to a range of operating points. In many situations, it is known how the dynamics of a system change with its operating points. It might even be possible to model the system in such a way that the operating points are parameterized by one or more variables, which we call *scheduling variables*. In such situations, we may linearize the system at several equilibrium points, design a linear feedback controller at each point, and implement the resulting family of linear controllers as a single controller whose parameters are changed by monitoring the scheduling variables. Such controller is called *gain-scheduled controller*.

The concept of gain scheduling originated in connection with flight control systems.[9] The nonlinear equations of motion of an airplane or a missile are linearized about selected operating points that capture the key modes of operation throughout the flight envelope. Linear controllers are designed to achieve the desired stability and performance requirements for the linearizations about the selected operating points. The parameters of the controllers are then interpolated as functions of gain scheduling variables; typical variables are dynamic pressure, Mach number, altitude, and angle of attack. Finally, the gain-scheduled controller is implemented on the nonlinear system. We start with a simple example to illustrate the idea of gain scheduling.

Example 12.5 Consider the tank system of Exercise 1.19 where the cross-sectional area A varies with height. The system can be modeled by the equation

$$\frac{d}{dt}\left(\int_0^h A(y)\, dy\right) = w_i - k\sqrt{\rho g h}$$

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[9] See [159] for a survey of research on gain scheduling and applications to flight control and automotive engine control.

where h is the liquid height in the tank, w_i is the input flow rate, ρ is the liquid density, g is the acceleration due to gravity, and k is a positive constant. Taking $x = h$ as the state variable and $u = w_i$ as the control input, the state model is given by

$$\dot{x} = \frac{1}{A(x)}\left(u - c\sqrt{x}\right) \overset{\text{def}}{=} f(x,u)$$

where $c = k\sqrt{\rho g}$ will be treated as an uncertain parameter. Suppose we want to design a controller such that x tracks a reference signal r. We define $y = x$ as the controlled output and use r as the scheduling variable. When $r = \alpha$ (a positive constant), the output y should be regulated to α. To cope with the uncertainty in c, we use integral control. The equilibrium equations (12.16) and (12.17) take the form

$$0 = u_{\text{ss}} - c\sqrt{x_{\text{ss}}}, \qquad \alpha = x_{\text{ss}}$$

Hence, $x_{\text{ss}} = \alpha$ and $u_{\text{ss}} = c\sqrt{\alpha}$. Augmenting the integrator $\dot{\sigma} = e = y - r$ with the state equation, we obtain

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{\sigma} &= x - r \end{aligned}$$

We use the PI controller

$$u = -k_1(\alpha)e - k_2(\alpha)\sigma$$

to stabilize the augmented state equation at $(x_{\text{ss}}, \sigma_{\text{ss}})$, where $\sigma_{\text{ss}} = -u_{\text{ss}}/k_2(\alpha)$, provided $k_2 \neq 0$. The closed-loop system is given by

$$\begin{aligned} \dot{x} &= f(x, -k_1(\alpha)(x-r) - k_2(\alpha)\sigma) \\ \dot{\sigma} &= x - r \end{aligned}$$

When $r = \alpha$, the system has an equilibrium point at $(x_{\text{ss}}, \sigma_{\text{ss}})$. Linearization of the closed-loop system about $(x,\sigma) = (x_{\text{ss}}, \sigma_{\text{ss}})$ and $r = \alpha$ yields

$$\dot{\xi}_\delta = \begin{bmatrix} a(\alpha) - b(\alpha)k_1(\alpha) & -b(\alpha)k_2(\alpha) \\ 1 & 0 \end{bmatrix} \xi_\delta + \begin{bmatrix} b(\alpha)k_1(\alpha) \\ -1 \end{bmatrix} r_\delta, \qquad y_\delta = x_\delta$$

where $\xi_\delta = [x_\delta \;\; \sigma_\delta]^T$, $x_\delta = x - \alpha$, $\sigma_\delta = \sigma - \sigma_{\text{ss}}$, $r_\delta = r - \alpha$,

$$\begin{aligned} a(\alpha) &= \left.\frac{\partial f}{\partial x}\right|_{x=\alpha, u=c\sqrt{\alpha}} = \left[\frac{1}{A(x)}\left(\frac{-c}{2\sqrt{x}}\right) - \frac{A'(x)}{A^2(x)}\left(u - c\sqrt{x}\right)\right]_{x=\alpha, u=c\sqrt{\alpha}} \\ &= -\frac{c\sqrt{\alpha}}{2\alpha A(\alpha)} \end{aligned}$$

and

$$b(\alpha) = \left.\frac{\partial f}{\partial u}\right|_{x=\alpha, u=c\sqrt{\alpha}} = \frac{1}{A(\alpha)}$$

Assuming we know an upper bound on c, we choose k_1 and k_2 as

$$k_1(\alpha) = \frac{2\zeta\omega_n}{b(\alpha)}, \quad k_2(\alpha) = \frac{\omega_n^2}{b(\alpha)}$$

where $0 < \zeta < 1$, and $2\zeta\omega_n \gg |a(\alpha)|$, to (approximately) assign the closed-loop eigenvalues at the roots of

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Thus, the linearization of the closed-loop system under the fixed-gain controller is

$$\dot{\xi}_\delta = A_f(\alpha)\xi_\delta + B_f r_\delta, \quad y_\delta = C_f \xi_\delta$$

where

$$A_f(\alpha) = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix}, \quad B_f = \begin{bmatrix} 2\zeta\omega_n \\ -1 \end{bmatrix}, \quad \text{and} \quad C_f = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The closed-loop transfer function from the command input r_δ to the output y_δ is

$$\frac{2\zeta\omega_n s + \omega_n^2}{s^2 + [2\zeta\omega_n - a(\alpha)]s + \omega_n^2}$$

Now, leaving aside the hypothetical situation where r was assumed to be constant, let us deal with time-varying r. A gain-scheduled PI controller is taken as

$$u = -k_1(r)e - k_2(r)\sigma, \quad \dot{\sigma} = e = x - r$$

where α is replaced by r so that the gains k_1 and k_2 vary directly with r. The closed-loop nonlinear system under the gain-scheduled controller is

$$\begin{aligned} \dot{x} &= f(x, -k_1(r)(x - r) - k_2(r)\sigma) \\ \dot{\sigma} &= x - r \end{aligned}$$

When $r = \alpha$, the system has an equilibrium point at $(x_{\rm ss}, \sigma_{\rm ss})$. This shows that the closed-loop nonlinear system under the gain-scheduled controller achieves the desired operating point for every α. Linearization about $(x, \sigma) = (x_{\rm ss}, \sigma_{\rm ss})$ and $r = \alpha$ yields

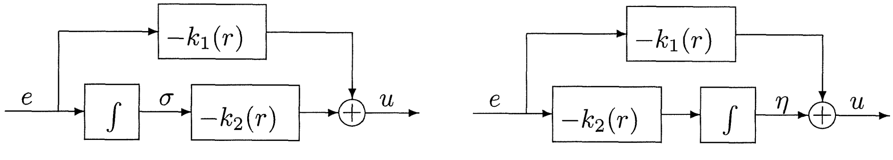
$$\dot{\xi}_\delta = A_s(\alpha)\xi_\delta + B_s(\alpha)r_\delta, \quad y_\delta = C_s \xi_\delta$$

where

$$A_s(\alpha) = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix}, \quad B_s(\alpha) = \begin{bmatrix} 2\zeta\omega_n + \gamma(\alpha) \\ -1 \end{bmatrix}, \quad C_s = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and $\gamma(\alpha) = -b(\alpha)k_2'(\alpha)\sigma_{\rm ss}(\alpha) = A'(\alpha)c\sqrt{\alpha}/A^2(\alpha)$. The closed-loop transfer function from the command input r_δ to the output y_δ is

$$\frac{[2\zeta\omega_n + \gamma(\alpha)]s + \omega_n^2}{s^2 + [2\zeta\omega_n - a(\alpha)]s + \omega_n^2}$$



Original

Modified

Figure 12.3: Modification of the gain-scheduled PI controller of Example 12.5.

Let us note the difference between the two linear models represented by (A_f, B_f, C_f) and (A_s, B_s, C_s). The first model is the linearization of the closed-loop system under the fixed-gain controller, while the second model is the linearization of the closed-loop system under the gain-scheduled controller. In both cases, the linearization is about the desired operating point. Ideally, we would like these two models to be equivalent, for then we know that the local behavior of the closed-loop system near the desired operating point matches the behavior predicted by the design model. Comparing the two models shows that $A_s = A_f$ and $C_s = C_f$, but $B_s \neq B_f$, resulting in a different zero location in the closed-loop transfer function. Despite this difference, the two transfer functions have the same poles and the property of zero steady-state regulation error to step inputs. If these are the only design objectives, we can say that the gain-scheduled controller is acceptable. On the other hand, if other performance issues are of concern, like the transient part of the step response, which is affected by the zero location, then we have to study the effect of the zero shift by linear analysis or simulation of the model (A_s, B_s, C_s) (or both). Alternatively, we may modify the gain-scheduled controller with the objective of arriving at a linear model that is equivalent to (A_f, B_f, C_f) for every α. This can be achieved by modifying the gain-scheduled controller to[10]

$$u = -k_1(r)e + \eta, \qquad \dot{\eta} = -k_2(r)e$$

For constant gain k_2, the modification can be interpreted as commuting the gain $-k_2$ with the integrator. (See Figure 12.3.) The closed-loop nonlinear system under the modified gain-scheduled controller is

$$\begin{aligned} \dot{x} &= f(x, -k_1(r)(x - r) + \eta) \\ \dot{\eta} &= -k_2(r)(x - r) \end{aligned}$$

When $r = \alpha$, the system has an equilibrium point at $x = x_{ss}$ and $\eta = u_{ss}$. Linearization about $(x, \eta) = (x_{ss}, u_{ss})$ and $r = \alpha$ yields

$$\dot{z}_\delta = A_{ms}(\alpha)z_\delta + B_{ms}(\alpha)r_\delta, \qquad y_\delta = C_{ms}z_\delta$$

where

$$A_{ms}(\alpha) = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & b(\alpha) \\ -\omega_n^2/b(\alpha) & 0 \end{bmatrix}, \quad B_{ms}(\alpha) = \begin{bmatrix} 2\zeta\omega_n \\ \omega_n^2/b(\alpha) \end{bmatrix}, \quad C_{ms} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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[10] This modification is the velocity algorithm of [96]. Another modification is given in [114].

$z_\delta = [x_\delta \ \ \eta_\delta]^T$, and $\eta_\delta = \eta - u_{\rm ss}$. The derivative k_2' does not appear in this model because k_2 is a multiple of e, which vanishes at the equilibrium point. It can be easily seen that the models (A_f, B_f, C_f) and (A_{ms}, B_{ms}, C_{ms}) are equivalent by the similarity transformation

$$\xi_\delta = \begin{bmatrix} 1 & 0 \\ 0 & -b(\alpha)/\omega_n^2 \end{bmatrix} z_\delta$$

Hence, both models have the same transfer function from r_δ to y_δ. △

In view of this example, we can describe the development of a gain-scheduled tracking controller for nonlinear systems by the following steps:

1. Linearize the nonlinear model about a family of operating (equilibrium) points, parameterized by the scheduling variables.
2. Using linearization, design a parameterized family of linear controllers to achieve the specified performance at each operating point.
3. Construct a gain-scheduled controller such that
   - for each constant value of the exogenous input, the closed-loop system under the gain-scheduled controller has the same equilibrium point as the closed-loop system under the fixed-gain controller;
   - the linearization of the closed-loop system under the gain-scheduled controller is equivalent to the linearization of the closed-loop system under the fixed-gain controller.
4. Check the nonlocal performance of the gain-scheduled controller by simulating the nonlinear closed-loop model.

The second step can be achieved by solving the design problem for a family of linear models that depend continuously on the scheduling variables, as we have done in the foregoing example, or by solving the problem only at a finite number of operating points by using the same controller structure for all of them, but allowing the controller parameters to change from one operating point to another. Then, the controller parameters are interpolated at intermediate operating points to produce the parameterized family of linear controllers. This interpolation process is usually ad hoc in nature and relies on physical insight.[11] In the forthcoming development we limit ourselves to the case where the design problem is solved for a family of linear models that depend continuously on the scheduling variables.

Consider the system

$$\dot{x} = f(x, u, v, w) \tag{12.28}$$

$$y = h(x, w) \tag{12.29}$$

$$y_m = h_m(x, w) \tag{12.30}$$

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[11] See [159] for further discussion of this interpolation process.

where f, h, and h_m are twice continuously differentiable functions in (x, u, v) and continuous in w in a domain $D_x \times D_u \times D_v \times D_w \subset R^n \times R^p \times R^q \times R^l$. Here, x is the state, u is the control input, v is a measured exogenous input, w is a vector of unknown constant parameters and disturbances, $y \in R^p$ is the controlled output, and $y_m \in R^m$ is the measured output. We assume that y can be measured; that is, y is a subset of y_m. Let $r \in D_r \subset R^p$ be a reference signal. We want to design an output feedback controller that achieves small tracking error $e = y - r$ in response to the exogenous input

$$\rho = \begin{bmatrix} r \\ v \end{bmatrix} \in D_\rho \stackrel{\text{def}}{=} D_r \times D_v$$

We use integral control to achieve zero steady-state error when $v = \alpha$ (a constant vector) and rely on gain scheduling to achieve small error for slowly varying ρ. We partition α as $\alpha = [\alpha_r^T,\ \alpha_v^T]^T$, where α_r and α_v are constant values for r and v, respectively. We use ρ as the scheduling variable.[12] For the design of integral control, we assume that there is a unique pair $(x_{\text{ss}}, u_{\text{ss}}) : D_\rho \times D_w \rightarrow D_x \times D_u$, continuously differentiable in α and continuous in w, such that

$$\begin{aligned} 0 &= f(x_{\text{ss}}(\alpha, w), u_{\text{ss}}(\alpha, w), \alpha_v, w) && (12.31) \\ \alpha_r &= h(x_{\text{ss}}(\alpha, w), w) && (12.32) \end{aligned}$$

for all $(\alpha, w) \in D_\rho \times D_w$. When $\rho = \alpha$, we can use linearization, as in the previous section, to design an integral controller of the form

$$\begin{aligned} \dot{\sigma} &= e = y - r && (12.33) \\ \dot{z} &= F(\alpha)z + G_1(\alpha)\sigma + G_2(\alpha)y_m && (12.34) \\ u &= L(\alpha)z + M_1(\alpha)\sigma + M_2(\alpha)y_m + M_3(\alpha)e && (12.35) \end{aligned}$$

where the controller gains F, G_1, G_2, L, M_1, M_2, and M_3 are continuously differentiable functions of α, designed such that

$$\mathcal{A}_c(\alpha, w) = \begin{bmatrix} A + BM_2C_m + BM_3C & BM_1 & BL \\ C & 0 & 0 \\ G_2C_m & G_1 & F \end{bmatrix}$$

is Hurwitz for all $(\alpha, w) \in D_\rho \times D_w$, where

$$A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}, \quad C = \frac{\partial h}{\partial x}, \quad \text{and} \quad C_m = \frac{\partial h_m}{\partial x}$$

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[12] In the gain scheduling literature, the scheduling variable is also allowed to depend on the measured output y_m. (See [159].)

with all Jacobian matrices evaluated at $(x, u, v) = (x_{ss}, u_{ss}, \alpha_v)$. The new element here is allowing the controller gains to depend on α (the frozen value of the scheduling variable ρ). In the state feedback case, we can drop z and its state equation (12.34) and take $y_m = x$, $L = 0$, $M_1 = -K_2$, $M_2 = -K_1$, and $M_3 = 0$, where $K = [K_1 \ K_2]$ is designed such that

$$\begin{bmatrix} A - BK_1 & -BK_2 \\ C & 0 \end{bmatrix}$$

is Hurwitz for every $(\alpha, w) \in D_\rho \times D_w$.

The closed-loop system under the fixed-gain controller (12.33)–(12.35) is

$$\begin{aligned} \dot{x} &= f(x, Lz + M_1\sigma + M_2 h_m(x, w) + M_3 e, v, w) && (12.36) \\ \dot{\sigma} &= e \ = \ h(x, w) - r && (12.37) \\ \dot{z} &= Fz + G_1\sigma + G_2 h_m(x, w) && (12.38) \\ y &= h(x, w) && (12.39) \end{aligned}$$

When $\rho = \alpha$, the system has an equilibrium point at $(x_{ss}, \sigma_{ss}, z_{ss})$ at which $e = 0$. Linearization about $(x, \sigma, z) = (x_{ss}, \sigma_{ss}, z_{ss})$ and $\rho = \alpha$ yields

$$\begin{aligned} \dot{\xi}_\delta &= A_f(\alpha, w)\xi_\delta + B_f(\alpha, w)\rho_\delta && (12.40) \\ y_\delta &= C_f(\alpha, w)\xi_\delta && (12.41) \end{aligned}$$

where

$$\xi_\delta = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \\ z - z_{ss} \end{bmatrix}, \quad \rho_\delta = \rho - \alpha = \begin{bmatrix} r_\delta \\ v_\delta \end{bmatrix}, \quad y_\delta = y - \alpha_r$$

$$A_f = \mathcal{A}_c, \quad B_f = \begin{bmatrix} -BM_3 & E \\ -I & 0 \\ 0 & 0 \end{bmatrix}, \quad C_f = \begin{bmatrix} C & 0 & 0 \end{bmatrix}$$

$$E = \frac{\partial f}{\partial v}(x, u, v, w)\bigg|_{x = x_{ss}, u = u_{ss}, v = \alpha_v}$$

Hence, when $\rho = \alpha$, the equilibrium point $(x_{ss}, \sigma_{ss}, z_{ss})$ is exponentially stable.

A gain-scheduled controller can be obtained from the fixed-gain controller (12.33)–(12.35) by scheduling the gains F, G_1, G_2 L, M_1, M_2, and M_3 as functions of the scheduling variable ρ, that is, replacing α by ρ. It can be verified that the closed-loop system under this controller will have the desired equilibrium point and its linearization $(A_s(\alpha, w), B_s(\alpha, w), C_s(\alpha, w))$ will have $A_s = A_f = \mathcal{A}_c$, $C_s = C_f$, but, in general, $B_s \neq B_f$ due to partial differentiation of the scheduled gains with respect to ρ. The fact that $\mathcal{A}_c(\alpha, w)$ is Hurwitz for all $(\alpha, w) \in D_\rho \times D_w$ shows that the gain-scheduled controller will produce an exponentially stable equilibrium point with zero steady-state tracking error when $\rho = \alpha$. However, the closed-loop transfer function from ρ_δ to y_δ could be different from the corresponding transfer function of the design model. Therefore, we must check the local performance of the gain-scheduled controller via analysis or simulation. If the performance is acceptable, we may go ahead and implement the gain-scheduled controller. It turns out, however, that we can do better than that. As in Example 12.5, we can modify the gain-scheduled controller to achieve equivalence between the linearized models of the closed-loop systems under the fixed-gain and gain-scheduled controllers. In the example, we commuted the gain k_2 and the integrator; that is, we moved the integrator from the input side of the controller to its output side, so that both gains k_1 and k_2 are multiples of e, which vanishes at steady-state. This is basically what we would like to do for the controller (12.33)–(12.35). However, the current situation is complicated by the presence of the dynamic equation (12.34), which has two driving inputs: σ and y_m. While σ is the output of an integrator and it makes sense to talk about moving the integrator to the output side of the controller, y_m is not the output of an integrator. This difficulty can be overcome if we can measure $\dot{y}_m$, the derivative of y_m. For, then, we can represent the controller (12.33)–(12.35) as

$$\begin{aligned}
\dot{\lambda} &= \psi \\
\dot{z} &= F(\alpha)z + G(\alpha)\lambda \\
u &= L(\alpha)z + M(\alpha)\lambda + M_3(\alpha)e
\end{aligned}$$

where

$$\psi = \begin{bmatrix} e \\ \dot{y}_m \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$$

The transfer function from ψ to u,

$$\{L(\alpha)[sI - F(\alpha)]^{-1}G(\alpha) + M(\alpha)\}\frac{1}{s}$$

is equivalent to

$$\frac{1}{s}\{L(\alpha)[sI - F(\alpha)]^{-1}G(\alpha) + M(\alpha)\}$$

Hence, the controller can be realized by

$$\begin{aligned}
\dot{\varphi} &= F(\alpha)\varphi + G(\alpha)\psi \\
\dot{\eta} &= L(\alpha)\varphi + M(\alpha)\psi \\
u &= \eta + M_3(\alpha)e
\end{aligned}$$

Figure 12.4 shows the original and modified realizations of the fixed-gain controller. Scheduling the gains F, G, L, M, and M_3 in the modified realization as functions of the scheduling variable ρ, we obtain the gain-scheduled controller

$$\dot{\varphi} = F(\rho)\varphi + G_1(\rho)e + G_2(\rho)\dot{y}_m \tag{12.42}$$

$$\dot{\eta} = L(\rho)\varphi + M_1(\rho)e + M_2(\rho)\dot{y}_m \tag{12.43}$$

$$u = \eta + M_3(\rho)e \tag{12.44}$$

When this controller is applied to the nonlinear system (12.28)–(12.30), it results in the closed-loop system

$$\begin{aligned} \dot{\mathcal{X}} &= g(\mathcal{X}, \rho, w) && (12.45)\\ y &= h(x, w) && (12.46) \end{aligned}$$

where

$$\mathcal{X} = \begin{bmatrix} x \\ \varphi \\ \eta \end{bmatrix}, \quad g(\mathcal{X}, \rho, w) = \begin{bmatrix} f(x, \eta + M_3(\rho)e, v, w) \\ F(\rho)\varphi + G_1(\rho)e + G_2(\rho)\dot{y}_m \\ L(\rho)\varphi + M_1(\rho)e + M_2(\rho)\dot{y}_m \end{bmatrix}$$

$$e = h(x, w) - r, \qquad \dot{y}_m = \frac{\partial h_m}{\partial x}(x, w) f(x, \eta + M_3(\rho)e, v, w)$$

When $\rho = \alpha$, the system (12.45)–(12.46) has a unique equilibrium point

$$\mathcal{X}_{\text{ss}}(\alpha, w) = \begin{bmatrix} x_{\text{ss}}(\alpha, w) \\ 0 \\ u_{\text{ss}}(\alpha, w) \end{bmatrix} \tag{12.47}$$

at which $y = \alpha_r$. Linearization of (12.45)–(12.46) about $\mathcal{X} = \mathcal{X}_{\text{ss}}$ and $\rho = \alpha$ yields[13]

$$\begin{aligned} \dot{\mathcal{X}}_\delta &= A_{ms}(\alpha, w)\mathcal{X}_\delta + B_{ms}(\alpha, w)\rho_\delta && (12.48)\\ y_\delta &= C_{ms}(\alpha, w)\mathcal{X}_\delta && (12.49) \end{aligned}$$

where

$$\mathcal{X}_\delta = \mathcal{X} - \mathcal{X}_{\text{ss}}, \qquad A_{ms} = \begin{bmatrix} A + BM_3C & 0 & B \\ G_1C + G_2C_m(A + BM_3C) & F & G_2C_mB \\ M_1C + M_2C_m(A + BM_3C) & L & M_2C_mB \end{bmatrix}$$

$$B_{ms} = \begin{bmatrix} -BM_3 & E \\ -G_1 - G_2C_mBM_3 & G_2C_mE \\ -M_1 - M_2C_mBM_3 & M_2C_mE \end{bmatrix}, \qquad C_{ms} = \begin{bmatrix} C & 0 & 0 \end{bmatrix}$$

We leave it as an exercise for the reader (Exercise 12.6) to verify that the matrix

$$P = \begin{bmatrix} I & 0 & 0 \\ G_2C_m & G_1 & F \\ M_2C_m & M_1 & L \end{bmatrix} \tag{12.50}$$

is nonsingular and

$$P^{-1}A_{ms}P = A_f, \qquad P^{-1}B_{ms} = B_f, \qquad \text{and} \qquad C_{ms}P = C_f \tag{12.51}$$

Hence, the linear model (12.48)–(12.49) is equivalent to the linear model (12.40)–(12.41).

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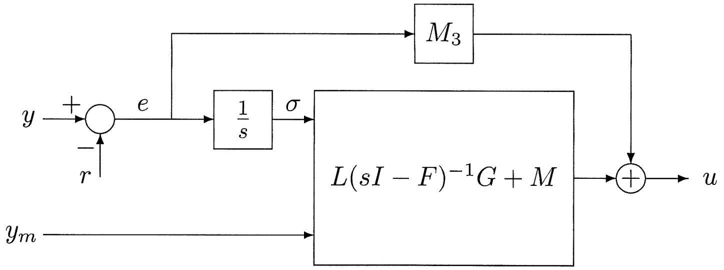
[13] While calculating the matrices A_{ms} and B_{ms}, note that partial derivatives that appear as coefficients of φ, e, or f vanish at the operating point.



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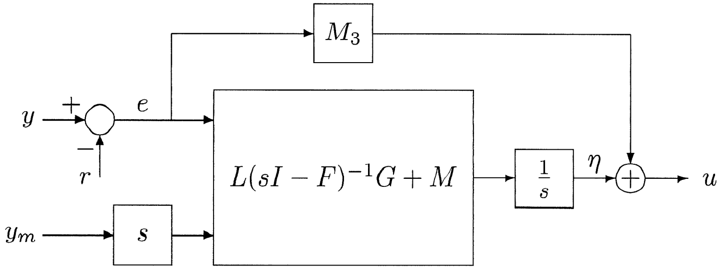

Modified

Figure 12.4: Modification of the gain-scheduled controller.

So far, our analysis of the closed-loop system under the gain-scheduled controller has focused on the local behavior in the neighborhood of a constant operating point. Can we say more about the behavior of the nonlinear system? What if the scheduling variable is not constant? In applications of gain scheduling, the practice has been that you can schedule on time-varying variables as long as they are slow enough relative to the dynamics of the system. This practice is justified by the next theorem.

Theorem 12.1 *Consider the closed-loop system* (12.45)–(12.46) *under the stated assumptions. Suppose $\rho(t)$ is continuously differentiable, $\rho(t) \in S$ (a compact subset of D_ρ), and $\|\dot{\rho}(t)\| \leq \mu$ for all $t \geq 0$. Then, there exist positive constants k_1, k_2, k, and T such that if $\mu < k_1$ and $\|\mathcal{X}(0) - \mathcal{X}_{\rm ss}(\rho(0), w)\| < k_2$, then $\mathcal{X}(t)$ will be uniformly bounded for all $t \geq 0$ and*

$$\|e(t)\| \leq k\mu, \qquad \forall\ t \geq T$$

Furthermore, if $\rho(t) \to \rho_{\rm ss}$ and $\dot{\rho}(t) \to 0$ as $t \to \infty$, then

$$e(t) \to 0 \quad \text{as} \quad t \to \infty$$

◇

Proof: See Appendix C.19.

The theorem shows that if the scheduling variable is slowly varying and the initial state is sufficiently close to the equilibrium point at the initial time, then the tracking error will, eventually, be of the order of the derivative of the scheduling variable and will tend to zero if the scheduling variable approaches a constant limit.

If measurement of $\dot{y}_m$ is not available, we can use the gain-scheduled controller

$$\begin{aligned} \dot{\varphi} &= F(\rho)\varphi + G_1(\rho)e + G_2(\rho)\vartheta & (12.52)\\ \dot{\eta} &= L(\rho)\varphi + M_1(\rho)e + M_2(\rho)\vartheta & (12.53)\\ u &= \eta + M_3(\rho)e & (12.54) \end{aligned}$$

where $\dot{y}_m$ is replaced by its estimate ϑ, provided by the filter

$$\begin{aligned} \varepsilon\dot{\zeta} &= -\zeta + y_m & (12.55)\\ \vartheta &= \frac{1}{\varepsilon}(-\zeta + y_m) & (12.56) \end{aligned}$$

where ε is a "sufficiently small" positive constant and the filter is always initiated at $\zeta(0)$ such that

$$\|\zeta(0) - y_m(0)\| \leq k\varepsilon \tag{12.57}$$

for some $k > 0$. Since y_m is measured, we can always meet this initial condition. Furthermore, whenever the system is initited from an equilibrium point, the condition (12.57) is automatically satisfied, since, at equilibrium, $y_m = \zeta$. The filter (12.55)–(12.56) acts as a derivative approximator when ε is sufficiently small, as it can be seen from its transfer function

$$\frac{s}{\varepsilon s + 1} I$$

which approximates the differentiator transfer function sI for frequencies much smaller than $1/\varepsilon$.[14] The closed-loop system under the gain-scheduled controller (12.52)–(12.56) takes the singularly perturbed form

$$\begin{aligned} \dot{\mathcal{X}} &= g(\mathcal{X}, \rho, w) + N(\rho)(\vartheta - \dot{y}_m) & (12.58)\\ \varepsilon\dot{\vartheta} &= -\vartheta + \dot{y}_m & (12.59)\\ y &= h(x, w) & (12.60) \end{aligned}$$

where

$$\dot{y}_m = \frac{\partial h_m}{\partial x}(x, w) f(x, \eta + M_3(\rho)e, v, w), \qquad N = \begin{bmatrix} 0 \\ G_2 \\ M_2 \end{bmatrix}$$

---

[14] Approximating the derivative $\dot{y}_m$ can be achieved by using the high-gain observer of Section 14.5. In fact, the filter (12.55)–(12.56) is a reduced-order high-gain observer for a second-order system whose output is y_m. The condition (12.57) eliminates peaking of the transient response. If this condition cannot be enforced, the estimate ϑ should be saturated to eliminate peaking, as discussed in Section 14.5.

Setting $\varepsilon = 0$ results in $\vartheta = \dot{y}_m$ and the system (12.58)–(12.60) reduces to the system (12.45)–(12.46). The next theorem justifies the use of the filter (12.55)–(12.56) for sufficiently small ε.

Theorem 12.2 *Consider the closed-loop system* (12.58)–(12.60) *under the stated assumptions. Suppose $\rho(t)$ is continuously differentiable, $\rho(t) \in S$ (a compact subset of D_ρ), and $\|\dot{\rho}(t)\| \leq \mu$ for all $t \geq 0$. Then, there exist positive constants k_1, k_2, k_3, k, and T such that if $\mu < k_1$, $\|\mathcal{X}(0) - \mathcal{X}_{\rm ss}(\rho(0), w)\| < k_2$, and $\varepsilon < k_3$, then $\mathcal{X}(t)$ will be uniformly bounded for all $t \geq 0$ and*

$$\|e(t)\| \leq k\mu, \qquad \forall\ t \geq T$$

Furthermore, if $\rho(t) \to \rho_{\rm ss}$ and $\dot{\rho}(t) \to 0$ as $t \to \infty$, then

$$e(t) \to 0 \ \text{ as } \ t \to \infty$$

◇

Proof: See Appendix C.20.

The theorem shows that if the scheduling variable is slowly varying, the initial state is sufficiently close to the equilibrium point at the initial time, and ε is sufficiently small, then the tracking error will, eventually, be of the order of the derivative of the scheduling variable and will tend to zero if the scheduling variable approaches a constant limit.

Example 12.6 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= \tan x_1 + x_2 \\ \dot{x}_2 &= x_1 + u \\ y &= x_2 \end{aligned}$$

where y is the only measured signal; that is, $y_m = y$. We want y to track a reference signal r. We use r as the scheduling variable. When $r = \alpha =$ constant, the equilibrium equations (12.31) and (12.32) have the unique solution

$$x_{\rm ss}(\alpha) = \begin{bmatrix} -\tan^{-1}\alpha \\ \alpha \end{bmatrix}, \quad u_{\rm ss}(\alpha) = \tan^{-1}\alpha$$

We use the observer-based integral controller

$$\dot{\sigma} = e = y - r \tag{12.61}$$

$$\dot{\hat{x}} = A(\alpha)\hat{x} + Bu + H(\alpha)(y - C\hat{x}) \tag{12.62}$$

$$u = -K_1(\alpha)\hat{x} - K_2(\alpha)\sigma \tag{12.63}$$

where

$$A(\alpha) = \begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$K_1(\alpha) = \left[\ (1+\alpha^2)(3+\alpha^2)+3+\tfrac{1}{1+\alpha^2} \qquad 3+\alpha^2\ \right], \quad K_2(\alpha) = -\frac{1}{1+\alpha^2}$$

$$H(\alpha) = \left[\begin{array}{c} 10+(4+\alpha^2)(1+\alpha^2) \\ (4+\alpha^2) \end{array}\right]$$

The feedback gains $K_1(\alpha)$ and $K_2(\alpha)$ are designed to assign the closed-loop eigenvalues at -1, $-(1/2) \pm j(\sqrt{3}/2)$. The observer gain $H(\alpha)$ is designed to assign the observer eigenvalues at $-(3/2) \pm j(3\sqrt{3}/2)$. We have chosen the eigenvalues independent of α for convenience, but we could have allowed them to depend on α as long as their real parts are less than a negative number independent of α. This fixed-gain controller is a special case of (12.33)–(12.35) with $z = \hat{x}$, $F = A - BK_1 - HC$, $G_1 = -BK_2$, $G_2 = H$, $L = -K_1$, $M_1 = -K_2$, $M_2 = 0$, and $M_3 = 0$. Since $\dot{y}$ is not available, we implement the gain-scheduled controller (12.52)–(12.56) with $\varepsilon = 0.01$. Figure 12.5 shows the response of the closed-loop system to a sequence of step changes in the reference signal. A step change in the reference signal resets the equilibrium point of the system, and the initial state of the system at time 0_+ is the equilibrium state at time 0_-. If the initial state is within the region of attraction of the new equilibrium point, the system reaches steady state at that point. Since our controller is based on linearization, it guarantees only local stabilization. Therefore, in general, step changes in the reference signal will have to be limited. Reaching a large value of the reference signal can be done by a sequence of step changes, as in the figure, allowing enough time for the system to settle down after each step change. Another method to change the reference set point is to move slowly from one set point to another. Figure 12.6 shows the response of the closed-loop system to a slow ramp that takes the set point from zero to one over a period of 100 seconds. This response is consistent with our conclusions about the behavior of gain-scheduled controllers under slowly varying scheduling variables. The same figure shows the response to a faster ramp signal. As the slope of the ramp increases, tracking performance deteriorates. If we keep increasing the slope

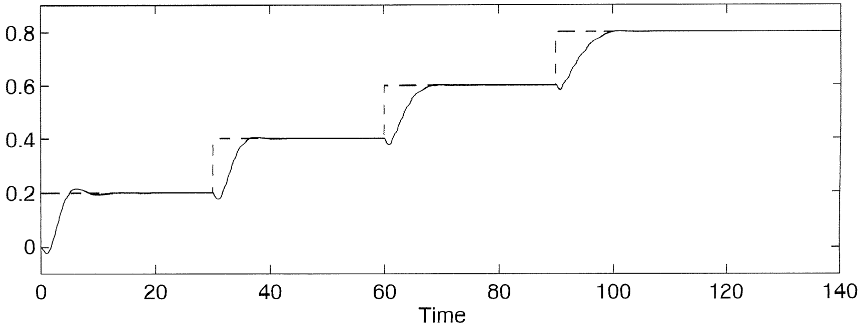

Figure 12.5: The reference (dashed) and output (solid) signals of the gain-scheduled controller of Example 12.6.

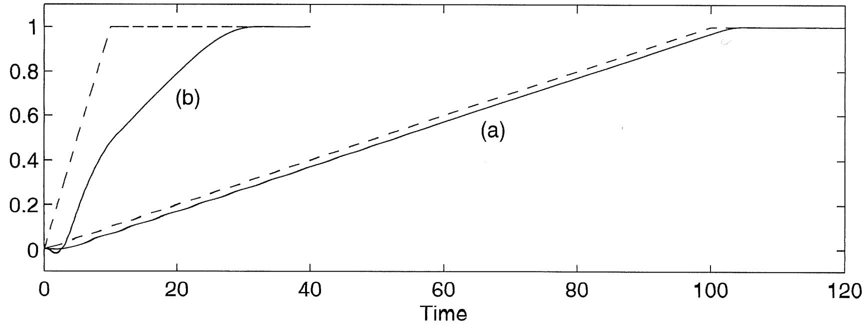

Figure 12.6: The reference (dashed) and output (solid) signals of the gain-scheduled controller of Example 12.6 with ramp reference: (a) ramp slope = 0.01; (b) ramp slope = 0.1.

of the ramp, the system will eventually go unstable. To appreciate what we gain by gain scheduling, Figure 12.7 shows the response of the closed-loop system to the same sequence of step changes of Figure 12.5 when a fixed-gain controller evaluated at $\alpha = 0$ is used. For small reference inputs, the response is as good as the one with the gain-scheduled controller, but as the reference signal increases, the performance deteriorates and the system goes unstable. Finally, to see why we may have to mod-

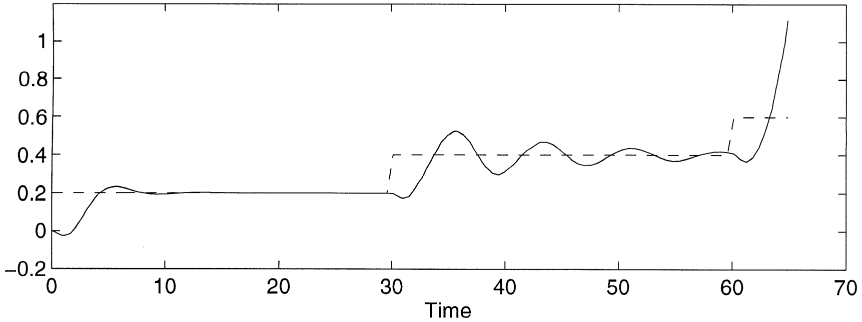

Figure 12.7: The reference (dashed) and output (solid) signals of the fixed gain controller of Example 12.6.

ify the gain-scheduled controller as in Figure 12.4, Figure 12.8 shows the response of the closed-loop system under an unmodified controller (obtained by simply replacing α by r in the foregoing controller equations) to the same sequence of step changes of Figure 12.5. While stability and zero steady-state tracking error are achieved, as predicted by our analysis, the transient response deteriorates rapidly as the reference signal increases. This is due to additional zeros in the closed-loop transfer function. Such bad transient behavior could lead to instability as it could

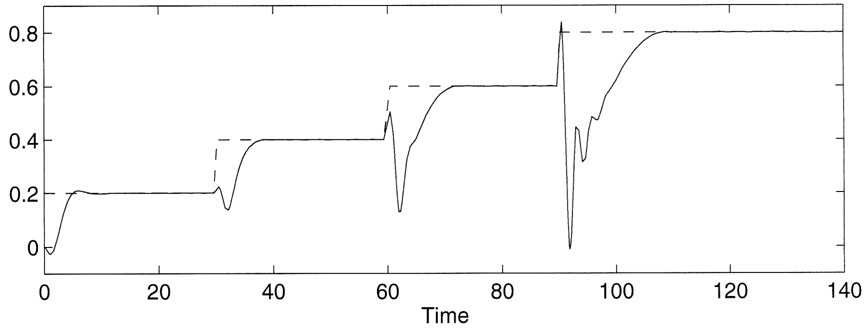

Figure 12.8: The reference (dashed) and output (solid) signals of the unmodified gain-scheduled controller of Example 12.6.

take the state of the system out of the finite region of attraction, although instability was not observed in this example. △

12.6 Exercises

12.1 Consider the closed-loop system of Example 12.2. Assume $a = c = 10$, $\delta = \pi/4$, $b = 0$, $k_1 = 2.5$, and $k_2 = 1$. Find a Lyapunov function for the system, and use it to estimate the region of attraction.

12.2 For each of the following systems, use linearization to

(a) design a state feedback controller to stabilize the origin.

(b) design an output feedback controller to stabilize the origin.

$$(1) \qquad \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_1 + x_2 \\ \dot{x}_2 & = & 3x_1^2 x_2 + x_1 + u \\ y & = & -x_1^3 + x_2 \end{array} \right.$$

$$(2) \qquad \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_1 + x_2 \\ \dot{x}_2 & = & x_1 x_2^2 - x_1 + x_3 \\ \dot{x}_3 & = & u \\ y & = & -x_1^3 + x_2 \end{array} \right.$$

$$(3) \qquad \left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_2 \\ \dot{x}_2 & = & x_1 - x_2 - x_1 x_3 + u \\ \dot{x}_3 & = & x_1 + x_1 x_2 - 2x_3 \\ y & = & x_1 \end{array} \right.$$

12.3 Let

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

where A, B, and C satisfy the rank condition (12.23). Show that $(\mathcal{A}, \mathcal{B})$ is controllable (respectively, stabilizable) if and only if (A, B) is controllable (respectively, stabilizable)

12.4 Consider the pendulum of Example 12.2.

(a) Assuming that you measure θ, but not $\dot{\theta}$, design, using linearization, an output feedback integral controller to stabilize the pendulum at an angle $\theta = \delta$.

(b) Assuming that you measure both θ and $\dot{\theta}$, design a gain-scheduled, state feedback, integral controller so that the angle θ tracks a reference angle θ_r. Study the performance of the gain-scheduled controller by computer simulation.

(c) Assuming that you measure θ, but not $\dot{\theta}$, design a gain-scheduled, observer-based, integral controller so that the angle θ tracks a reference angle θ_r. Study the performance of the gain-scheduled controller by computer simulation.

Use the following numerical data: a $= 10$, $b = 0.1$, and $c = 10$.

12.5 Consider the linear system

$$\dot{x} = A(\alpha)x + B(\alpha)u$$

where $A(\alpha)$ and $B(\alpha)$ are continuously differentiable functions of the constant vector α and $\alpha \in \Gamma$, a compact subset of R^m. Let $W(\alpha)$ be the controllability Gramian, defined by

$$W(\alpha) = \int_0^{\tau} \exp[-A(\alpha)\sigma]B(\alpha)B^T(\alpha)\exp[-A^T(\alpha)\sigma]\; d\sigma$$

for some $\tau > 0$, independent of α. Suppose (A, B) is controllable, uniformly in α, in the sense that there are positive constants c_1 and c_2, independent of α, such that

$$c_1 I \leq W(\alpha) \leq c_2 I, \quad \forall\; \alpha \in \Gamma$$

Let

$$Q(\alpha) = \int_0^{\tau} e^{-2c\sigma} \exp[-A(\alpha)\sigma]B(\alpha)B^T(\alpha)\exp[-A^T(\alpha)\sigma]\; d\sigma, \quad c > 0$$

(a) Show that

$$c_1 e^{-2c\tau} I \leq Q(\alpha) \leq c_2 I, \quad \forall\; \alpha \in \Gamma$$

(b) Let

$$u = -K(\alpha)x \overset{\text{def}}{=} -\tfrac{1}{2}B^T(\alpha)P(\alpha)x$$

where $P(\alpha) = Q^{-1}(\alpha)$. Using $V = x^T P(\alpha)x$ as a Lyapunov function candidate for

$$\dot{x} = [A(\alpha) - B(\alpha)K(\alpha)]x$$

show that $\dot{V} \le -2cV$.

(c) Show that $[A(\alpha) - B(\alpha)K(\alpha)]$ is Hurwitz uniformly in α for all $\alpha \in \Gamma$.

12.6 Show that $P(\alpha)$, defined by (12.50), is nonsingular and satisfies (12.51).

12.7 A simplified model of the low-frequency motion of a ship is given by [60]

$$\tau\ddot{\psi} + \dot{\psi} = k\delta$$

where ψ is the heading angle of the ship and δ is the rudder angle, viewed here as the control input. The time constant τ and the gain k depend on the forward speed of the ship v, according to the expressions $\tau = \tau_0 v_0/v$ and $k = k_0 v/v_0$, where τ_0, k_0, and v_0 are constants.

(a) Assuming a constant forward speed, design a state feedback integral controller so that ψ tracks a desired angle ψ_r.

(b) Use gain scheduling to compensate for varying forward speed.

12.8 The magnetic suspension system of Exercise 1.18 is modeled by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a + x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a + x_1)^2} + u\right]
\end{aligned}$$

where $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$, and $u = v$. Use the following numerical data: $m = 0.1$ kg, $k = 0.001$ N/m/sec, $g = 9.81$ m/sec^2, $a = 0.05$ m, $L_0 = 0.01$ H, $L_1 = 0.02$ H, and $R = 1\ \Omega$.

(a) Find the steady-state values I_{ss} and V_{ss} of i and v, respectively, which are needed to balance the ball at a desired position $y = r > 0$.

(b) Show that the equilibrium point obtained by taking $u = V_{ss}$ is unstable.

(c) Using linearization, design a state feedback control law to stabilize the ball at $y = 0.05$ m.

(d) Assume the permissible range of y is 0 to 0.1 m and the permissible range of the input voltage is 0 to 15 V. Starting with the ball at equilibrium, move it a small distance up (and then down) and let it go. Repeat this experiment, gradually increasing the amount of initial disturbance. Using simulation, determine the largest range of initial disturbance for which the ball will return to the equilibrium point without violating the constraints on y and v. To account for the constraint on v, include a limiter in your simulation.

(e) Using simulation, investigate the effect of perturbations in the mass m. Simulate the closed-loop system with the nominal controller, but with the mass changing from its nominal value. Find the range of m for which the controller will still balance the ball and investigate the steady-state error.

(f) Repeat the design of part (c) using integral control. Repeat parts (d) and (e) for this design. Comment on the effect of integral control on the transient response and steady-state error.

(g) Repeat the design of part (c) assuming you can only measure y. Repeat parts (d) and (e) for this design.

(h) Repeat the design of part (c) assuming you can only measure y and i. Repeat parts (d) and (e) for this design.

(i) Repeat the integral control design of part (f) assuming you can only measure y. Repeat parts (d) and (e) for this design.

(j) Repeat the integral control design of part (f) assuming you can only measure y and i. Repeat parts (d) and (e) for this design.

(k) Design a gain-scheduled, observer-based, integral controller so that the ball position y tracks a reference position r. Assume you can measure y and i. Using simulation, study the performance of the gain-scheduled controller when r changes slowly from 0.03 to 0.07.

(l) If you can only measure i, can you design a linear output feedback control law to stabilize the ball at $y = r$? Can you design a linear output feedback integral controller?

12.9 A field-controlled DC motor is described in Exercise 1.17. When the field circuit is driven by a current source, we can view the field current as the control input and model the system by the second-order state model

$$\begin{aligned} \dot{x}_1 &= -\theta_1 x_1 - \theta_2 x_2 u + \theta_3 \\ \dot{x}_2 &= -\theta_4 x_2 + \theta_5 x_1 u \\ y &= x_2 \end{aligned}$$

where x_1 is the armature current, x_2 is the speed, u is the field current, and θ_1 to θ_5 are positive constants. It is required to design a speed control system so that y asymptotically tracks a constant speed reference r. It is assumed that $r^2 < \theta_3^2\theta_5/4\theta_1\theta_2\theta_4$ and the domain of operation is restricted to $x_1 > \theta_3/2\theta_1$.

(a) Find the steady-state input u_{ss} needed to maintain the output r. Verify that that the open-loop control $u = u_{ss}$ results in an exponentially stable equilibrium point.

(b) Starting with the motor at rest ($y = 0$), apply a small step change in the reference signal and simulate the response. Repeat this experiment, gradually increasing the amount of step change. Determine the largest range of initial step for which the motor will reach steady-state at the desired speed.

(c) Using computer simulation, study the performance of the system when the rotor inertia changes by ±50%.

(d) Using linearization, design a state feedback integral controller to achieve the desired speed regulation. Repeat parts (b) and (c) for this controller and compare its performance with the open-loop controller of part (a).

(e) Suppose you measure the speed x_2, but not the armature current x_1. Repeat part (d) by using an observer to estimate the armature current. Repeat parts (b) and (c) for this controller and compare its performance with the one designed in part (d).

(f) Design a gain-scheduled, observer-based, integral controller so that the speed x_2 tracks a reference speed r

In parts (b) through (e), use the following numerical data: $\theta_1 = 60$, $\theta_2 = 0.5$, $\theta_3 = 40$, $\theta_4 = 6$, and $\theta_5 = 4 \times 10^4$.

12.10 Consider the inverted pendulum of Exercise 1.15.

(a) Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as state variables and $u = F$ as control input, write down the state equation.

(b) Show that the open-loop system has an equilibrium set.

(c) Suppose we want to stabilize the pendulum at the vertical position ($\theta = 0$). Find an open-loop equilibrium point at which $\theta = 0$, and show that it is unstable.

(d) Linearize the nonlinear state equation at the desired equilibrium point, and verify that the linearized state equation is controllable.

(e) Using linearization, design a state feedback control law to stabilize the system at the desired equilibrium point.

(f) Using computer simulation, study the transient behavior and the effect of ±20% perturbation in the mass of the pendulum and its moment of inertia.

(g) Starting with the pendulum at equilibrium, move it a small angle to the right (and then to the left) and let it go. Repeat this experiment, gradually increasing the amount of initial disturbance. Using simulation, determine the largest range of initial disturbance for which the pendulum will return to equilibrium.

(h) Suppose you can only measure the angle θ and the cart position y. Using linearization, design an output feedback controller to stabilize the pendulum at $\theta = 0$. Repeat parts (f) and (g) for this controller.

(i) Repeat part (h) if it is desired to stabilize the pendulum at an angle $\theta = \theta_r$, where $-\pi/2 < \theta_r < \pi/2$.

In parts (e) to (i), use the following numerical data: $m = 0.1$ kg, $M = 1$ kg, $k = 0.1$ N/m/sec, $I = 0.025/3$ kg m^2, $g = 9.81$ m/sec^2, and $L = 0.5$ m.

Chapter 13

Feedback Linearization

We consider a class of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned}$$

and pose the question of whether there exist a state feedback control

$$u = \alpha(x) + \beta(x)v$$

and a change of variables

$$z = T(x)$$

that transform the nonlinear system into an equivalent linear system. In Section 13.1, we motivate the idea by simple examples and introduce the notions of *full-state linearization*, where the state equation is completely linearized, and *input–output linearization*, where the input–output map is linearized, while the state equation may be only partially linearized. In Section 13.2, we study input–output linearization, introducing the notions of relative degree, zero dynamics, and minimum phase systems. In Section 13.3, we characterize the class of nonlinear systems that can be feedback linearized. To simplify the presentation, Sections 13.2 and 13.3 deal only with single-input–single-output systems. State feedback control of feedback (or partially feedback) linearizable systems is discussed in Section 13.4, where we deal with both stabilization and tracking.

13.1 Motivation

To introduce the idea of feedback linearization, let us start with the problem of stabilizing the origin of the pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

Inspection of the state equation shows that we can choose u as

$$u = \frac{a}{c}[\sin(x_1 + \delta) - \sin\delta] + \frac{v}{c}$$

to cancel the nonlinear term $a[\sin(x_1 + \delta) - \sin\delta]$. This cancellation results in the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 + v \end{aligned}$$

Thus, the stabilization problem for the nonlinear system has been reduced to a stabilization problem for a controllable linear system. We can proceed to design a stabilizing linear state feedback control

$$v = -k_1x_1 - k_2x_2$$

to locate the eigenvalues of the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1x_1 - (k_2 + b)x_2 \end{aligned}$$

in the open left-half plane. The overall state feedback control law is given by

$$u = \left(\frac{a}{c}\right)[\sin(x_1 + \delta) - \sin\delta] - \frac{1}{c}(k_1x_1 + k_2x_2)$$

How general is this idea of nonlinearity cancellation? Clearly, we should not expect to be able to cancel nonlinearities in every nonlinear system. There must be a certain structural property of the system that allows us to perform such cancellation. It is not hard to see that to cancel a nonlinear term $\alpha(x)$ by subtraction, the control u and the nonlinearity $\alpha(x)$ must always appear together as a sum $u + \alpha(x)$. To cancel a nonlinear term $\gamma(x)$ by division, the control u and the nonlinearity $\gamma(x)$ must always appear as a product $\gamma(x)u$. If the matrix $\gamma(x)$ is nonsingular in the domain of interest, then it can be cancelled by $u = \beta(x)v$, where $\beta(x) = \gamma^{-1}(x)$ is the inverse of the matrix $\gamma(x)$. Therefore, the ability to use feedback to convert a nonlinear state equation into a controllable linear state equation by cancelling nonlinearities requires the nonlinear state equation to have the structure

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \tag{13.1}$$

where A is $n \times n$, B is $n \times p$, the pair (A, B) is controllable, the functions $\alpha : R^n \rightarrow R^p$ and $\gamma : R^n \rightarrow R^{p\times p}$ are defined in a domain $D \subset R^n$ that contains the origin, and the matrix $\gamma(x)$ is nonsingular for every $x \in D$. If the state equation takes the form (13.1), then we can linearize it via the state feedback

$$u = \alpha(x) + \beta(x)v \tag{13.2}$$

where $\beta(x) = \gamma^{-1}(x)$, to obtain the linear state equation

$$\dot{x} = Ax + Bv \tag{13.3}$$

For stabilization, we design $v = -Kx$ such that $A - BK$ is Hurwitz. The overall nonlinear stabilizing state feedback control is

$$u = \alpha(x) - \beta(x)Kx \tag{13.4}$$

Suppose the nonlinear state equation does not have the structure of (13.1). Does this mean we cannot linearize the system via feedback? The answer is no. Recall that the state model of a system is not unique. It depends on the choice of the state variables. Even if the state equation does not have the structure of (13.1) for one choice of state variables, it might do so for another choice. Consider, for example, the system

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= -x_1^2 + u \end{aligned}$$

We cannot simply choose u to cancel the nonlinear term $a \sin x_2$. However, if we first change the variables by the transformation

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= a \sin x_2 = \dot{x}_1 \end{aligned}$$

then z_1 and z_2 satisfy

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a \cos x_2 \, (-x_1^2 + u) \end{aligned}$$

and the nonlinearities can be cancelled by the control

$$u = x_1^2 + \frac{1}{a \cos x_2} v$$

which is well defined for $-\pi/2 < x_2 < \pi/2$. The state equation in the new coordinates (z_1, z_2) can be found by inverting the transformation to express (x_1, x_2) in terms of (z_1, z_2); that is,

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \sin^{-1}\left(\frac{z_2}{a}\right) \end{aligned}$$

which is well defined for $-a < z_2 < a$. The transformed state equation is given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a \cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right) (-z_1^2 + u) \end{aligned}$$

When a change of variables $z = T(x)$ is used to transform the state equation from the x-coordinates to the z-coordinates, the map T must be invertible; that is, it must have an inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$, where D is the domain of T. Moreover, because the derivatives of z and x should be continuous, we require both $T(\cdot)$ and $T^{-1}(\cdot)$ to be continuously differentiable. A continuously differentiable map with a continuously differentiable inverse is known as a *diffeomorphism*. If the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at a point $x_0 \in D$, then it follows from the inverse function theorem[1] that there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N. A map T is said to be a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$.[2] Now we have all the elements we need to define feedback linearizable systems.

Definition 13.1 *A nonlinear system*

$$\dot{x} = f(x) + G(x)u \tag{13.5}$$

where $f : D \to R^n$ and $G : D \to R^{n\times p}$ are sufficiently smooth[3] *on a domain $D \subset R^n$, is said to be feedback linearizable (or input–state linearizable) if there exists a diffeomorphism $T : D \to R^n$ such that $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms the system* (13.5) *into the form*

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)] \tag{13.6}$$

with (A, B) controllable and $\gamma(x)$ nonsingular for all $x \in D$.

When certain output variables are of interest, as in tracking control problems, the state model is described by state and output equations. Linearizing the state equation does not necessarily linearize the output equation. For example, if the system

$$\begin{aligned}\dot{x}_1 &= a\sin x_2\\ \dot{x}_2 &= -x_1^2 + u\end{aligned}$$

has an output $y = x_2$, then the change of variables and state feedback control

$$z_1 = x_1, \qquad z_2 = a\sin x_2, \qquad \text{and} \quad u = x_1^2 + \frac{1}{a\cos x_2}v$$

yield

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= v\\ y &= \sin^{-1}\left(\frac{z_2}{a}\right)\end{aligned}$$

---

[1] See [10, Theorem 7-5].

[2] T is a global diffeomorphism if and only if $[\partial T/\partial x]$ is nonsingular for all $x \in R^n$ and T is proper; that is, $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$. (See [165] or [212] for a proof of this statement.)

[3] By "sufficiently smooth," we mean that all the partial derivatives, that will appear later on, are defined and continuous.

While the state equation is linear, solving a tracking control problem for y is still complicated by the nonlinearity of the output equation. Inspection of both the state and output equations in the x-coordinates shows that, if we use the state feedback control $u = x_1^2 + v$, we can linearize the input–output map from u to y, which will be described by the linear model

$$\begin{aligned} \dot{x}_2 &= v \\ y &= x_2 \end{aligned}$$

We can now proceed to solve the tracking control problem using linear control theory. This discussion shows that sometimes it is more beneficial to linearize the input–output map even at the expense of leaving part of the state equation nonlinear. In this case, the system is said to be input–output linearizable. One catch about input–output linearization is that the linearized input–output map may not account for all the dynamics of the system. In the foregoing example, the full system is described by

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= v \\ y &= x_2 \end{aligned}$$

Note that the state variable x_1 is not connected to the output y. In other words, the linearizing feedback control has made x_1 unobservable from y. When we design tracking control, we should make sure that the variable x_1 is well behaved; that is, stable or bounded in some sense. A naive control design that uses only the linear input–output map may result in an ever-growing signal $x_1(t)$. For example, suppose we design a linear control to stabilize the output y at a constant value r. Then, $x_1(t) = x_1(0) + t\ a \sin r$ and, for $\sin r \neq 0$, $x_1(t)$ will grow unbounded. This internal stability issue will be addressed by using the concept of zero dynamics.

13.2 Input–Output Linearization

Consider the single-input–single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u && (13.7) \\ y &= h(x) && (13.8) \end{aligned}$$

where f, g, and h are sufficiently smooth in a domain $D \subset R^n$. The mappings $f : D \rightarrow R^n$ and $g : D \rightarrow R^n$ are called vector fields on D. The derivative $\dot{y}$ is given by

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\ u$$

where

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is called the *Lie Derivative* of h with respect to f or along f. This is the familiar notion of the derivative of h along the trajectories of the system $\dot{x} = f(x)$. The new notation is convenient when we repeat the calculation of the derivative with respect to the same vector field or a new one. For example, the following notation is used:

$$\begin{aligned} L_gL_fh(x) &= \frac{\partial(L_fh)}{\partial x}g(x) \\ L_f^2h(x) &= L_fL_fh(x) = \frac{\partial(L_fh)}{\partial x}f(x) \\ L_f^kh(x) &= L_fL_f^{k-1}h(x) = \frac{\partial(L_f^{k-1}h)}{\partial x}f(x) \\ L_f^0h(x) &= h(x) \end{aligned}$$

If $L_gh(x) = 0$, then $\dot{y} = L_fh(x)$, independent of u. If we continue to calculate the second derivative of y, denoted by $y^{(2)}$, we obtain

$$y^{(2)} = \frac{\partial(L_fh)}{\partial x}[f(x) + g(x)u] = L_f^2h(x) + L_gL_fh(x)\ u$$

Once again, if $L_gL_fh(x) = 0$, then $y^{(2)} = L_f^2h(x)$, independent of u. Repeating this process, we see that if $h(x)$ satisfies

$$L_gL_f^{i-1}h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_gL_f^{\rho-1}h(x) \neq 0$$

then u does not appear in the equations of y, $\dot{y}$, $\ldots$, $y^{(\rho-1)}$ and appears in the equation of $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)} = L_f^\rho h(x) + L_gL_f^{\rho-1}h(x)\ u$$

The foregoing equation shows clearly that the system is input–output linearizable, since the state feedback control

$$u = \frac{1}{L_gL_f^{\rho-1}h(x)}\left[-L_f^\rho h(x) + v\right]$$

reduces the input–output map to

$$y^{(\rho)} = v$$

which is a chain of ρ integrators. In this case, the integer ρ is called the *relative degree* of the system, according to the following definition:

Definition 13.2 *The nonlinear system* (13.7)–(13.8) *is said to have relative degree* ρ, $1 \leq \rho \leq n$, *in a region* $D_0 \subset D$ *if*

$$L_gL_f^{i-1}h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_gL_f^{\rho-1}h(x) \neq 0 \tag{13.9}$$

for all $x \in D_0$.

Example 13.1 Consider the controlled van der Pol equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad \varepsilon > 0\end{aligned}$$

with output $y = x_1$. Calculating the derivatives of the output, we obtain

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2\\ \ddot{y} &= \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u\end{aligned}$$

Hence, the system has relative degree two in R^2. For the output $y = x_2$,

$$\dot{y} = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$

and the system has relative degree one in R^2. For the output $y = x_1 + x_2^2$,

$$\dot{y} = x_2 + 2x_2[-x_1 + \varepsilon(1 - x_1^2)x_2 + u]$$

and the system has relative degree one in $D_0 = \{x \in R^2 \mid x_2 \neq 0\}$. △

Example 13.2 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2 + u\\ y &= x_1\end{aligned}$$

Calculating the derivatives of y, we obtain

$$\dot{y} = \dot{x}_1 = x_1 = y$$

Consequently, $y^{(n)} = y = x_1$ for all $n \geq 1$. In this case, the system does not have a well-defined relative degree. Because the example is simple, it is not difficult to see why this is so: The output $y(t) = x_1(t) = e^t x_1(0)$ is independent of the input u. △

Example 13.3 A field-controlled DC motor with negligible shaft damping can be modeled by the state equation (Exercise 1.17)

$$\begin{aligned}\dot{x}_1 &= -ax_1 + u\\ \dot{x}_2 &= -bx_2 + k - cx_1x_3\\ \dot{x}_3 &= \theta x_1 x_2\end{aligned}$$

where x_1, x_2, and x_3 are the field current, armature current, and angular velocity, respectively, and a, b, c, k, and θ are positive constants. For speed control, we choose the output as $y = x_3$. The derivatives of the output are given by

$$\begin{aligned}\dot{y} &= \dot{x}_3 = \theta x_1 x_2\\ \ddot{y} &= \theta x_1 \dot{x}_2 + \theta \dot{x}_1 x_2 = (\cdot) + \theta x_2 u\end{aligned}$$

where $(\cdot)$ contains terms which are functions of x. The system has relative degree two in the region $D_0 = \{x \in R^3 \mid x_2 \neq 0\}$. △

Example 13.4 Consider a linear system represented by the transfer function

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

where $m < n$ and $b_m \neq 0$. A state model for the system can be taken as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & \ldots & & & \ldots & 0 \\ 0 & 0 & 1 & \ldots & & & \ldots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_m & \ldots & \ldots & -a_{n-1} \end{array} \right]_{n\times n}, \quad B = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \\ 1 \end{array} \right]_{n\times 1}$$

$$C = \left[\begin{array}{ccccccccc} b_0 & b_1 & \ldots & \ldots & b_m & 0 & \ldots & 0 \end{array} \right]_{1\times n}$$

This linear state model is a special case of (13.7)–(13.8), where $f(x) = Ax$, $g = B$, and $h(x) = Cx$. To check the relative degree of the system, we calculate the derivatives of the output. The first derivative is

$$\dot{y} = CAx + CBu$$

If $m = n - 1$, then $CB = b_{n-1} \neq 0$ and the system has relative degree one. Otherwise, $CB = 0$ and we continue to calculate the second derivative $y^{(2)}$. Noting that CA is a row vector obtained by shifting the elements of C one position to the right, while CA^2 is obtained by shifting the elements of C two positions to the right, and so on, we see that

$$CA^{i-1}B = 0, \quad \text{for } i = 1, 2, \ldots, n-m-1, \quad \text{and} \quad CA^{n-m-1}B = b_m \neq 0$$

Thus, u appears first in the equation of $y^{(n-m)}$, given by

$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu$$

and the relative degree of the system is $n - m$ (the difference between the degrees of the denominator and numerator polynomials of $H(s)$).[4] △

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[4] The terminology "relative degree" of a nonlinear system is consistent with the use of the term relative degree in linear control theory, which is defined as $n - m$.

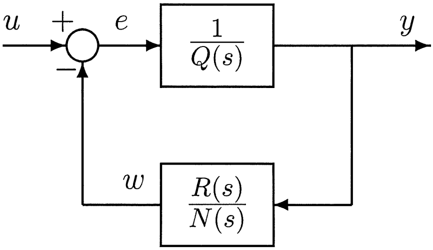

Figure 13.1: Feedback representation of $H(s)$.

To probe further into the control of input–output linearizable systems and issues of internal stability, let us start with the linear system of the foregoing example. The transfer function $H(s)$ can be written as

$$H(s) = \frac{N(s)}{D(s)}$$

where $\deg D = n$ and $\deg N = m < n$. The relative degree $\rho = n - m$. By Euclidean division, we can write $D(s)$ as

$$D(s) = Q(s)N(s) + R(s)$$

where $Q(s)$ and $R(s)$ are the quotient and remainder polynomials, respectively. From Euclidean division rules, we know that

$$\deg Q = n - m = \rho, \qquad \deg R < m$$

and the leading coefficient of $Q(s)$ is $1/b_m$. With this representation of $D(s)$, we can rewrite $H(s)$ as

$$H(s) = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)}\ \frac{R(s)}{N(s)}}$$

Thus, $H(s)$ can be represented as a negative feedback connection with $1/Q(s)$ in the forward path and $R(s)/N(s)$ in the feedback path. (See Figure 13.1.) The ρth-order transfer function $1/Q(s)$ has no zeros and can be realized by the ρth-order state vector

$$\xi = [\ y, \ \ \dot{y}, \ \ \ldots, \ \ y^{(\rho-1)}\ ]^T$$

to obtain the state model

$$\begin{aligned} \dot{\xi} &= (A_c + B_c\lambda^T)\xi + B_c b_m e \\ y &= C_c\xi \end{aligned}$$

where (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators; that is,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{13.10}$$

and $\lambda \in R^\rho$. Let (A_0, B_0, C_0) be a minimal realization of the transfer function $R(s)/N(s)$; that is,

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 y \\ w &= C_0\eta \end{aligned}$$

The eigenvalues of A_0 are the zeros of the polynomial $N(s)$, which are the zeros of the transfer function $H(s)$. From the feedback connection, we see that $H(s)$ can be realized by the state model

$$\dot{\eta} = A_0\eta + B_0 C_c \xi \tag{13.11}$$

$$\dot{\xi} = A_c\xi + B_c(\lambda^T\xi - b_m C_0\eta + b_m u) \tag{13.12}$$

$$y = C_c\xi \tag{13.13}$$

Using the special structure of (A_c, B_c, C_c), it is straightforward to verify that

$$y^{(\rho)} = \lambda^T\xi - b_m C_0\eta + b_m u$$

The (input–output linearizing) state feedback control

$$u = \frac{1}{b_m}[-\lambda^T\xi + b_m C_0\eta + v]$$

results in the system

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 C_c\xi \\ \dot{\xi} &= A_c\xi + B_c v \\ y &= C_c\xi \end{aligned}$$

whose input–output map is a chain of ρ integrators, and whose state subvector η is unobservable from the output y. Suppose we want to stabilize the output at a constant reference r. This requires stabilizing ξ at $\xi^* = (r, 0, \dots, 0)^T$. Shifting the equilibrium point to the origin by the change of variables $\zeta = \xi - \xi^*$ reduces the problem to a stabilization problem for $\dot{\zeta} = A_c\zeta + B_c v$. Taking $v = -K\zeta = -K(\xi - \xi^*)$, where $A_c - B_c K$ is Hurwitz, completes the design of the control law as

$$u = \frac{1}{b_m}[-\lambda^T\xi + b_m C_0\eta - K(\xi - \xi^*)]$$

The corresponding closed-loop system is given by

$$\begin{aligned}\dot{\eta} &= A_0\eta + B_0C_c(\xi^* + \zeta)\\ \dot{\zeta} &= (A_c - B_cK)\zeta\end{aligned}$$

Because $A_c - B_cK$ is Hurwitz, for any initial state $\zeta(0)$, we have $\zeta(t) \to 0$ as $t \to \infty$. Consequently, $y(t) \to r$ as $t \to \infty$. What about η? Equation (13.11) is driven by $y = C_c\xi$ as input. To ensure that $\eta(t)$ will be bounded for all possible waveforms of $y(t)$ and all possible initial states $\eta(0)$, we must require A_0 to be Hurwitz. Equivalently, the zeros of $H(s)$ must lie in the open left-half plane. A transfer function having all zeros in the open-left half plane is called *minimum phase.* From a pole placement viewpoint, the state feedback control, we have just designed via input–output linearization, assigns the closed-loop eigenvalues into two groups: ρ eigenvalues are assigned in the open-left half plane as the eigenvalues of $A_c - B_cK$, and $n - \rho$ eigenvalues are assigned at the open-loop zeros.[5]

Our analysis of the linear system of Example 13.4 sheds some light on the meaning of the state feedback control that reduces the input–output map to a chain of integrators and how to characterize internal stability. The key tool that allowed us to develop this understanding is the state model (13.11)–(13.13). Our next task is to develop a nonlinear version of (13.11)–(13.13) for the nonlinear system (13.7)–(13.8) when it has relative degree ρ. The ξ variables are taken the same as in the linear case, since the input–output map will still be a chain of ρ integrators. We would like to choose the η variables to produce a nonlinear version of (13.11). The key feature of (13.11) is the absence of the control input u. A change of variables that would transform (13.7)–(13.8) into a nonlinear version of (13.11)–(13.13) can be taken as

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{13.14}$$

where ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_0 \subset D$ and

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho, \ \forall\ x \in D_0 \tag{13.15}$$

The next theorem shows that ϕ_1 to $\phi_{n-\rho}$ exist, at least locally.

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[5] It should be noted that stabilizing the output at a constant reference does not require the system to be minimum phase. This requirement is a consequence of our choice to assign some of the closed-loop eigenvalues at the open-loop zeros.

Theorem 13.1 *Consider the system* (13.7)–(13.8)*, and suppose it has relative degree* $\rho \leq n$ *in* D*. If* $\rho = n$*, then for every* $x_0 \in D$*, a neighborhood* N *of* x_0 *exists such that the map*

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

restricted to N*, is a diffeomorphism on* N*. If* $\rho < n$*, then, for every* $x_0 \in D$*, a neighborhood* N *of* x_0 *and smooth functions* $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ *exist such that* (13.15) *is satisfied for all* $x \in N$ *and the map* $T(x)$ *of* (13.14)*, restricted to* N*, is a diffeomorphism on* N*.* ◇

Proof: See Appendix C.21.

The condition (13.15) ensures that when we calculate

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u]$$

the u term cancels out. It is now easy to verify that the change of variables (13.14) transforms (13.7)–(13.8) into

$$\dot{\eta} = f_0(\eta, \xi) \tag{13.16}$$

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)] \tag{13.17}$$

$$y = C_c\xi \tag{13.18}$$

where $\xi \in R^\rho$, $\eta \in R^{n-\rho}$, (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators,

$$f_0(\eta, \xi) = \left. \frac{\partial \phi}{\partial x} f(x) \right|_{x=T^{-1}(z)} \tag{13.19}$$

$$\gamma(x) = L_g L_f^{\rho-1} h(x) \quad \text{and} \quad \alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)} \tag{13.20}$$

We have kept α and γ in (13.17), expressed in the original coordinates. These functions are uniquely determined by (13.20) in terms of f, g, and h. They are independent of the choice of ϕ. They can be expressed in the new coordinates by setting

$$\alpha_0(\eta, \xi) = \alpha\left(T^{-1}(z)\right) \quad \text{and} \quad \gamma_0(\eta, \xi) = \gamma\left(T^{-1}(z)\right)$$

which, of course, will depend on the choice of ϕ. In this case, (13.17) can be rewritten as

$$\dot{\xi} = A_c\xi + B_c\gamma_0(\eta, \xi)[u - \alpha_0(\eta, \xi)]$$

If x^* is an open-loop equilibrium point of (13.7), then (η^*, ξ^*), defined by

$$\eta^* = \phi(x^*), \quad \xi^* = \begin{bmatrix} h(x^*) & 0 & \cdots & 0 \end{bmatrix}$$

is an equilibrium point of (13.16)–(13.17). If y vanishes at $x = x^*$, that is, $h(x^*) = 0$, we can transform x^* into the origin point ($\eta = 0$, $\xi = 0$) by choosing $\phi(x)$ such that $\phi(x^*) = 0$.

Equations (13.16) through (13.18) are said to be in the *normal form.* This form decomposes the system into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$u = \alpha(x) + \beta(x)v$$

where $\beta(x) = \gamma^{-1}(x)$, while the internal part is made unobservable by the same control. The internal dynamics are described by (13.16). Setting $\xi = 0$ in that equation results in

$$\dot{\eta} = f_0(\eta, 0) \tag{13.21}$$

which is called the *zero dynamics*, a name that matches nicely with the fact that for linear systems, (13.21) is given by $\dot{\eta} = A_0\eta$, where the eigenvalues of A_0 are the zeros of the transfer function $H(s)$. The system is said to be *minimum phase* if (13.21) has an asymptotically stable equilibrium point in the domain of interest. In particular, if $T(x)$ is chosen such that the origin ($\eta = 0$, $\xi = 0$) is an equilibrium point of (13.16)–(13.18), then the system is said to be minimum phase if the origin of the zero dynamics (13.21) is asymptotically stable. It is useful to know that the zero dynamics can be characterized in the original coordinates. Noting that

$$y(t) \equiv 0 \Rightarrow \xi(t) \equiv 0 \Rightarrow u(t) \equiv \alpha(x(t))$$

we see that if the output is identically zero, the solution of the state equation must be confined to the set

$$Z^* = \{x \in D_0 \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

and the input must be

$$u = u^*(x) \overset{\text{def}}{=} \alpha(x)|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \overset{\text{def}}{=} [f(x) + g(x)\alpha(x)]_{x \in Z^*}$$

In the special case $\rho = n$, the normal form (13.16)–(13.18) reduces

$$\dot{z} = A_c z + B_c \gamma(x)[u - \alpha(x)] \tag{13.22}$$

$$y = C_c z \tag{13.23}$$

where $z = \xi = [h(x), \cdots, L_f^{n-1} h(x)]^T$ and the η variable does not exist. In this case, the system has no zero dynamics and, by default, is said to be minimum phase.

Example 13.5 Consider the controlled van der Pol equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1+\varepsilon(1-x_1^2)x_2+u\\ y &= x_2\end{aligned}$$

We have seen in Example 13.1 that the system has relative degree one in R^2. Taking $\xi = y$ and $\eta = x_1$, we see that the system is already in the normal form. The zero dynamics are given by $\dot{x}_1 = 0$, which does not have an asymptotically stable equilibrium point. Hence, the system is not minimum phase. △

Example 13.6 The system

$$\begin{aligned}\dot{x}_1 &= -x_1+\frac{2+x_3^2}{1+x_3^2}\,u\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_1x_3+u\\ y &= x_2\end{aligned}$$

has an open-loop equilibrium point at the origin. The derivatives of the output are

$$\begin{aligned}\dot{y} &= \dot{x}_2 = x_3\\ \ddot{y} &= \dot{x}_3 = x_1x_3+u\end{aligned}$$

Therefore, the system has relative degree two in R^3. Using $L_gL_fh(x) = 1$ and $L_f^2h(x) = x_1x_3$ in (13.20), we obtain

$$\gamma = 1 \quad\text{and}\quad \alpha(x) = -x_1x_3$$

To characterize the zero dynamics, restrict x to

$$Z^* = \{x \in R^3 \mid x_2 = x_3 = 0\}$$

and take $u = u^*(x) = 0$. This process yields

$$\dot{x}_1 = -x_1$$

which shows that the system is minimum phase. To transform it into the normal form, we want to choose a function $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial\phi}{\partial x}g(x) = 0$$

and

$$T(x) = \left[\begin{array}{ccc}\phi(x) & x_2 & x_3\end{array}\right]^T$$

is a diffeomorphism on some domain containing the origin. The partial differential equation

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2+x_3^2}{1+x_3^2} + \frac{\partial \phi}{\partial x_3} = 0$$

can be solved by separating variables to obtain

$$\phi(x) = -x_1 + x_3 + \tan^{-1} x_3$$

which satisfies the condition $\phi(0) = 0$. The mapping $T(x)$ is a global diffeomorphism, as can be seen by the fact that for any $z \in R^3$, the equation $T(x) = z$ has a unique solution. Thus, the normal form

$$\begin{aligned}
\dot{\eta} &= \underbrace{\left(-\eta + \xi_2 + \tan^{-1} \xi_2\right)}_{-x_1} \left(1 + \frac{2+\xi_2^2}{1+\xi_2^2}\xi_2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \left(-\eta + \xi_2 + \tan^{-1} \xi_2\right) \xi_2 + u \\
y &= \xi_1
\end{aligned}$$

is defined globally. △

Example 13.7 The field-controlled DC motor of Example 13.3 has relative degree two in $D_0 = \{x \in R^3 \mid x_2 \neq 0\}$. Using (13.20), we obtain

$$\gamma = \theta x_2 \quad \text{and} \quad \alpha(x) = -\frac{\theta x_2(-a x_1) + \theta x_1(-b x_2 + k - c x_1 x_3)}{\theta x_2}$$

To characterize the zero dynamics, restrict x to

$$Z^* = \{x \in D_0 \mid x_3 = 0 \text{ and } x_1 x_2 = 0\} = \{x \in D_0 \mid x_3 = 0 \text{ and } x_1 = 0\}$$

and take $u = u^*(x) = 0$, to obtain

$$\dot{x}_2 = -b x_2 + k$$

The zero dynamics have an asymptotically stable equilibrium point at $x_2 = k/b$. Hence, the system is minimum phase. To transform it into the normal form, we want to find a function $\phi(x)$ such that $[\partial\phi/\partial x]g = \partial\phi/\partial x_1 = 0$ and $T = [\phi(x), x_3, \theta x_1 x_2]^T$ is a diffeomorphism on some domain $D_x \subset D_0$. The choice $\phi(x) = x_2 - k/b$ satisfies $\partial\phi/\partial x_1 = 0$, makes $T(x)$ a diffeomorphism on $D_x = \{x \in R^3 \mid x_2 > 0\}$, and transforms the equilibrium point of the zero dynamics to the origin. △

Example 13.8 Consider a single-input–single-output nonlinear system represented by the nth-order differential equation

$$\begin{aligned}
y^{(n)} &= p\left(z, z^{(1)}, \ldots, z^{(m-1)}, y, y^{(1)}, \ldots, y^{(n-1)}\right) \\
&\quad + q\left(z, z^{(1)}, \ldots, z^{(m-1)}, y, y^{(1)}, \ldots, y^{(n-1)}\right) z^{(m)}, \quad m < n \qquad (13.24)
\end{aligned}$$

where z is the input, y is the output, $p(\cdot)$ and $q(\cdot)$ are sufficiently smooth functions in a domain of interest, and $q(\cdot) \neq 0$. This nonlinear input–output model reduces to the transfer function model of Example 13.4 for linear systems. We extend the dynamics of the system by adding a series of m integrators at the input side and define $u = z^{(m)}$ as the control input of the extended system.[6] The extended system is of order $(n+m)$. A state model of the extended system can be obtained by taking the state variables as

$$\zeta = \begin{bmatrix} z \\ z^{(1)} \\ \vdots \\ z^{(m-1)} \end{bmatrix}, \quad \xi = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ \vdots \\ y^{(n-1)} \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} \zeta \\ \xi \end{bmatrix}$$

The state model is given by

$$\begin{aligned} \dot{\zeta} &= A_u \zeta + B_u u \\ \dot{\xi} &= A_c \xi + B_c [p(x) + q(x) u] \\ y &= C_c \xi \end{aligned}$$

where (A_c, B_c, C_c) is a canonical form representation of a chain of n integrators and (A_u, B_u) is a controllable canonical pair that represents a chain of m integrators. Let $D \subset R^{n+m}$ be a domain over which p and q are sufficiently smooth and $q \neq 0$. Using the special structure of (A_c, B_c, C_c), it can be easily seen that

$$y^{(i)} = C_c A_c^i \xi, \quad \text{for} \quad 1 \le i \le n-1, \quad \text{and} \quad y^{(n)} = p(x) + q(x) u$$

Hence, the system has relative degree n. To find the zero dynamics, notice that $L_f^{i-1} h(x) = \xi_i$. Consequently, $Z^* = \{x \in R^{n+m} \mid \xi = 0\}$ and $u^*(x) = -p(x)/q(x)$ evaluated at $\xi = 0$. Thus, the zero dynamics are given by

$$\dot{\zeta} = A_u \zeta + B_u u^*(x)$$

Recalling the definition of ζ, it can be easily seen that $\zeta_1 = z$ satisfies the mth-order differential equation

$$0 = p\left(z, z^{(1)}, \ldots, z^{(m-1)}, 0, 0, \ldots, 0\right) + q\left(z, z^{(1)}, \ldots, z^{(m-1)}, 0, 0, \ldots, 0\right) z^{(m)} \tag{13.25}$$

which is the same equation obtained from (13.24) upon setting $y(t) \equiv 0$. For linear systems, (13.25) reduces to a linear differential equation that corresponds to the numerator polynomial of the transfer function. The minimum phase property of the system can be determined by studying (13.25). To transform the system into the normal form, we note that ξ is already a vector of y and its derivatives up to $y^{(n-1)}$. So, we only need to find a function $\phi = \phi(\zeta, \xi) : R^{n+m} \rightarrow R^m$ such that

---

[6] In this example, we show that the extended system is input–output linearizable, which allows us to design feedback control using input–output linearization techniques. When such control is applied to the original system, the m integrators become part of the dynamics of the controller.

$$\frac{\partial \phi}{\partial \zeta} B_u + \frac{\partial \phi}{\partial \xi} B_c q(x) = 0$$

which is equivalent to

$$\frac{\partial \phi_i}{\partial \zeta_m} + \frac{\partial \phi_i}{\partial \xi_n} q(x) = 0, \quad \text{for } 1 \leq i \leq m \tag{13.26}$$

In some special cases, there are obvious solutions for these partial differential equations. For example, if q is constant, ϕ can be taken as

$$\phi_i = \zeta_i - \frac{1}{q} \xi_{n-m+i}, \quad \text{for } 1 \leq i \leq m$$

Another case is pursued in Exercise 13.5. △

13.3 Full-State Linearization

The single-input system

$$\dot{x} = f(x) + g(x)u \tag{13.27}$$

where f and g are sufficiently smooth in a domain $D \subset R^n$, is feedback linearizable if a sufficiently smooth function $h : D \rightarrow R$ exists such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u && (13.28) \\ y &= h(x) && (13.29) \end{aligned}$$

has relative degree n in a region $D_0 \subset D$. This statement follows from the fact that for systems with relative degree n, the normal form reduces to

$$\begin{aligned} \dot{z} &= A_c z + B_c \gamma(x)[u - \alpha(x)] && (13.30) \\ y &= C_c z && (13.31) \end{aligned}$$

On the other hand, if the system (13.27) is feedback linearizable per Definition 13.1, then there is a change of variables $\zeta = S(x)$ that transforms the system into

$$\dot{\zeta} = A\zeta + B\bar{\gamma}(x)[u - \bar{\alpha}(x)]$$

where (A, B) is controllable and $\bar{\gamma}(x) \neq 0$ in some domain. For any controllable pair (A, B), we can find a nonsingular matrix M that transforms (A, B) into a controllable canonical form;[7] that is, $MAM^{-1} = A_c + B_c\lambda^T$ and $MB = B_c$, where (A_c, B_c) represents a chain of n integrators. The change of variables

$$z = M\zeta = MS(x) \stackrel{\text{def}}{=} T(x)$$

transforms the system (13.27) into

$$\dot{z} = A_c z + B_c\gamma(x)[u - \alpha(x)]$$

where $\gamma(x) = \bar{\gamma}(x)$ and $\alpha(x) = \bar{\alpha}(x) - \lambda^T MS(x)/\gamma(x)$. Because

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x}$$

the equality

$$A_cT(x) + B_c\gamma(x)[u - \alpha(x)] = \frac{\partial T}{\partial x}[f(x) + g(x)u]$$

must hold for all x and u in the domain of interest. By taking $u = 0$, we split the foregoing equation into two:

$$\frac{\partial T}{\partial x}f(x) = A_cT(x) - B_c\alpha(x)\gamma(x) \tag{13.32}$$

$$\frac{\partial T}{\partial x}g(x) = B_c\gamma(x) \tag{13.33}$$

Equation (13.32) is equivalent to

$$\begin{aligned}
\frac{\partial T_1}{\partial x}f(x) &= T_2(x)\\
\frac{\partial T_2}{\partial x}f(x) &= T_3(x)\\
&\vdots\\
\frac{\partial T_{n-1}}{\partial x}f(x) &= T_n(x)\\
\frac{\partial T_n}{\partial x}f(x) &= -\alpha(x)\gamma(x)
\end{aligned}$$

and (13.33) is equivalent to

$$\begin{aligned}
\frac{\partial T_1}{\partial x}g(x) &= 0\\
\frac{\partial T_2}{\partial x}g(x) &= 0\\
&\vdots\\
\frac{\partial T_{n-1}}{\partial x}g(x) &= 0\\
\frac{\partial T_n}{\partial x}g(x) &= \gamma(x) \neq 0
\end{aligned}$$

---

[7] See, for example, [158].

Setting $h(x) = T_1(x)$, we see that

$$T_{i+1}(x) = L_f T_i(x) = L_f^i h(x), \quad i = 1, 2, \ldots, n-1$$

$h(x)$ satisfies the partial differential equations

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, n-1 \tag{13.34}$$

subject to the condition

$$L_g L_f^{n-1} h(x) \neq 0 \tag{13.35}$$

and α, γ are given by

$$\gamma(x) = L_g L_f^{n-1} h(x), \quad \alpha(x) = -\frac{L_f^n h(x)}{L_g L_f^{n-1} h(x)} \tag{13.36}$$

In summary, the system (13.27) is feedback linearizable if and only if a function $h(x)$ exists such that the system (13.28)–(13.29) has relative degree n, or, equivalently, h satisfies the partial differential equations (13.34) subject to the condition (13.35). The existence of h can be characterized by necessary and sufficient conditions on the vector fields f and g. These conditions use the notions of *Lie brackets* and *invariant distributions*, which we introduce next.

For two vector fields f and g on $D \subset R^n$, the *Lie bracket* $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

where $[\partial g/\partial x]$ and $[\partial f/\partial x]$ are Jacobian matrices. We may repeat bracketing of g with f. The following notation is used to simplify this process:

$$\begin{aligned} ad_f^0 g(x) &= g(x) \\ ad_f g(x) &= [f, g](x) \\ ad_f^k g(x) &= [f, ad_f^{k-1} g](x), \quad k \geq 1 \end{aligned}$$

It is obvious that $[f, g] = -[g, f]$ and for constant vector fields f and g, $[f, g] = 0$.

Example 13.9 Let

$$f(x) = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

Then,

$$\begin{aligned} [f, g](x) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \overset{\text{def}}{=} ad_f g \end{aligned}$$

$$
\begin{aligned}
ad_f^2 g &= [f, ad_f g] \\
&= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \\
&= \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix}
\end{aligned}
$$

△

Example 13.10 If $f(x) = Ax$ and g is a constant vector field, then

$$ad_f g(x) = [f, g](x) = -Ag$$

$$ad_f^2 g = [f, ad_f g] = -A(-Ag) = A^2 g$$

and

$$ad_f^k g = (-1)^k A^k g$$

△

For vector fields f_1, f_2, ..., f_k on $D \subset R^n$, let

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}$$

be the subspace of R^n spanned by the vectors $f_1(x)$, $f_2(x)$, ..., $f_k(x)$ at any fixed $x \in D$. The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a distribution and referred to by

$$\Delta = \text{span}\{f_1, f_2, \ldots, f_k\}$$

The dimension of $\Delta(x)$, defined by

$$\dim(\Delta(x)) = \text{rank}\ [f_1(x), f_2(x), \ldots, f_k(x)]$$

may vary with x, but if $\Delta = \text{span}\{f_1, \ldots, f_k\}$, where $\{f_1(x), \ldots, f_k(x)\}$ are linearly independent for all $x \in D$, then $\dim(\Delta(x)) = k$ for all $x \in D$. In this case, we say that Δ is a nonsingular distribution on D, generated by f_1, ..., f_k. A distribution Δ is *involutive* if

$$g_1 \in \Delta \ \text{ and } \ g_2 \in \Delta \ \Rightarrow \ [g_1, g_2] \in \Delta$$

If Δ is a nonsingular distribution on D, generated by f_1, ..., f_k, then it can be verified (Exercise 13.9) that Δ is involutive if and only if

$$[f_i, f_j] \in \Delta, \ \ \forall\, 1 \leq i, j \leq k$$

Example 13.11 Let $D = R^3$ and $\Delta = \text{span}\{f_1, f_2\}$, where

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}$$

It can be verified that $\dim(\Delta(x)) = 2$ for all $x \in D$ and

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$[f_1, f_2] \in \Delta$ if and only if rank $[f_1(x), f_2(x), [f_1, f_2](x)] = 2$, for all $x \in D$. However,

$$\text{rank } [f_1(x), f_2(x), [f_1, f_2](x)] = \text{rank} \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \quad \forall\, x \in D$$

Hence, Δ is not involutive. △

Example 13.12 Let $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}$ and $\Delta = \text{span}\{f_1, f_2\}$, where

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}$$

It can be verified that $\dim(\Delta(x)) = 2$ for all $x \in D$,

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

and

$$\text{rank } [f_1(x), f_2(x), [f_1, f_2](x)] = \text{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \quad \forall\, x \in D$$

Therefore, $[f_1, f_2] \in \Delta$. Since $[f_2, f_1] = -[f_1, f_2]$, we conclude that Δ is involutive. △

We are now ready to characterize the class of feedback linearizable systems.

Theorem 13.2 *The system* (13.27) *is feedback linearizable if and only if there is a domain $D_0 \subset D$ such that*

1. *the matrix $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_0$;*

2. the distribution $\mathcal{D} = \text{span}\ \{g, ad_f g, \ldots, ad_f^{n-2} g\}$ *is involutive in* D_0. ◇

Proof: See Appendix C.22.

In the next three examples, we illustrate the application of Theorem 13.2 and the solution of the partial differential equations (13.34). In all examples, we assume that the system (13.27) has an equilibrium point x^* when $u = 0$. We choose $h(x)$ such that $h(x^*) = 0$. Consequently, the change of variables $z = T(x)$ maps the equilibrium point $x = x^*$ into the origin $z = 0$.

Example 13.13 Reconsider the system

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \stackrel{\text{def}}{=} f(x) + gu$$

from Section 13.1. We have

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

The matrix

$$\mathcal{G} = [g, ad_f g] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$$

has rank two for all x such that $\cos x_2 \neq 0$. The distribution $\mathcal{D} = \text{span}\{g\}$ is involutive. Hence, the conditions of Theorem 13.2 are satisfied in the domain $D_0 = \{x \in R^2 \mid \cos x_2 \neq 0\}$. To find the change of variables that transforms the system into the form (13.6), we want to find $h(x)$ that satisfies

$$\frac{\partial h}{\partial x} g = 0; \quad \frac{\partial (L_f h)}{\partial x} g \neq 0, \quad \text{and} \quad h(0) = 0$$

From the condition $[\partial h/\partial x]g = 0$, we have

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$$

Thus, h must be independent of x_2. Therefore,

$$L_f h(x) = \frac{\partial h}{\partial x_1} a \sin x_2$$

The condition

$$\frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0$$

is satisfied in the domain D_0 by any choice of h for which $(\partial h/\partial x_1) \neq 0$. Taking $h(x) = x_1$ results in the transformation we used earlier. Other choices of h can be made. For example, $h(x) = x_1 + x_1^3$ gives another change of variables that transforms the system into the form (13.6). △

Example 13.14 A single link manipulator with flexible joints and negligible damping can be represented by a fourth-order model of the form (Exercise 1.5)

$$\dot{x} = f(x) + gu$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

and a, b, c, and d are positive constants. The unforced system has equilibrium at $x = 0$. We have

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} 0 \\ 0 \\ -d \\ 0 \end{bmatrix}$$

$$ad_f^2 g = [f, ad_f g] = -\frac{\partial f}{\partial x} ad_f g = \begin{bmatrix} 0 \\ bd \\ 0 \\ -cd \end{bmatrix}$$

$$ad_f^3 g = [f, ad_f^2 g] = -\frac{\partial f}{\partial x} ad_f^2 g = \begin{bmatrix} -bd \\ 0 \\ cd \\ 0 \end{bmatrix}$$

The matrix

$$\mathcal{G} = [g, ad_f g, ad_f^2 g, ad_f^3 g] = \begin{bmatrix} 0 & 0 & 0 & -bd \\ 0 & 0 & bd & 0 \\ 0 & -d & 0 & cd \\ d & 0 & -cd & 0 \end{bmatrix}$$

has full rank for all $x \in R^4$. The distribution $\Delta = \text{span}(g, ad_f g, ad_f^2)$ is involutive, since g, $ad_f g$, and $ad_f^2 g$ are constant vector fields. Thus, the conditions of Theorem 13.2 are satisfied for all $x \in R^4$. To find the change of variables that transforms the state equation into the form (13.6), we want to find $h(x)$ that satisfies

$$\frac{\partial (L_f^{i-1} h)}{\partial x} g = 0, \quad i = 1, 2, 3, \quad \frac{\partial (L_f^3 h)}{\partial x} g \neq 0, \quad \text{and} \quad h(0) = 0$$

From the condition $[\partial h/\partial x]g = 0$, we have $(\partial h/\partial x_4) = 0$, so we must choose h independent of x_4. Therefore,

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)] + \frac{\partial h}{\partial x_3} x_4$$

From the condition $[\partial(L_f h)/\partial x]g = 0$, we have

$$\frac{\partial(L_f h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_3} = 0$$

So, we choose h independent of x_3. Therefore, $L_f h$ simplifies to

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)]$$

and

$$L_f^2 h(x) = \frac{\partial(L_f h)}{\partial x_1} x_2 + \frac{\partial(L_f h)}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)] + \frac{\partial(L_f h)}{\partial x_3} x_4$$

Finally,

$$\frac{\partial(L_f^2 h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial(L_f h)}{\partial x_3} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_2} = 0$$

and we choose h independent of x_2. Hence,

$$L_f^3 h(x) = \frac{\partial(L_f^2 h)}{\partial x_1} x_2 + \frac{\partial(L_f^2 h)}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)] + \frac{\partial(L_f^2 h)}{\partial x_3} x_4$$

and the condition $[\partial(L_f^3 h)/\partial x]g \neq 0$ is satisfied whenever $(\partial h/\partial x_1) \neq 0$. Therefore, we take $h(x) = x_1$. The change of variables

$$\begin{aligned}
z_1 &= h(x) &&= x_1 \\
z_2 &= L_f h(x) &&= x_2 \\
z_3 &= L_f^2 h(x) &&= -a\sin x_1 - b(x_1 - x_3) \\
z_4 &= L_f^3 h(x) &&= -a x_2 \cos x_1 - b(x_2 - x_4)
\end{aligned}$$

transforms the state equation into

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -(a\cos z_1 + b + c)z_3 + a(z_2^2 - c)\sin z_1 + bdu
\end{aligned}$$

which is of the form (13.6). Unlike the previous example, in the current one the state equation in the z-coordinates is valid globally because $z = T(x)$ is a global diffeomorphism. △

Example 13.15 In Examples 13.3 and 13.7 we considered a field-controlled DC motor represented by the third-order model

$$\dot{x} = f(x) + gu$$

where

$$f(x) = \begin{bmatrix} -ax_1 \\ -bx_2 + k - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and a, b, c, θ, and k are positive constants. We saw that, with the output $y = x_3$, the system has relative degree two and hence is partially feedback linearizable. Let us investigate whether the state equation is fully linearizable. We have

$$ad_f g = [f, g] = \begin{bmatrix} a \\ cx_3 \\ -\theta x_2 \end{bmatrix}; \quad ad_f^2 g = [f, ad_f g] = \begin{bmatrix} a^2 \\ (a+b)cx_3 \\ (b-a)\theta x_2 - \theta k \end{bmatrix}$$

The determinant of

$$\mathcal{G} = [g, ad_f g, ad_f^2 g] = \begin{bmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a+b)cx_3 \\ 0 & -\theta x_2 & (b-a)\theta x_2 - \theta k \end{bmatrix}$$

is given by

$$\det \mathcal{G} = c\theta(-k + 2bx_2)x_3$$

Hence, $\mathcal{G}$ has rank three for $x_2 \neq k/2b$ and $x_3 \neq 0$. The distribution $\mathcal{D} = \text{span}\{g, ad_f g\}$ is involutive if $[g, ad_f g] \in \mathcal{D}$. We have

$$[g, ad_f g] = \frac{\partial(ad_f g)}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $\mathcal{D}$ is involutive and the conditions of Theorem 13.2 are satisfied in the domain

$$D_0 = \{x \in R^3 \mid x_2 > \frac{k}{2b} \text{ and } x_3 > 0\}$$

We proceed now to find a function h that satisfies (13.34) and (13.35). The unforced system has an equilibrium set at $x_1 = 0$ and $x_2 = k/b$. We take the desired operating point as $x^* = [0, k/b, \omega_0]^T$, where ω_0 is a desired set point for the angular velocity x_3. We want to find $h(x)$ that satisfies

$$\frac{\partial h}{\partial x} g = 0; \quad \frac{\partial(L_f h)}{\partial x} g = 0; \quad \frac{\partial(L_f^2 h)}{\partial x} g \neq 0$$

with $h(x^*) = 0$, From the condition

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_1} = 0$$

we see that h must be independent of x_1. Therefore,

$$L_f h(x) = \frac{\partial h}{\partial x_2}[-bx_2 + k - cx_1x_3] + \frac{\partial h}{\partial x_3}\theta x_1 x_2$$

From the condition $[\partial(L_f h)/\partial x]g = 0$, we have

$$cx_3 \frac{\partial h}{\partial x_2} = \theta x_2 \frac{\partial h}{\partial x_3}$$

which is satisfied if h takes the form

$$h = c_1[\theta x_2^2 + cx_3^2] + c_2$$

for some constants c_1 and c_2. We choose $c_1 = 1$ and, to satisfy the condition $h(x^*) = 0$, we take

$$c_2 = -\theta\,(x_2^*)^2 - c\,(x_3^*)^2 = -\theta(k/b)^2 - c\omega_0^2$$

With this choice of h, $L_f h$ and $L_f^2 h$ are given by

$$L_f h(x) = 2\theta x_2(k - bx_2), \qquad L_f^2 h(x) = 2\theta(k - 2bx_2)(-bx_2 + k - cx_1x_3)$$

Hence,

$$\frac{\partial(L_f^2 h)}{\partial x} g = \frac{\partial(L_f^2 h)}{\partial x_1} = -2c\theta(k - 2bx_2)x_3$$

and the condition $[\partial(L_f^2 h)/\partial x]g \neq 0$ is satisfied whenever $x_2 \neq k/2b$ and $x_3 \neq 0$. Assuming $x_3^* > 0$, it can be easily verified (Exercise 13.15) that the map $z = T(x)$ is a diffeomorphism on D_0 and the state equation in the z-coordinates is well defined in the domain

$$D_z = T(D_0) = \left\{ z \in R^3 \mid z_1 > \theta\phi^2(z_2) - \theta(k/b)^2 - c\omega_0^2 \text{ and } z_2 < \frac{\theta k^2}{2b} \right\}$$

where $\phi(\cdot)$ is the inverse of the map $2\theta x_2(k - bx_2)$, which is well defined for $x_2 > k/2b$. The domain D_z contains the origin $z = 0$. △

13.4 State Feedback Control

13.4.1 Stabilization

Consider a partially feedback linearizable system of the form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) && (13.37) \\ \dot{\xi} &= A\xi + B\gamma(x)[u - \alpha(x)] && (13.38) \end{aligned}$$

where

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

$T(x)$ is a diffeomorphism on a domain $D \subset R^n$, $D_z = T(D)$ contains the origin, (A, B) is controllable, $\gamma(x)$ is nonsingular for all $x \in D$, $f_0(0,0) = 0$, and $f_0(\eta, \xi)$,

$\alpha(x)$, and $\gamma(x)$ are continuously differentiable. Our goal is to design a state feedback control law to stabilize the origin $z = 0$. The form (13.37)–(13.38) is clearly motivated by the normal form (13.16)–(13.18) of input–output linearizable systems. However, (13.18) is dropped since the output y plays no role in the state feedback stabilization problem. The system (13.37)–(13.38) includes also feedback linearizable systems by dropping equation (13.37). We do not restrict our discussions to single-input systems or to a pair (A, B) in the controllable canonical form. We proceed to discuss the more general system (13.37)–(13.38) and our conclusions will apply to the normal form (13.16)–(13.18) or to feedback linearizable systems as special cases.

The state feedback control

$$u = \alpha(x) + \beta(x)v$$

where $\beta(x) = \gamma^{-1}(x)$, reduces (13.37)–(13.38) to the "triangular" system

$$\dot{\eta} = f_0(\eta, \xi) \tag{13.39}$$

$$\dot{\xi} = A\xi + Bv \tag{13.40}$$

Equation (13.40) can be easily stabilized by $v = -K\xi$, where K is designed such that $(A - BK)$ is Hurwitz. Asymptotic stability of the origin of the full closed-loop system

$$\dot{\eta} = f_0(\eta, \xi) \tag{13.41}$$

$$\dot{\xi} = (A - BK)\xi \tag{13.42}$$

follows from asymptotic stability of the origin of $\dot{\eta} = f_0(\eta, 0)$, as shown in the next lemma.

Lemma 13.1 *The origin of* (13.41)–(13.42) *is asymptotically stable if the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is asymptotically stable.* ◇

Proof: By (the converse Lyapunov) Theorem 4.16, there is a continuously differentiable Lyapunov function $V_1(\eta)$ such that

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \le -\alpha_3(\|\eta\|)$$

in some neighborhood of $\eta = 0$, where α_3 is a class $\mathcal{K}$ function. Let $P = P^T > 0$ be the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$ and use $V(\eta, \xi) = V_1(\eta) + k\sqrt{\xi^T P \xi}$, with $k > 0$, as a Lyapunov function candidate for (13.41)–(13.42).[8] The derivative $\dot{V}$ is given by

$$\dot{V} = \frac{\partial V_1}{\partial \eta} f_0(\eta, \xi) + \frac{k}{2\sqrt{\xi^T P \xi}}\, \xi^T [P(A - BK) + (A - BK)^T P]\xi$$

---

[8] The function $V(\eta, \xi)$ is continuously differentiable everywhere around the origin, except on the manifold $\xi = 0$. Both $V(\eta, \xi)$ and $\dot{V}(\eta, \xi)$ are defined and continuous around the origin. It can be easily seen that the statement of Theorem 4.1 is still valid.

$$= \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) + \frac{\partial V_1}{\partial \eta}[f_0(\eta,\xi) - f_0(\eta,0)] - \frac{k\xi^T\xi}{2\sqrt{\xi^T P \xi}}$$

On any bounded neighborhood of the origin, we can use continuous differentiability of V_1 and f_0 to obtain

$$\dot{V} \le -\alpha_3(\|\eta\|) + k_1\|\xi\| - kk_2\|\xi\|$$

for some positive constants k_1 and k_2. Choosing $k > k_1/k_2$ ensures that $\dot{V}$ is negative definite. Hence, the origin is asymptotically stable. □

The foregoing discussion shows that a minimum phase input–output linearizable system can be stabilized by the state feedback control

$$u = \alpha(x) - \beta(x)KT_2(x) \tag{13.43}$$

The control (13.43) is independent of $T_1(x)$. Therefore, it is independent of the function ϕ that satisfies the partial differential equation (13.15).

The proof of Lemma 13.1 is valid only on bounded sets. Hence, it cannot be extended to show global asymptotic stability. We can show global asymptotic stability by requiring the system $\dot{\eta} = f_0(\eta,\xi)$ to be input-to-state stable when ξ is viewed as the input.

Lemma 13.2 *The origin of* (13.41)–(13.42) *is globally asymptotically stable if the system* $\dot{\eta} = f_0(\eta,\xi)$ *is input-to-state stable* ◇

Proof: Apply Lemma 4.7. □

Input-to-state stability of $\dot{\eta} = f_0(\eta,\xi)$ does not follow from global asymptotic, or even exponential, stability of the origin of $\dot{\eta} = f_0(\eta,0)$, as we saw in Section 4.10. Consequently, knowing that an input–output linearizable system is "globally" minimum phase does not automatically guarantee that the control (13.43) will globally stabilize the system. It will be globally stabilizing if the origin of $\dot{\eta} = f_0(\eta,0)$ is globally exponentially stable and $f_0(\eta,\xi)$ is globally Lipschitz in (η,ξ), since in that case Lemma 4.6 confirms that the system $\dot{\eta} = f_0(\eta,\xi)$ will be input-to-state stable. Otherwise, we have to establish input-to-state stability by further analysis. Global Lipschitz conditions are sometimes referred to as *linear growth conditions.* The next two examples illustrate some of the difficulties that may arise in the absence of linear growth conditions.

Example 13.16 Consider the second-order system

$$\begin{aligned}\dot{\eta} &= -\eta + \eta^2\xi \\ \dot{\xi} &= v\end{aligned}$$

While the origin of $\dot{\eta} = -\eta$ is globally exponentially stable, the system $\dot{\eta} = -\eta + \eta^2\xi$ is not input-to-state stable. This fact can be seen by noting that $\xi(t) \equiv 1$ and

$\eta(0) \geq 2$ imply that $\dot{\eta}(t) \geq 2$. Therefore, η grows unbounded. On the other hand, by Lemma 13.1, we see that the linear control $v = -k\xi$, with $k > 0$, stabilizes the origin of the full system. In fact, the origin will be exponentially stable. However, this linear control does not make the origin globally asymptotically stable. Taking $\nu = \eta\xi$ and noting that

$$\dot{\nu} = \eta\dot{\xi} + \dot{\eta}\xi = -k\eta\xi - \eta\xi + \eta^2\xi^2 = -(1+k)\nu + \nu^2$$

we see that the set $\{\eta\xi < 1+k\}$ is positively invariant. On the boundary $\eta\xi = 1+k$, the trajectory is given by $\eta(t) = e^{kt}\eta(0)$ and $\xi(t) = e^{-kt}\xi(0)$. Thus, $\eta(t)\xi(t) \equiv 1+k$. Inside the set $\{\eta\xi < 1+k\}$, $\nu(t)$ will be strictly decreasing and after a finite time T, $\nu(t) \leq 1/2$ for all $t \geq T$. Then, $\eta\dot{\eta} \leq -(1/2)\eta^2$, for all $t \geq T$, which shows that the trajectory approaches the origin as t tends to infinity. Hence, the set $\{\eta\xi < 1+k\}$ is the exact region of attraction. While this conclusion shows that the origin is not globally asymptotically stable, it also shows that the region of attraction expands as k increases. In fact by choosing k large enough, we can include any compact set in the region of attraction. Thus, the linear feedback control $v = -k\xi$ can achieve semiglobal stabilization. △

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, one might think that the triangular system (13.39)–(13.40) can be globally stabilized, or at least semiglobally stabilized, by designing the linear feedback control $v = -K\xi$ to assign the eigenvalues of $(A - BK)$ far to the left in the complex plane so that the solution of $\dot{\xi} = (A - BK)\xi$ decays to zero arbitrarily fast. Then, the solution of $\dot{\xi} = f_0(\eta, \xi)$ will quickly approach the solution of $\dot{\eta} = f_0(\eta, 0)$, which is well behaved, because its origin is globally asymptotically stable. It may even appear that this strategy is the one used to achieve semiglobal stabilization in the preceding example. The next example shows why such strategy may fail.[9]

Example 13.17 Consider the third-order system

$$\begin{aligned} \dot{\eta} &= -\tfrac{1}{2}(1+\xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \end{aligned}$$

The linear feedback control

$$v = -k^2\xi_1 - 2k\xi_2 \overset{\text{def}}{=} -K\xi$$

assigns the eigenvalues of

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$$

---

[9] See, however, Exercise 13.20 for a special case where this strategy will work.

at $-k$ and $-k$. The exponential matrix

$$e^{(A-BK)t} = \begin{bmatrix} (1+kt)e^{-kt} & te^{-kt} \\ \\ -k^2te^{-kt} & (1-kt)e^{-kt} \end{bmatrix}$$

shows that as $k \rightarrow \infty$, the solution $\xi(t)$ will decay to zero arbitrarily fast. Notice, however, that the coefficient of the (2,1) element of the exponential matrix is a quadratic function of k. It can be shown that the absolute value of this element reaches a maximum value k/e at $t = 1/k$. While this term can be made to decay to zero arbitrarily fast by choosing k large, its transient behavior exhibits a peak of the order of k. The phenomenon is known as *the peaking phenomenon.*[10] The interaction of peaking with nonlinear growth could destabilize the system. In particular, for the initial states $\eta(0) = \eta_0$, $\xi_1(0) = 1$, and $\xi_2(0) = 0$, we have $\xi_2(t) = -k^2te^{-kt}$ and

$$\dot{\eta} = -\ \tfrac{1}{2}\left(1 - k^2te^{-kt}\right)\eta^3$$

During the peaking period, the coefficient of η^3 is positive, causing $|\eta(t)|$ to grow. Eventually, the coefficient of η^3 will become negative, but that might not happen soon enough, since the system might have a finite escape time. Indeed, the solution

$$\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2[t + (1+kt)e^{-kt} - 1]}$$

shows that if $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough. △

We will come back to the triangular system (13.39)–(13.40) in Sections 14.3 and 14.4 and show how to design v as a nonlinear function of ξ and η to achieve global stabilization. This will be done by using backstepping in Section 14.3 and passivity-based control in Section 14.4. We will even deal with cases where $\dot{\eta} = f_0(\eta, \xi)$ is not input-to-state stable.

While feedback linearization provides a simple and systematic procedure for stabilizing a class of nonlinear systems, there are legitimate concerns about the robustness and efficiency of such design. In the remainder of this section, we shed some light on these two issues.

Feedback linearization is based on exact mathematical cancellation of the nonlinear terms α and γ, which requires exact knowledge of α, $\beta = \gamma^{-1}$, and T_2. This is almost impossible for several practical reasons such as model simplification, parameter uncertainty, and computational errors. Most likely, the controller will be implementing functions $\hat{\alpha}$, $\hat{\beta}$, and $\hat{T}_2$, which are approximations of α, β, and T_2; that is to say, the actual controller will be implementing the feedback control law

$$u = \hat{\alpha}(x) - \hat{\beta}(x)K\hat{T}_2(x)$$

---

[10] To read more about the peaking phenomenon, see [188]. For an illustration of the peaking phenomenon in high-gain observers, see Section 14.5.

The closed-loop system under this feedback control is

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + B\gamma(x)[\hat{\alpha}(x) - \hat{\beta}(x)K\hat{T}_2(x) - \alpha(x)]\end{aligned}$$

By adding and subtracting the term $BK\xi$ to the right-hand side of the second equation, we can rewrite the closed-loop system as

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) && (13.44) \\ \dot{\xi} &= (A - BK)\xi + B\delta(z) && (13.45)\end{aligned}$$

where

$$\delta(z) = \gamma(x)\{\hat{\alpha}(x) - \alpha(x) + [\beta(x) - \hat{\beta}(x)]KT_2(x) + \hat{\beta}(x)K[T_2(x) - \hat{T}_2(x)]\}\Big|_{x=T^{-1}(z)}$$

Thus, the closed-loop system appears as a perturbation of the nominal system

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= (A - BK)\xi\end{aligned}$$

In view of the perturbation results of Chapter 10, we do not expect a serious problem from a small error $\delta(z)$. The next two lemmas confirm this expectation. We start with feedback linearizable systems where the closed-loop equation simplifies to

$$\dot{z} = (A - BK)z + B\delta(z) \qquad (13.46)$$

Lemma 13.3 *Consider the closed-loop system* (13.46)*, where* $(A - BK)$ *is Hurwitz. Let* $P = P^T > 0$ *be the solution of the Lyapunov equation*

$$P(A - BK) + (A - BK)^T P = -I$$

and k *be a nonnegative constant less than* $1/(2\|PB\|_2)$.

- *If* $\|\delta(z)\| \le k\|z\|$ *for all* z*, the origin of* (13.46) *will be globally exponentially stable.*
- *If* $\|\delta(z)\| \le k\|z\| + \varepsilon$ *for all* z*, the state* z *will be globally ultimately bounded by* εc *for some* $c > 0$.

Proof: Let $V(z) = z^T P z$. Then

$$\begin{aligned}\dot{V} &= z^T[P(A - BK) + (A - BK)^T P]z + 2z^T PB\delta(z) \\ &\le -\|z\|_2^2 + 2\|PB\|_2\|z\|_2\|\delta(z)\|_2\end{aligned}$$

If $\|\delta(z)\|_2 \le k\|z\|_2 + \varepsilon$, we have

$$\begin{aligned}\dot{V} &\le -\|z\|_2^2 + 2k\|PB\|_2\|z\|_2^2 + 2\varepsilon\|PB\|_2\|z\|_2 \\ &= -(1 - \theta_1)\|z\|_2^2 - \theta_1\|z\|_2^2 + 2k\|PB\|_2\|z\|_2^2 + 2\varepsilon\|PB\|_2\|z\|_2\end{aligned}$$

where $\theta_1 \in (0,1)$ is chosen close enough to one such that $k < \theta_1/(2\|PB\|_2)$. Consequently,

$$\dot{V} \leq -(1-\theta_1)\|z\|_2^2 + 2\varepsilon\|PB\|_2\|z\|_2$$

If $\|\delta(z)\|_2 \leq k\|z\|_2$, we set $\varepsilon = 0$ in the preceding inequality and conclude that the origin is globally exponentially stable. If $\varepsilon > 0$,

$$\dot{V} \leq -(1-\theta_1)(1-\theta_2)\|z\|_2^2, \quad \forall\ \|z\|_2 \geq \frac{2\varepsilon\|PB\|_2}{(1-\theta_1)\theta_2} \stackrel{\text{def}}{=} \varepsilon c_0$$

where $\theta_2 \in (0,1)$. Application of Theorem 4.18 shows that $z(t)$ is globally ultimately bounded by $\varepsilon c_0\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$. □

It is clear from the proof that if the bound on $\delta(z)$ is satisfied only in a neighborhood of the origin, we can prove a local version of the lemma.

Example 13.18 Consider the pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1+\delta_1) - bx_2 + cu \end{aligned}$$

where $x_1 = \theta - \delta_1$, $x_2 = \dot{\theta}$, and $u = T$ is a torque input. The goal is to stabilize the pendulum at the angle $\theta = \delta_1$. A linearizing–stabilizing feedback control is given by

$$u = \left(\frac{a}{c}\right)\sin(x_1+\delta_1) - \left(\frac{1}{c}\right)(k_1x_1 + k_2x_2)$$

where k_1 and k_2 are chosen such that

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2+b) \end{bmatrix}$$

is Hurwitz. Suppose that, due to uncertainties in the parameters a and c, the actual control is

$$u = \left(\frac{\hat{a}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{1}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

where $\hat{a}$ and $\hat{c}$ are estimates of a and c. The closed-loop system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1x_1 - (k_2+b)x_2 + \delta(x) \end{aligned}$$

where

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{c-\hat{c}}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

The error term $\delta(x)$ satisfies the bound $|\delta(x)| \leq k\|x\|_2 + \varepsilon$ globally, where

$$k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c-\hat{c}}{\hat{c}}\right|\sqrt{k_1^2+k_2^2}, \qquad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right|\ |\sin\delta_1|$$

The constants k and ε are measures of the size of the error in estimating the parameters a and c. Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

be the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. If

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

then the solutions of the system are globally ultimately bounded by a bound that is proportional to ε. If $\sin \delta_1 = 0$, the foregoing bound on k ensures global exponential stability of the origin. △

We turn now to the more general closed-loop system (13.44)–(13.45).

Lemma 13.4 *Consider the closed-loop system* (13.44)–(13.45), *where* $A - BK$ *is Hurwitz.*

- *If* $\|\delta(z)\| \le \varepsilon$ *for all* z *and* $\dot{\eta} = f_0(\eta, \xi)$ *is input-to-state stable, then the state* z *is globally ultimately bounded by a class* $\mathcal{K}$ *function of* ε.
- *If* $\|\delta(z)\| \le k\|z\|$ *in some neighborhood of* $z = 0$, *with sufficiently small* k, *and the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is exponentially stable, then* $z = 0$ *is an exponentially stable equilibrium point of the system* (13.44)–(13.45).

◇

Proof: Let $V(\xi) = \xi^T P \xi$, where $P = P^T > 0$ is the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. Then

$$\begin{aligned} \dot{V} &= \xi^T[P(A - BK) + (A - BK)^T P]\xi + 2\xi^T PB\delta(z) \\ &\le -\|\xi\|_2^2 + 2\|PB\|_2\|\xi\|_2\|\delta(z)\|_2 \end{aligned}$$

If $\|\delta(z)\|_2 \le \varepsilon$, we have

$$\dot{V} \le -\|\xi\|_2^2 + 2\varepsilon\|PB\|_2\|\xi\|_2 \le -\tfrac{1}{2}\|\xi\|_2^2, \quad \forall\ \|\xi\|_2 \ge 4\varepsilon\|PB\|_2$$

Hence, applying Theorem 4.18 shows that a finite time t_0 and a positive constant c exist such that

$$\|\xi(t)\|_2 \le c\varepsilon, \quad \forall\ t \ge t_0$$

By input-to-state stability of $\dot{\eta} = f_0(\eta, \xi)$, we have

$$\|\eta(t)\|_2 \le \beta_0(\|\eta(t_0)\|_2, t - t_0) + \gamma_0(\sup_{t \ge t_0} \|\xi(t)\|_2) \le \beta_0(\|\eta(t_0)\|_2, t - t_0) + \gamma_0(c\varepsilon)$$

where β_0 and γ_0 are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively. The term $\beta_0(\|\eta(t_0)\|_2, t - t_0)$ satisfies $\beta_0 \le \varepsilon$ after some finite time. Therefore, $\|z(t)\|_2$ is ultimately bounded by $c\varepsilon + \varepsilon + \gamma_0(c\varepsilon)$, which is a class $\mathcal{K}$ function of ε. To prove the second case of the lemma, recall from Theorem 4.14 that a Lyapunov function $V_1(\eta)$ exists in some neighborhood of $\eta = 0$ such that

$$c_1\|\eta\|_2^2 \le V_1(\eta) \le c_2\|\eta\|_2^2, \quad \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \le -c_3\|\eta\|_2^2, \quad \left\|\frac{\partial V_1}{\partial \eta}\right\|_2 \le c_4\|\eta\|_2$$

Using $V(z) = bV_1(\eta) + \xi^T P\xi$, with $b > 0$, as a Lyapunov function candidate for (13.44)–(13.45), we obtain

$$\begin{aligned}
\dot{V} &= b\frac{\partial V_1}{\partial \eta} f_0(\eta, 0) + b\frac{\partial V_1}{\partial \eta}[f_0(\eta, \xi) - f_0(\eta, 0)] \\
&\quad + \xi^T[P(A - BK) + (A - BK)^T P]\xi + 2\xi^T PB\delta(z) \\
&\le -bc_3\|\eta\|_2^2 + bc_4L\|\eta\|_2\|\xi\|_2 - \|\xi\|_2^2 + 2k\|PB\|_2\|\xi\|_2^2 + 2k\|PB\|_2\|\xi\|_2\|\eta\|_2 \\
&= -\begin{bmatrix} \|\eta\|_2 \\ \|\xi\|_2 \end{bmatrix}^T \begin{bmatrix} bc_3 & -(k\|PB\|_2 + bc_4L/2) \\ -(k\|PB\|_2 + bc_4L/2) & 1 - 2k\|PB\|_2 \end{bmatrix} \begin{bmatrix} \|\eta\|_2 \\ \|\xi\|_2 \end{bmatrix} \\
&\overset{\text{def}}{=} -\begin{bmatrix} \|\eta\|_2 \\ \|\xi\|_2 \end{bmatrix}^T Q \begin{bmatrix} \|\eta\|_2 \\ \|\xi\|_2 \end{bmatrix}
\end{aligned}$$

where L is a Lipschitz constant of f_0 with respect to ξ. Taking $b = k$, it can be verified that Q is positive definite for sufficiently small k. Therefore, the origin is exponentially stable. □

In Exercises 13.22 through 13.24, we present a few variations of Lemma 13.4. If $\dot{\eta} = f_0(\eta, \xi)$ is not input-to-state stable, but the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable, we can prove a local version of the first case of the lemma (Exercise 13.22). If $f(\eta, \xi)$ is globally Lipschitz and the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally exponentially stable, we can prove a global version of the second case (Exercise 13.23). If the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically, but not exponentially, stable, we can prove asymptotic stability of the origin of the closed-loop system by restricting the dependence of δ on η (Exercise 13.24).

The feedback control $u = \alpha(x) - \beta(x)K\xi$ has a linearizing component $u = \alpha(x) + \beta(x)v$ and a stabilizing component $v = -K\xi$. The foregoing Lyapunov analysis shows that the stabilizing component achieves a certain degree of robustness to model uncertainty.[11] We will see in Chapter 14 that the stabilizing component can be designed to achieve a much higher degree of robustness by exploiting the fact that the perturbation term $B\delta(z)$ in (13.45) belongs to the range space of the input matrix B. Such perturbation is said to satisfy the *matching condition.* The techniques of Chapter 14 can guarantee robustness to any $\delta(z)$ provided an upper bound on δ is known.

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[11] Another type of model uncertainty that is not considered here is the sensitivity of the relative degree and the minimum phase property to parameter perturbations. To read more about this issue, see [92] and [169].

The basic philosophy of feedback linearization is to cancel the nonlinear terms of the system. Aside from the issues of whether or not we can cancel the nonlinear terms, effect of uncertainties, implementation factors, and so on, we should examine the philosophy itself: Is it a good idea to cancel nonlinear terms? Our motivation to do so has been mathematically driven. We wanted to linearize the system to make it more tractable and to use the relatively well-developed linear control theory. From a performance viewpoint, however, a nonlinear term could be "good" or "bad" and the decision whether we should use feedback to cancel a nonlinear term is, in reality, problem dependent. Let us use a couple of examples to illustrate this point.

Example 13.19 Consider the scalar system

$$\dot{x} = ax - bx^3 + u$$

where a and b are positive constants. A linearizing–stabilizing feedback control can be taken as

$$u = -(k+a)x + bx^3, \quad k > 0$$

which results in the closed-loop system $\dot{x} = -kx$. This feedback control cancels the nonlinear term $-bx^3$, but this term provides "nonlinear damping." In fact, without any feedback control, such nonlinear damping would guarantee boundedness of the solutions despite the fact that the origin is unstable. So, why should we cancel it? If we simply use the linear control

$$u = -(k+a)x, \quad k > 0$$

we will obtain the closed-loop system

$$\dot{x} = -kx - bx^3$$

whose origin is globally exponentially stable and its trajectories approach the origin faster than the trajectories of $\dot{x} = -kx$. Moreover, the linear control is simpler and uses less control effort. △

Example 13.20 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) + u \end{aligned}$$

where $h(0) = 0$ and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$. The system is clearly feedback linearizable and a linearizing–stabilizing feedback control can be taken as

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

where k_1 and k_2 are chosen to assign the closed-loop eigenvalues at desired locations in the left-half complex plane. On the other hand, our study of passive systems in Chapter 7 shows that with the feedback control

$$u = -\sigma(x_2)$$

where σ is any locally Lipschitz function that satisfies $\sigma(0) = 0$ and $y\sigma(y) > 0$ for $y \neq 0$, the closed-loop system will be passive, and the derivative of the Lyapunov function $V = \int_0^{x_1} h(z)\ dz + (1/2)x_2^2$ is given by

$$\dot{V} = -x_2\sigma(x_2)$$

Because

$$x_2(t) \equiv 0 \ \Rightarrow \ \dot{x}_2(t) \equiv 0 \ \Rightarrow \ h(x_1(t)) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$$

asymptotic stability of the origin follows from the invariance principle. The control $u = -\sigma(x_2)$ has two advantages over the linearizing feedback control. First, it does not use a model of the nonlinear function h. Hence, it is robust to uncertainty in modeling h. Second, the flexibility in choosing the function σ can be used to reduce the control effort. For example, we can meet any constraint of the form $|u| \leq k$, by choosing $u = -k \ \mathrm{sat}(x_2)$. However, the control $u = -\sigma(x_2)$ cannot arbitrarily assign the rate of decay of $x(t)$. Linearization of the closed-loop system at the origin yields the characteristic equation

$$s^2 + \sigma'(0)s + h'(0) = 0$$

One of the two roots of the foregoing equation cannot be moved to the left of $\mathrm{Re}[s] = -\sqrt{h'(0)}$. Feedback control laws that exploit passivity properties will be discussed in Section 14.4. △

These two examples make the point that there are situations where nonlinearities are beneficial and cancelling them should not be an automatic choice. We should try our best to understand the effect of the nonlinear terms and decide whether or not cancellation is appropriate. Admittedly, this is not an easy task.

The robustness and efficiency concerns we raised regarding feedback linearization as a design procedure should not undermine the feedback linearization theory we developed in this chapter. The theory provides us with valuable tools to characterize a class of nonlinear systems whose structure is open to feedback control design, with or without nonlinearity cancellation. The concepts of relative degree and zero dynamics of nonlinear systems bring into focus the common input–output structure of linear and nonlinear systems and play a crucial role in extending to nonlinear systems some of the feedback design procedures that were successfully used for linear systems, such as high-gain feedback. The ability to transform a system into a normal form, where nonlinear terms enter the state equation at the same point as the control input, brings in the matching condition structure that will be used in Chapter 14 to develop some useful robust control techniques.

13.4.2 Tracking

Consider a single-input–single-output, input–output linearizable system represented in the normal form (13.16)–(13.18):

$$\dot{\eta} \ = \ f_0(\eta, \xi)$$

$$\begin{aligned}\dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)]\\ y &= C_c\xi\end{aligned}$$

Without loss of generality, we assume that $f_0(0, 0) = 0$. We want to design a state feedback control law such that the output y asymptotically tracks a reference signal $r(t)$. When the system has relative degree $\rho = n$, it has no nontrivial zero dynamics. In this case, the η variable and its equation are dropped, but the rest of the development remains the same. We assume that

- $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$ and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t;
- the signals $r, \ldots, r^{(\rho)}$ are available on-line.

The reference signal $r(t)$ could be specified, together with its derivatives, as some given functions of time, or it could be the output of a *reference model* driven by some input signal $w(t)$. In the latter case, the assumptions on r can be met by appropriately choosing the reference model. For example, for a relative degree two system, a reference model could be a second-order linear time-invariant system represented by the transfer function

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where the positive constants ζ and ω_n are chosen to shape the reference signal $r(t)$ for a given input signal $w(t)$. The signal $r(t)$ can be generated on-line by using the state model

$$\begin{aligned}\dot{y}_1 &= y_2\\ \dot{y}_2 &= -\omega_n^2 y_1 - 2\zeta\omega_n y_2 + \omega_n^2 w\\ r &= y_1\end{aligned}$$

Therefore, $r(t)$, $\dot{r}(t)$, and $\ddot{r}(t)$ will be available on-line. If $w(t)$ is a piecewise continuous bounded function of t, then $r(t)$, $\dot{r}(t)$, and $\ddot{r}(t)$ will satisfy the required assumptions.

Let

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}$$

The change of variables $e = \xi - \mathcal{R}$ yields

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e + \mathcal{R})\\ \dot{e} &= A_c e + B_c\left\{\gamma(x)[u - \alpha(x)] - r^{(\rho)}\right\}\end{aligned}$$

The state feedback control

$$u = \alpha(x) + \beta(x)\left[v + r^{(\rho)}\right]$$

where $\beta(x) = 1/\gamma(x)$, reduces the normal form to the cascade system

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e + \mathcal{R})\\ \dot{e} &= A_c e + B_c v\end{aligned}$$

Our control objective can be met by any design of v that stabilizes the second equation while maintaining η bounded for all $t \geq 0$. With $v = -Ke$, where $A_c - B_c K$ is Hurwitz, the complete state feedback control is given by[12]

$$u = \alpha(x) + \beta(x)\left\{-K[T_2(x) - \mathcal{R}] + r^{(\rho)}\right\} \tag{13.47}$$

and the closed-loop system is given by

$$\dot{\eta} = f_0(\eta, e + \mathcal{R}) \tag{13.48}$$

$$\dot{e} = (A_c - B_c K)e \tag{13.49}$$

For minimum phase systems, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. It follows from (the converse Lyapunov function) Theorem 4.16 and Theorem 4.18 that for sufficiently small $e(0)$, $\eta(0)$, and $\mathcal{R}(t)$, the state $\eta(t)$ will be bounded for all $t \geq 0$. Thus, the state feedback control (13.47) solves the local tracking problem. To extend the validity of the control to global tracking, where $\mathcal{R}(t)$ can be any bounded function of t, we face the same issues we encountered in global stabilization. A sufficient condition to ensure global tracking is input-to-state stability of the system $\dot{\eta} = f_0(\eta, \xi)$.

Example 13.21 Consider the pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - bx_2 + cu\\ y &= x_1\end{aligned}$$

The system has relative degree two in R^2 and is already represented in the normal form. It has no nontrivial zero dynamics, so it is minimum phase by default. We want the output y to track a reference signal $r(t)$, with bounded derivatives $\dot{r}(t)$ and $\ddot{r}(t)$. Taking

$$e_1 = x_1 - r, \quad e_2 = x_2 - \dot{r}$$

we obtain

$$\begin{aligned}\dot{e}_1 &= e_2\\ \dot{e}_2 &= -a\sin x_1 - bx_2 + cu - \ddot{r}\end{aligned}$$

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[12] As in Section 13.2, T_2 comprises the last ρ components of the diffeomorphism $T(x)$ that transforms the system into the normal form.

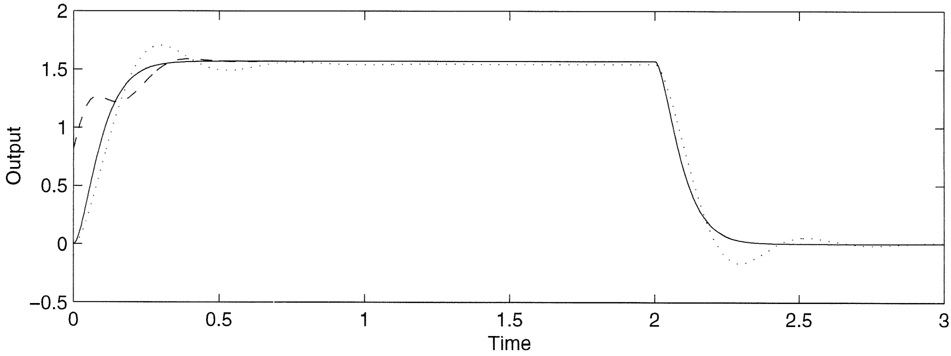

Figure 13.2: Simulation of the tracking control of Example 13.21.

The state feedback control (13.47) is given by

$$u = \frac{1}{c}[a \sin x_1 + b x_2 + \ddot{r} - k_1 e_1 - k_2 e_2]$$

where $K = [k_1, k_2]$ is designed to assign the eigenvalues of $A_c - B_c K$ at desired locations in the open left-half complex plane. Because all the assumptions hold globally, this control achieves global tracking. Figure 13.2 shows the response of the system when $a = c = 10$, $b = 1$, $k_1 = 400$, and $k_2 = 20$ to some reference signal. The solid curve is both the reference signal and output signal in the nominal case; they are identical. Here, tracking is achieved for all t and not just asymptotically, because $x(0) = \mathcal{R}(0)$. If $x(0) \neq \mathcal{R}(0)$, tracking will be achieved asymptotically, which is shown by the dashed curve. Finally, the dotted curve shows the response of the system when b and c are perturbed to $b = 0.5$ and $c = 5$, which correspond to doubling the mass. △

In many control problems, the designer has some freedom in choosing the reference signal r. For example, one of the typical problems in controlling robot manipulators is moving the manipulator from an initial to a final point within some time interval. The first task in approaching this problem is planning the path between the two points, which has to comply with any physical constraints due to the presence of obstacles. Then, the motion trajectory is planned by specifying velocities and accelerations of the moving parts as functions of time. The outcome of this trajectory planning process is the reference signal that the output variable has to track.[13] The freedom in choosing the reference signal can be used to improve the performance of the system, especially in the presence of constraints on the control signal. The next example illustrates this point.

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[13] To read about trajectory planning in robot manipulators, see [171].

Example 13.22 Reconsider the pendulum equation of the previous example with the nominal parameters $a = c = 10$ and $b = 1$. Suppose the pendulum is resting at the open-loop equilibrium point $x = 0$ and we want to move it to a new equilibrium point at $x_1 = \pi/2$ and $x_2 = 0$. Taking the reference signal r as the output of the second-order transfer function $1/(\tau s+1)^2$ driven by a step input w will provide the desired motion if the jump in w is taken as $\pi/2$. The tracking control is taken as

$$u = 0.1(10 \sin x_1 + x_2 + \ddot{r} - k_1 e_1 - k_2 e_2)$$

where $k_1 = 400$ and $k_2 = 20$. Taking the initial conditions of the reference model to be zero, we find that the tracking error $e(t) = x(t) - \mathcal{R}(t)$ will be identically zero and the motion of the pendulum will track the desired reference signal for all t. The choice of the time constant τ determines the speed of motion from the initial to the final position. If there were no constraint on the magnitude of the control u, we could have chosen τ arbitrarily small and achieved arbitrarily fast transition from $x_1 = 0$ to $x_1 = \pi/2$. However, the control input u is the torque of a motor and there is a maximum torque that the motor can supply. This constraint puts a limit on how quick we can move the pendulum. By choosing τ to be compatible with the torque constraint, we can achieve better performance. Figure 13.3 shows two different choices of τ when the control is constrained to $|u| \leq 2$. For $\tau = 0.05$ sec, the output $y(t)$ deviates from the reference $r(t)$, reflecting the fact that the reference signal demands a control effort that cannot be delivered by the motor. On the other hand, with $\tau = 0.25$ sec, the output signal achieves a good tracking of the reference signal. In both cases, we could not achieve a settling time better than about 1.2 seconds, but by choosing $\tau = 0.25$, we were able to avoid the overshoot that took place when $\tau = 0.05$. △

13.5 Exercises

13.1 Consider the third-order model of a synchronous generator connected to an infinite bus from Exercise 1.8. Consider two possible choices of the output:

$$\textbf{(1)}\ \ y = \delta; \qquad \textbf{(2)}\ \ y = \delta + \gamma\dot{\delta},\ \ \gamma \neq 0$$

In each case, study the relative degree of the system and transform it into the normal form. Specify the region over which the transformation is valid. If there are nontrivial zero dynamics, find whether or not the system is minimum phase.

13.2 Consider the system

$$\dot{x}_1 = -x_1 + x_2 - x_3, \qquad \dot{x}_2 = -x_1 x_3 - x_2 + u, \qquad \dot{x}_3 = -x_1 + u, \qquad y = x_3$$

(a) Is the system input–output linearizable?

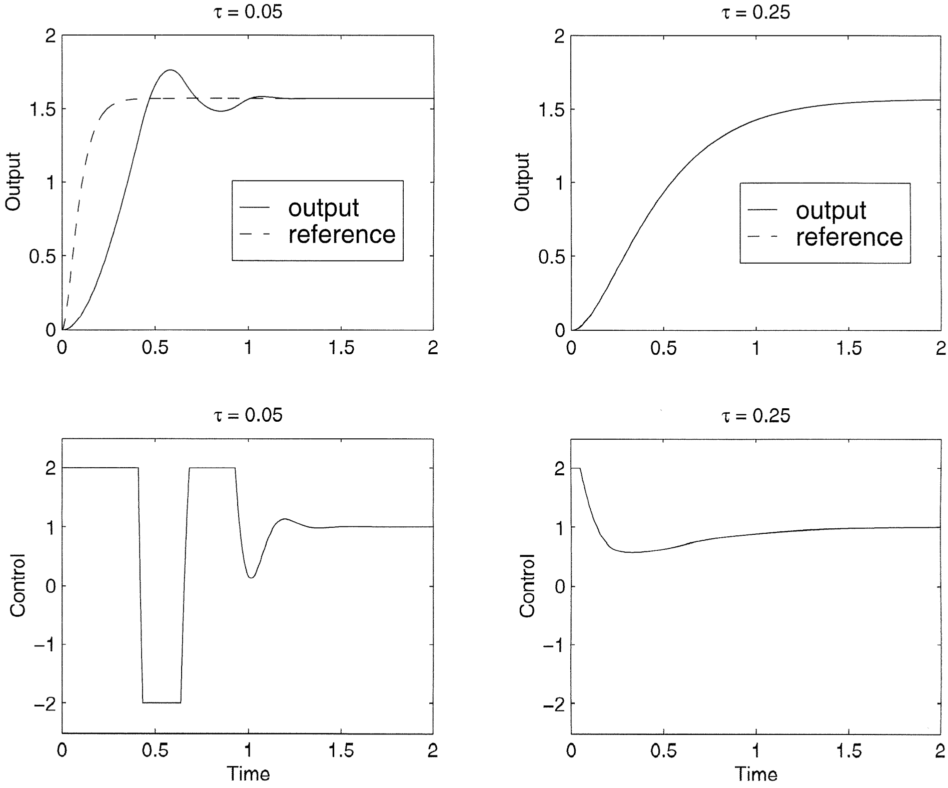

Figure 13.3: Simulation of the tracking control of Example 13.22.

(b) If yes, transform it into the normal form and specify the region over which the transformation is valid.

(c) Is the system minimum phase?

13.3 Consider the inverted pendulum of Exercise 1.15 and let θ be the output. Is the system input–output linearizable? Is it minimum phase?

13.4 Consider the system of Example 12.6. Is the system input–output linearizable? Is it minimum phase?

13.5 With reference to Example 13.8, consider the partial differential equations (13.26). Suppose $q(x)$ is independent of ζ_m and ξ_n. Show that $\phi_i = \zeta_i$ for $1 \le i \le m-1$ and $\phi_m = \zeta_m - \xi_n/q(x)$ satisfy the partial differential equations.

13.6 Show that the state equation of Exercise 6.11 is feedback linearizable.

13.7 Show that the state equation of the m-link robot of Exercise 1.4 is feedback linearizable.

13.8 Prove the Jacobi identity

$$L_{[f,g]}h(x) = L_f L_g h(x) - L_g L_f h(x)$$

where f and g are vector fields and h is a real-valued function.

13.9 Let Δ be a nonsingular distribution on D, generated by $f_1, \ldots, f_r$. Show that Δ is involutive if and only if $[f_i, f_j] \in \Delta$, $\forall\ 1 \leq i, j \leq r$.

13.10 Let

$$f_1(x) = \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -e^{x_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$D = R^4$ and $\Delta = \text{span}\{f_1, f_2\}$. Show that Δ is involutive.

13.11 Consider the system

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = 3x_1^2 x_2 + x_1 + u, \qquad y = -x_1^3 + x_2$$

(a) Is the system input–output linearizable?

(b) If yes, transform it into the normal form and specify the region over which the transformation is valid.

(c) Is the system minimum phase?

(d) Is the system feedback linearizable?

(e) If yes, find a feedback control law and a change of variables that linearize the state equation.

13.12 Repeat the previous exercise for the system

$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = x_2 + x_3, \qquad \dot{x}_3 = \delta(x) + u, \qquad y = x_1 + x_2$$

where $\delta(x)$ is a locally Lipschitz function of x.

13.13 An articulated vehicle (a semitrailer-like vehicle) can be modeled by the state equation

$$\begin{aligned}
\dot{x}_1 &= \tan(x_3) \\
\dot{x}_2 &= -\frac{\tan(x_2)}{a\cos(x_3)} + \frac{1}{b\cos(x_2)\cos(x_3)}\tan(u) \\
\dot{x}_3 &= \frac{\tan(x_2)}{a\cos(x_3)}
\end{aligned}$$

where a and b are positive constants. Show that system is feedback linearizable. Find the domain of validity of the exact linear model.

13.14 Consider the system

$$\dot{x}_1 = -x_1 + x_2 - x_3, \qquad \dot{x}_2 = -x_1 x_3 - x_2 + u, \qquad \dot{x}_3 = -x_1 + u$$

(a) Is the system feedback linearizable?

(b) If yes, find a feedback control law and a change of variables that linearize the state equation.

13.15 Verify that the map $z = T(x)$ in Example 13.15 is a diffeomorphism on D_0, and the state equation in the z coordinates is well defined on $D_z = T(D_0)$.

13.16 Consider the pendulum of Example 12.2 with the numerical data of Exercise 12.1. Design a stabilizing state feedback control law via feedback linearization, locating the closed-loop eigenvalues at the same locations used in Exercise 12.1. Compare the performance of the closed-loop system with that of Exercise 12.1.

13.17 Show that the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = x_1 - x_2 - x_1 x_3 + u, \qquad \dot{x}_3 = x_1 + x_1 x_2 - 2x_3$$

is feedback linearizable and design a state feedback control law to globally stabilize the origin.

13.18 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a \sin x_1 - bu \cos x_1$$

where a and b are positive constants.

(a) Show that the system is feedback linearizable.

(b) Using feedback linearization, design a state feedback controller to stabilize the system at $x_1 = \theta$, where $0 \le \theta < \pi/2$. Can you make this equilibrium point globally asymptotically stable?

13.19 Consider the link manipulator of Example 13.14. Suppose the parameters a, b, c, and d are not known exactly, but we know their estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$. Design a linearizing state feedback control law in terms of $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ and represent the closed-loop system as a perturbation of a nominal linear system.

13.20 Consider a special case of the system (13.37)–(13.38), where $f_0(\eta, \xi)$ depends only on ξ_1, and $(A, B) = (A_c, B_c)$ is a controllable canonical form that represents a chain of ρ integrators. Such system is said to be in a special normal form. Assume that the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable and there is a radially unbounded Lyapunov function $V_0(\eta)$ such that

$$\frac{\partial V_0}{\partial \eta} f_0(\eta, 0) \le -W(\eta)$$

for all η, where $W(\eta)$ is a positive definite function.

(a) Show that the control $u = \alpha(x) + \beta(x)v$ where $\beta(x) = \gamma^{-1}(x)$, and the change of variables

$$z_1 = \xi_1, \; z_2 = \varepsilon\xi_2, \ldots, z_\rho = \varepsilon^{\rho-1}\xi_\rho, \quad w = \varepsilon^\rho v$$

brings the system into the form

$$\dot{\eta} = f_0(\eta, z_1), \qquad \varepsilon\dot{z} = A_c z + B_c w$$

(b) Let K be chosen such that $A_c - B_c K$ is Hurwitz and P be the positive definite solution of the Lyapunov equation $P(A_c - B_c K) + (A_c - B_c K)^T P = -I$. Taking $w = -Kz$ and using $V(\eta, z) = V_0(\eta) + \sqrt{z^T P z}$ as a Lyapunov function candidate for the closed-loop system, show that, for sufficiently small ε, the origin $(\eta = 0, \; z = 0)$ is asymptotically stable and the set $\{V(\eta, z) \le c\}$, with an arbitrary $c > 0$, is included in the region of attraction.

(c) Show that the feedback control achieves semiglobal stabilization; that is, initial states (η_0, ξ_0) in any compact subset of R^n can be included in the region of attraction.

(d) In view of Example 13.17, investigate whether the current controller exhibits a peaking phenomenon, and if so, explain why is it possible to achieve semiglobal stabilization despite the presence of peaking.

13.21 Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1 x_2 - x_2^2 + u \\
\dot{x}_2 &= x_1 x_2 - x_2^2 + u \\
\dot{x}_3 &= x_1 + x_1 x_2 - x_2^2 - (x_3 - x_1)^3 + u \\
y &= x_1 - x_2
\end{aligned}$$

(a) Show that the system has a globally defined special normal form.

(b) Show that the origin of the zero dynamics is globally asymptotically stable.

(c) Design a semiglobally stabilizing state feedback control law.

Hint: See Exercise 13.20.

13.22 Consider the system (13.44)–(13.45), where $A - BK$ is Hurwitz, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable, and $\|\delta(z)\| \le \varepsilon$. Show that there is a neighborhood D of $z = 0$ and $\varepsilon^* > 0$ such that for every every $z(0) \in D$ and $\varepsilon \le \varepsilon^*$, the state z is ultimately bounded by a class $\mathcal{K}$ function of ε.

13.23 Consider the system (13.44)–(13.45), where $A - BK$ is Hurwitz, the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally exponentially stable, f_0 is globally Lipschitz, and $\|\delta\| \le k\|z\|$ for all z. Show that, for sufficiently small k, the origin $z = 0$ is globally exponentially stable.

13.24 Consider the system (13.44)–(13.45), where $A - BK$ is Hurwitz, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable with a Lyapunov function $V_0(\eta)$ such that $[\partial V_0/\partial\eta] f_0(\eta, 0) \leq -W(\eta)$ for some positive definite function $W(\eta)$. Suppose $\|\delta\| \leq k[\|\xi\| + W(\eta)]$. Using a composite Lyapunov function of the form $V = V_0(\eta) + \lambda\sqrt{\xi^T P \xi}$, where P is the solution of $P(A - BK) + (A - BK)^T P = -I$, show that, for sufficiently small k, the origin $z = 0$ is asymptotically stable

13.25 Consider the system

$$\dot{x}_1 = x_2 + 2x_1^2, \quad \dot{x}_2 = x_3 + u, \quad \dot{x}_3 = x_1 - x_3, \quad y = x_1$$

Design a state feedback control law such that the output y asymptotically tracks the reference signal $r(t) = \sin t$.

13.26 Repeat the previous exercise for the system

$$\dot{x}_1 = x_2 + x_1 \sin x_1, \quad \dot{x}_2 = x_1 x_2 + u, \quad y = x_1$$

13.27 The magnetic suspension system of Exercise 1.18 is modeled by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a + x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a + x_1)^2} + u\right]
\end{aligned}$$

where $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$, and $u = v$. Use the following numerical data: $m = 0.1$ kg, $k = 0.001$ N/m/sec, $g = 9.81$ m/sec^2, $a = 0.05$ m, $L_0 = 0.01$ H, $L_1 = 0.02$ H, and $R = 1\ \Omega$.

(a) Show that the system is feedback linearizable.

(b) Using feedback linearization, design a state feedback control law to stabilize the ball at $y = 0.05$ m. Repeat parts (d) and (e) of Exercise 12.8 and compare the performance of this controller with the one designed in part (c) of that exercise.

(c) Show that, with the ball position y as the output, the system is input–output linearizable.

(d) Using feedback linearization, design a state feedback control law so that the output y asymptotically tracks $r(t) = 0.05 + 0.01 \sin t$. Simulate the closed-loop system.

13.28 A field-controlled DC motor is described in Exercise 1.17. When the field circuit is driven by a current source, we can view the field current as the control input and model the system by the second-order state model

$$\begin{aligned} \dot{x}_1 &= -\theta_1 x_1 - \theta_2 x_2 u + \theta_3 \\ \dot{x}_2 &= -\theta_4 x_2 + \theta_5 x_1 u \\ y &= x_2 \end{aligned}$$

where x_1 is the armature current, x_2 is the speed, u is the field current, and θ_1 to θ_5 are positive constants. It is desired to design a state feedback controller such that the output y asymptotically tracks a time-varying reference signal $r(t)$, where both $r(t)$ and $\dot{r}(t)$ are continuous and bounded for all $t \geq 0$. Assume that the domain of operation is restricted to $x_1 > \theta_3/2\theta_1$.

(a) Show that the system is input–output linearizable and has relative degree one.

(b) Show that it is minimum phase.

(c) Using feedback linearization, design a state feedback control to achieve the desired tracking.

(d) Using computer simulation, study the performance of the system when r is the output of a first-order filter $1/(\tau s + 1)$ driven by a step command w. The time constant τ can be chosen to adjust the rate of change of r. In the simulation, take the initial conditions as $x_1(0) = \theta_3/\theta_1$ and $x_2(0) = 0$ and use the following numerical data: $\theta_1 = 60$, $\theta_2 = 0.5$, $\theta_3 = 40$, $\theta_4 = 6$, and $\theta_5 = 4 \times 10^4$. Let the step command w change from 0 to 100 at $t = 1$. Also, add saturation at the input of the plant such that the control signal is limited to ± 0.05.

- **(i)** Adjust τ and the feedback controller parameters to achieve a settling time of 0.5.
- **(ii)** Adjust τ and the feedback controller parameters to achieve a settling time of 0.1.
- **(iii)** Go back to the values used in part (i) and study the performance of the system when the rotor inertia changes by ±50%.
- **(iv)** Can you adjust the feedback controller parameters to improve its robustness to the parameter perturbations described in the previous part?

Chapter 14

Nonlinear Design Tools

The complexity of nonlinear feedback control challenges us to come up with systematic design procedures to meet control objectives and design specifications. Faced with such challenge, it is clear that we cannot expect one particular procedure to apply to all nonlinear systems. It is also unlikely that the whole design of a nonlinear feedback controller can be based on one particular tool. What a control engineer needs is a set of analysis and design tools that cover a wide range of situations. When working with a particular application, the engineer will need to employ the tools that are most appropriate for the problem in hand. We have already covered several such tools in the earlier chapters. In this chapter, we assemble five nonlinear design tools, which are simple enough to be presented in an introductory textbook, and practical enough to have been used in real-world problems.[1]

In the first two sections, we deal with robust control under the matching condition; that is, when uncertain terms enter the state equation at the same point as the control input. In the sliding mode control of Section 14.1, trajectories are forced to reach a sliding manifold in finite time and to stay on the manifold for all future time. Motion on the manifold is independent of matched uncertainties. By using a lower order model, the sliding manifold is designed to achieve the control objective. The Lyapunov redesign of Section 14.2 uses a Lyapunov function of a nominal system to design an additional control component that makes the design robust to large matched uncertainties. Both sliding mode control and Lyapunov redesign produce discontinuous controllers, which could suffer from chattering in the presence of delays or unmodeled high-frequency dynamics. Therefore, we develop "continuous" versions of the controllers. In Section 14.1, we use a second-order example to motivate the main elements of the sliding-mode-control technique, then we proceed to present stabilization, tracking, and integral control results. In Section 14.2, we show how Lyapunov redesign can be used to achieve stabilization, and introduce nonlinear damping— a technique that guarantees boundedness of trajectories even when no upper bound on the uncertainty is known.

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[1] For more nonlinear design tools, see [88], [89], [103], [124], [153], [167], and [172].

The matching condition can be relaxed via the backstepping technique, introduced in Section 14.3. Backstepping is a recursive procedure that interlaces the choice of a Lyapunov function with the design of feedback control. It breaks a design problem for the full system into a sequence of design problems for lower order (even scalar) subsystems. By exploiting the extra flexibility that exists with lower order and scalar subsystems, backstepping can often solve stabilization, tracking, and robust control problems under conditions less restrictive than those encountered in other methods.

Passivity-based control exploits passivity of the open-loop system in the design of feedback control. Stabilizing a passive system at an equilibrium point amounts to damping injection. In Section 14.4, we describe the basic idea of passivity-based control. We also describe feedback passivation–a technique that uses feedback control to convert a nonpassive system into a passive one.

Most of the design tools presented in Chapters 12 through 14 require state feedback. In Section 14.5, we introduce high-gain observers, which allow us to extend many of those tools to output feedback for a particular class of nonlinear systems.[2] The main idea of Section 14.5 is that the performance under globally bounded state feedback control can be recovered by output feedback control when the observer gain is sufficiently high.

14.1 Sliding Mode Control

14.1.1 Motivating Example

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= h(x) + g(x)u\end{aligned}$$

where h and g are unknown nonlinear functions and $g(x) \geq g_0 > 0$ for all x. We want to design a state feedback control law to stabilize the origin. Suppose we can design a control law that constrains the motion of the system to the manifold (or surface) $s = a_1x_1 + x_2 = 0$. On this manifold, the motion is governed by $\dot{x}_1 = -a_1x_1$. Choosing $a_1 > 0$ guarantees that $x(t)$ tends to zero as t tends to infinity and the rate of convergence can be controlled by choice of a_1. The motion on the manifold $s = 0$ is independent of h and g. How can we bring the trajectory to the manifold $s = 0$ and maintain it there? The variable s satisfies the equation

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1x_2 + h(x) + g(x)u$$

Suppose h and g satisfy the inequality

$$\left|\frac{a_1x_2 + h(x)}{g(x)}\right| \leq \varrho(x), \quad \forall\, x \in R^2$$

---

[2] Other output feedback control tools are described in Exercises 14.47 through 14.49.

for some known function $\varrho(x)$. With $V = (1/2)s^2$ as a Lyapunov function candidate for $\dot{s} = a_1x_2 + h(x) + g(x)u$, we have

$$\dot{V} = s\dot{s} = s[a_1x_2 + h(x)] + g(x)su \le g(x)|s|\varrho(x) + g(x)su$$

Taking

$$u = -\beta(x)\ \text{sgn}(s)$$

(It should be noted that $u = -\beta(x)\text{sgn}(s)$ is used only for $s \neq 0$ since in ideal sliding mode control u is not defined on the sliding surface $s = 0$. Alternatively, we can write,$u = -\beta\text{sgn}(s)$ for all s if $\text{sgn}(s)$ is not defined at $s = 0$. The same remark applies throughout the chapter to ideal sliding mode control.) where $\beta(x) \ge \varrho(x) + \beta_0$, $\beta_0 > 0$, and

$$\text{sgn}(s) = \begin{cases} 1, & s > 0 \\ 0, & s = 0 \\ -1, & s < 0 \end{cases}$$

yields

$$dotV \le g(x)|s|\varrho(x) - g(x)[\varrho(x) + \beta_0]s\ \text{sgn}(s) = -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

Thus, $W = \sqrt{2V} = |s|$ satisfies the differential inequality

$$D^+W \le -g_0\beta_0$$

and the comparison lemma shows that

$$W(s(t)) \le W(s(0)) - g_0\beta_0 t$$

Therefore, the trajectory reaches the manifold $s = 0$ in finite time and, once on the manifold, it cannot leave it, as seen from the inequality $\dot{V} \le -g_0\beta_0|s|$. In summary, the motion consists of a *reaching phase* during which trajectories starting off the manifold $s = 0$ move toward it and reach it in finite time, followed by a *sliding phase* during which the motion is confined to the manifold $s = 0$ and the dynamics of the system are represented by the reduced-order model $\dot{x}_1 = -a_1x_1$. A sketch of the phase portrait is shown in Figure 14.1. The manifold $s = 0$ is called the *sliding manifold* and the control law $u = -\beta(x)\ \text{sgn}(s)$ is called *sliding mode control.* The striking feature of sliding mode control is its robustness with respect to h and g. We only need to know the upper bound $\varrho(x)$ and during the sliding phase, the motion is completely independent of h and g.

The sliding mode controller simplifies if, in some domain of interest, h and g satisfy the inequality

$$\left| \frac{a_1x_2 + h(x)}{g(x)} \right| \le k_1$$

for some known nonnegative constant k_1. In this case, we can take

$$u = -k\ \text{sgn}(s), \qquad k > k_1$$

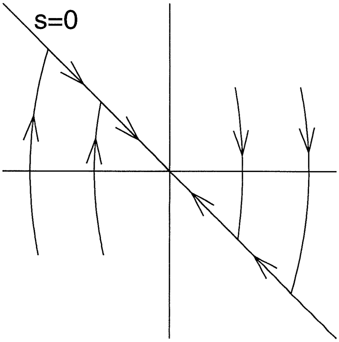

Figure 14.1: A typical phase portrait under sliding mode control.

which takes the form of a simple relay. This form, however, usually leads to a finite region of attraction, which can be estimated as follows: The condition $s\dot{s} \le 0$ in the set $\{|s| \le c\}$ makes it positively invariant. From the equation

$$\dot{x}_1 = x_2 = -a_1 x_1 + s$$

and the function $V_1 = (1/2)x_1^2$, we have

$$\dot{V}_1 = x_1 \dot{x}_1 = -a_1 x_1^2 + x_1 s \le -a_1 x_1^2 + |x_1| c \le 0, \quad \forall\ |x_1| \ge \frac{c}{a_1}$$

Thus,

$$|x_1(0)| \le \frac{c}{a_1} \ \Rightarrow\ |x_1(t)| \le \frac{c}{a_1}, \quad \forall\ t \ge 0$$

and the set

$$\Omega = \left\{ |x_1| \le \frac{c}{a_1},\ |s| \le c \right\}$$

sketched in Figure 14.2, is positively invariant if

$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \le k_1, \qquad \forall\ x \in \Omega$$

Moreover, every trajectory starting in Ω approaches the origin as t tends to infinity. By choosing c large enough, any compact set in the plane can be made a subset of Ω. Therefore, if k can be chosen arbitrarily large, the foregoing control law can achieve semiglobal stabilization.

Because of imperfections in switching devices and delays, sliding mode control suffers from *chattering.* The sketch of Figure 14.3 shows how delays can cause chattering. It depicts a trajectory in the region $s > 0$ heading toward the sliding

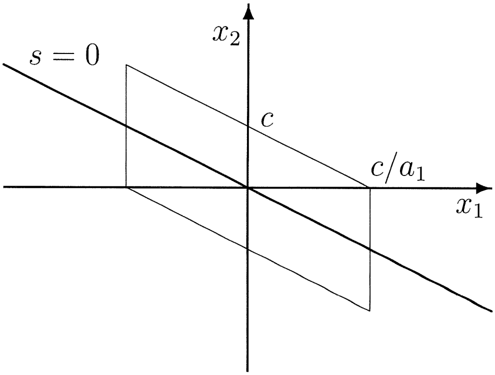

Figure 14.2: Estimate of the region of attraction.

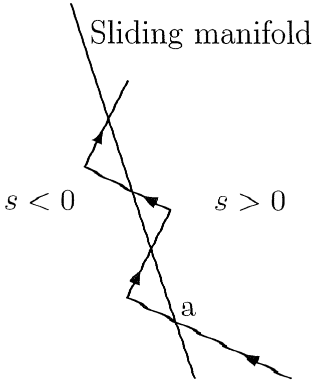

Figure 14.3: Chattering due to delay in control switching.

manifold $s=0$. It first hits the manifold at point a. In ideal sliding mode control, the trajectory should start sliding on the manifold from point a. In reality, there will be a delay between the time the sign of s changes and the time the control switches. During this delay period, the trajectory crosses the manifold into the region $s<0$. When the control switches, the trajectory reverses its direction and heads again toward the manifold. Once again it crosses the manifold, and repetition of this process creates the "zig-zag" motion (oscillation) shown in the sketch, which is known as chattering. Chattering results in low control accuracy, high heat losses in electrical power circuits, and high wear of moving mechanical parts. It may also excite unmodeled high-frequency dynamics, which degrades the performance of the system and may even lead to instability.

To get a better feel for chattering, we simulate the sliding mode control of the

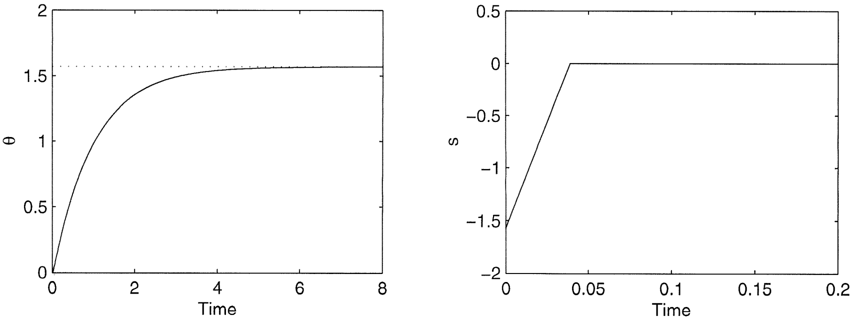

Figure 14.4: Ideal sliding mode control.

pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 + (1/m\ell^2)u\\ u &= -k\ \mathrm{sgn}(s) \ = \ -k\ \mathrm{sgn}(a_1x_1+x_2)\end{aligned}$$

to stabilize the pendulum at $\delta_1 = \pi/2$, where $x_1 = \theta - \delta_1$ and $x_2 = \dot{\theta}$. The constants m, ℓ, k_0, and g_0 are the mass, length, coefficient of friction, and acceleration due to gravity, respectively. We take $a_1 = 1$ and $k = 4$. The gain $k = 4$ is chosen by using

$$\begin{aligned}\left|\frac{a_1x_2+h(x)}{g}\right| &= \left|\ell^2(m-k_0)x_2 - mg_0\ell\cos(x_1)\right|\\ &\leq \ \ell^2|m-k_0|(2\pi) + mg_0\ell \ \leq \ 3.68\end{aligned}$$

where the bound is calculated over the set $\{|x_1| \leq \pi,\ |x_1+x_2| \leq \pi\}$ for $0.05 \leq m \leq 0.2$, $0.9 \leq \ell \leq 1.1$, and $0 \leq k_0 \leq 0.05$. The simulation is performed by using $m = 0.1$, $\ell = 1$, and $k_0 = 0.02$. Figure 14.4 shows ideal sliding mode control, while Figure 14.5 shows a nonideal case where switching is delayed by unmodeled actuator dynamics having the transfer function $1/(0.01s+1)^2$.

We will present two ideas for reducing or eliminating chattering. The first idea is to divide the control into continuous and switching components so as to reduce the amplitude of the switching one. Let $\hat{h}(x)$ and $\hat{g}(x)$ be nominal models of $h(x)$ and $g(x)$, respectively. Taking

$$u = -\ \frac{[a_1x_2+\hat{h}(x)]}{\hat{g}(x)} + v$$

results in

$$\dot{s} = a_1\left[1-\frac{g(x)}{\hat{g}(x)}\right]x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x) + g(x)v \overset{\text{def}}{=} \delta(x) + g(x)v$$

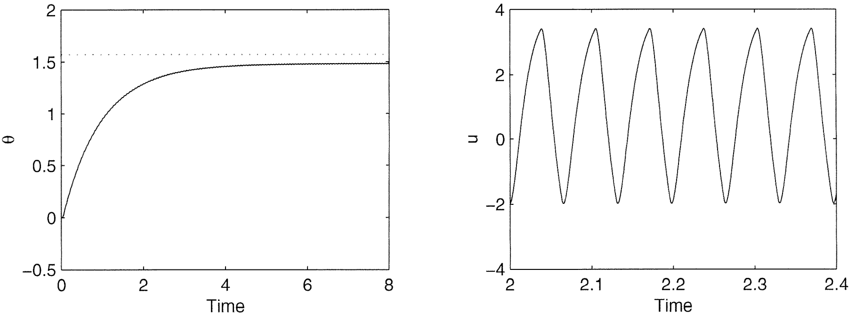

Figure 14.5: Sliding mode control with unmodeled actuator dynamics.

If the perturbation term $\delta(x)$ satisfies the inequality

$$\left|\frac{\delta(x)}{g(x)}\right| \leq \varrho(x)$$

we can take

$$v = -\beta(x)\ \mathrm{sgn}(s)$$

where $\beta(x) \geq \varrho(x)+\beta_0$. Because ϱ is an upper bound on the perturbation term, it is likely to be smaller than an upper bound on the whole function. Consequently, the amplitude of the switching component would be smaller. For example, returning to the pendulum equation and taking $\hat{m} = 0.125$, $\hat{\ell} = 1$, $\hat{k}_0 = 0.025$ to be nominal values of m, ℓ, k_0, we have

$$\left|\frac{\delta(x)}{g}\right| = \left|\left(a_1 m\ell^2 - a_1\hat{m}\hat{\ell}^2 - k_0\ell^2 + \hat{k}_0\hat{\ell}^2\right) x_2 - g_0(m\ell - \hat{m}\hat{\ell})\cos x_1\right| \leq 1.83$$

where the bound is calculated over the same set as before. The modified sliding mode control is taken as

$$u = -0.1x_2 + 1.2263\cos x_1 - 2\ \mathrm{sgn}(s)$$

which shows a reduction in the switching term amplitude from 4 to 2. Figure 14.6 shows simulation of this modified control in the presence of unmodeled actuator dynamics. The reduction in the amplitude of chattering is clear.

The second idea to eliminate chattering is to replace the signum function by a high-slope saturation function; that is, the control law is taken as

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

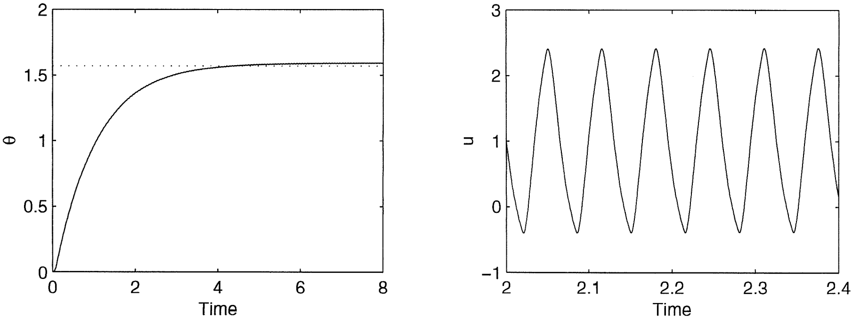

Figure 14.6: Modified sliding mode control with unmodeled actuator dynamics.

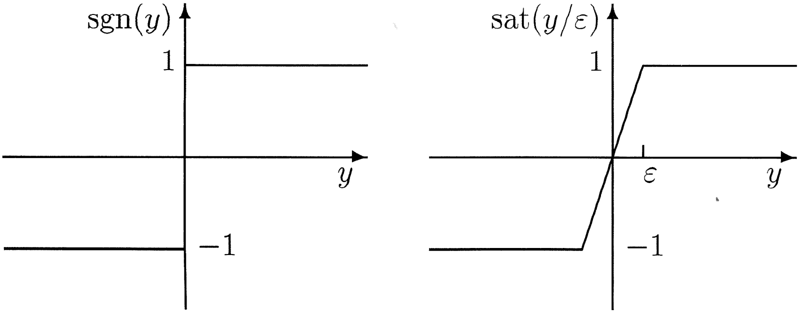

Figure 14.7: The signum nonlinearity and its saturation function approximation.

where sat(·) is the saturation function defined by

$$\operatorname{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \operatorname{sgn}(y), & \text{if } |y| > 1 \end{cases}$$

and ε is a positive constant. The signum and saturation functions are shown in Figure 14.7. The slope of the linear portion of $\operatorname{sat}(s/\varepsilon)$ is $1/\varepsilon$. Good approximation requires the use of small ε. In the limit, as $\varepsilon \to 0$, the saturation function $\operatorname{sat}(s/\varepsilon)$ approaches the signum function $\operatorname{sgn}(s)$. To analyze the performance of the "continuous" sliding mode controller, we examine the reaching phase by using the function $V = (1/2)s^2$ whose derivative satisfies the inequality

$$\dot{V} \le -g_0\beta_0|s|$$

when $|s| \geq \varepsilon$; that is, outside the boundary layer $\{|s| \leq \varepsilon\}$. Therefore, whenever $|s(0)| > \varepsilon$, $|s(t)|$ will be strictly decreasing, until it reaches the set $\{|s| \leq \varepsilon\}$ in finite time and remains inside thereafter. Inside the boundary layer, we have

$$\dot{x}_1 = -a_1 x_1 + s$$

where $|s| \leq \varepsilon$. The derivative of $V_1 = (1/2)x_1^2$ satisfies

$$\dot{V}_1 = -a_1 x_1^2 + x_1 s \leq -a_1 x_1^2 + |x_1|\varepsilon \leq -(1-\theta_1)a_1 x_1^2, \quad \forall\ |x_1| \geq \frac{\varepsilon}{a_1\theta_1}$$

where $0 < \theta_1 < 1$. Thus, the trajectory reaches the set $\Omega_\varepsilon = \{|x_1| \leq \varepsilon/(a_1\theta_1),\ |s| \leq \varepsilon\}$ in finite time. In general, we do not stabilize the origin, but we achieve ultimate boundedness with an ultimate bound that can be reduced by decreasing ε. What happens inside Ω_ε is problem dependent. Let us consider again the pendulum equation and see what happens inside Ω_ε in that case. Inside the boundary layer $\{|s| \leq \varepsilon\}$, the control reduces to the linear feedback law $u = -ks/\varepsilon$, and the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 - (k/m\ell^2\varepsilon)(a_1x_1 + x_2) \end{aligned}$$

has a unique equilibrium point at $(\bar{x}_1, 0)$, where $\bar{x}_1$ satisfies the equation

$$\varepsilon m g_0 \ell \sin(\bar{x}_1 + \delta_1) + k a_1 \bar{x}_1 = 0$$

and can be approximated for small ε by $\bar{x}_1 \approx -(\varepsilon m g_0 \ell / k a_1)\sin\delta_1$. Shifting the equilibrium point to the origin by the change of variables

$$y_1 = x_1 - \bar{x}_1, \quad y_2 = x_2$$

results in

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\sigma(y_1) - \left(\frac{k_0}{m} + \frac{k}{m\ell^2\varepsilon}\right) y_2 \end{aligned}$$

where

$$\sigma(y_1) = (g_0/\ell)[\sin(y_1 + \bar{x}_1 + \delta_1) - \sin(\bar{x}_1 + \delta_1)] + (k a_1/m\ell^2\varepsilon)y_1$$

Using

$$\tilde{V} = \int_0^{y_1} \sigma(s)\ ds + (1/2)y_2^2$$

as a Lyapunov function candidate, it can be verified that

$$\tilde{V} \geq -(g_0/2\ell)y_1^2 + (k a_1/2m\ell^2\varepsilon)y_1^2 + (1/2)y_2^2$$

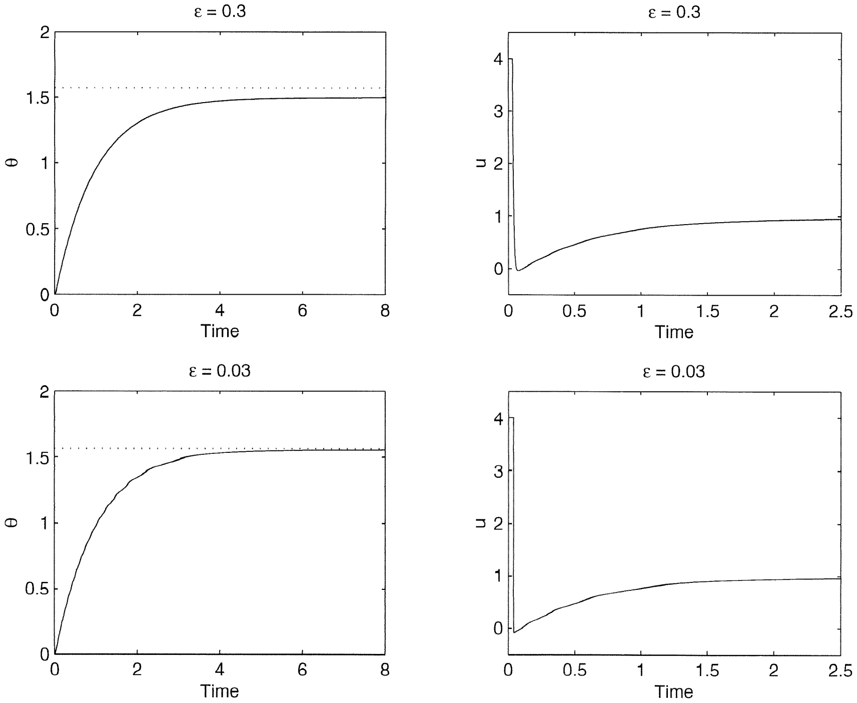

Figure 14.8: "Continuous" sliding mode control.

is positive definite for $(k/\varepsilon) > (m\ell g_0/a_1)$ and its derivative satisfies

$$\dot{\tilde{V}} = -\left(\frac{k_0}{m} + \frac{k}{m\ell^2\varepsilon}\right) y_2^2$$

Application of the invariance principle shows that the equilibrium point $(\bar{x}_1, 0)$ is asymptotically stable and attracts every trajectory in Ω_ε.

For better accuracy, we need to choose ε as small as possible, but a too small value of ε will induce chattering in the presence of time delays or unmodeled fast dynamics. Figure 14.8 shows the performance of the "continuous" sliding mode controller when applied to the pendulum equation for two different values of ε. Figure 14.9 shows the performance in the presence of the unmodeled actuator $1/(0.01s + 1)^2$. The striking observation here is that, while reducing ε improves accuracy in the ideal case, it may not have the same effect in the presence of delays due to chattering.

One special case where we can stabilize the origin without pushing ε too small arises when $h(0) = 0$. In this case, the system, represented inside the boundary

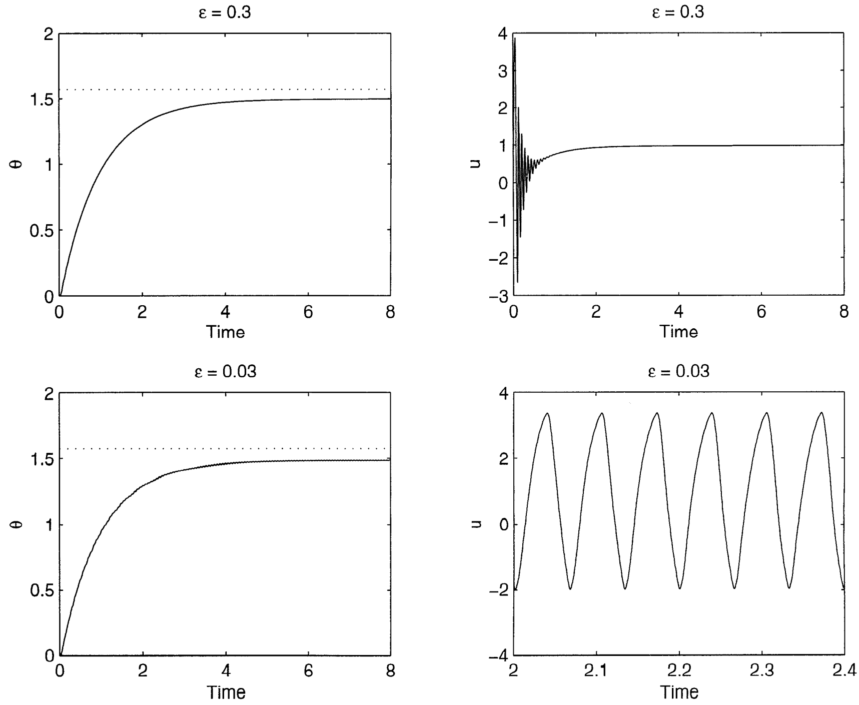

Figure 14.9: "Continuous" sliding mode control with unmodeled actuator dynamics.

layer by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x) - \left[\frac{g(x)k}{\varepsilon}\right](a_1x_1 + x_2) \end{aligned}$$

has an equilibrium point at the origin. We need to choose ε small enough to stabilize the origin and make Ω_ε a subset of its region of attraction. For the pendulum equation with $\delta_1 = \pi$, repeating the foregoing stability analysis confirms that we can achieve our goal if $k/\varepsilon > mg_0\ell/a_1$. For $\ell \leq 1.1$, $m \leq 0.2$, $k = 4$ and $a_1 = 1$, we need $\varepsilon < 1.8534$. Figure 14.10 shows simulation results for $\varepsilon = 1$.

If δ_1 is any angle other than 0 or π (the open-loop equilibrium points), the system will stabilize at an equilibrium point other than the origin, leading to a steady-state error, which was approximated earlier by $(\varepsilon mg_0\ell/ka_1)\sin\delta_1$. We can still achieve zero steady-state error by using integral action. Let $x_0 = \int x_1$ and

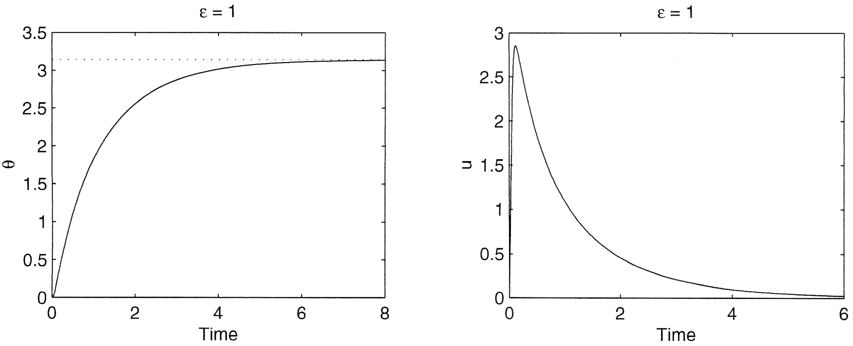

Figure 14.10: "Continuous" sliding mode control when $\delta_1 = \pi$.

write the augmented system as

$$\begin{aligned}
\dot{x}_0 &= x_1 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 + (1/m\ell^2)u
\end{aligned}$$

Take $s = a_0 x_0 + a_1 x_1 + x_2$, where the matrix

$$A_0 = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

is Hurwitz. If

$$m\ell^2 \left| a_0 x_1 + a_1 x_2 - (g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 \right| \le k_1$$

over the domain of interest, we can take the "continuous" sliding mode control as

$$u = -k \ \mathrm{sat}\left(\frac{s}{\varepsilon}\right), \qquad k > k_1$$

which ensures that s will reach the boundary-layer $\{|s| \le \varepsilon\}$ in finite time, since

$$s\dot{s} \le -(k-k_1)|s|, \qquad \text{for } |s| \ge \varepsilon$$

Inside the boundary layer, the system is represented by

$$\dot{\eta} = A_0 \eta + B_0 s, \qquad \text{where } \eta = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Taking $V_1 = \eta^T P_0 \eta$, where P_0 is the solution of the Lyapunov equation $P_0 A_0 + A_0 P_0^T = -I$, it can be verified that

$$\dot{V}_1 = -\eta^T \eta + 2\eta^T P_0 B_0 s \le -(1-\theta_1)\|\eta\|_2^2, \ \ \forall \ \|\eta\|_2 \ge 2\|P_0 B_0\|_2 \varepsilon/\theta_1$$

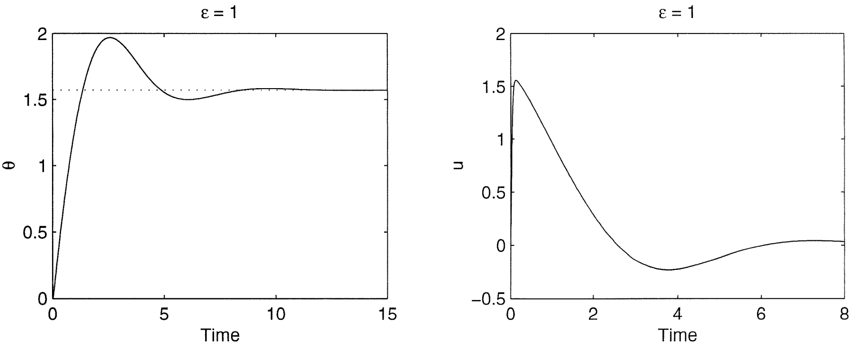

Figure 14.11: "Continuous" sliding mode control with integral action when $\delta_1 = \pi/2$.

where $0 < \theta_1 < 1$. Thus, all trajectories reach the set

$$\Omega_\varepsilon = \left\{ V_1(\eta) \le \frac{4\|P_0B_0\|_2^2\varepsilon^2\|P_0\|_2}{\theta_1^2}, \ |s| \le \varepsilon \right\}$$

in finite time. Inside Ω_ε, the system

$$\begin{aligned}
\dot{x}_0 &= x_1 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 - (k/m\ell^2\varepsilon)(a_0x_0 + a_1x_1 + x_2)
\end{aligned}$$

has a unique equilibrium point at $\bar{x} = [-(\varepsilon m g_0 \ell/k a_0)\sin\delta_1, \ 0, \ 0]^T$. Repeating the previous stability analysis, it can be shown that for sufficiently small ε the equilibrium point $\bar{x}$ is asymptotically stable and every trajectory in Ω_ε converges to $\bar{x}$ as t tends to infinity. Hence, θ converges to the desired position δ_1. Simulation results for $m = 0.1$, $\ell = 1$, $k_0 = 0.02$, $\delta_1 = \pi/2$, $a_0 = a_1 = 1$, $k = 4$, and $\varepsilon = 1$ are shown in Figure 14.11.

14.1.2 Stabilization

Consider the system

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(t,x,u)] \tag{14.1}$$

where $x \in R^n$ is the state, $u \in R^p$ is the control input, and f, B, G, and E are sufficiently smooth functions in a domain $D \subset R^n$ that contains the origin. The function δ is piecewise continuous in t and sufficiently smooth in (x, u) for $(t, x, u) \in [0, \infty) \times D \times R^p$. We assume that f, B, and E are known, while G and

δ could be uncertain. Furthermore, we assume that $E(x)$ is a nonsingular matrix and $G(x)$ is a diagonal matrix whose elements are positive and bounded away from zero; that is, $g_i(x) \geq g_0 > 0$, for all $x \in D$.[3] Suppose $f(0) = 0$ so that, in the absence of δ, the origin is an open-loop equilibrium point. Our goal is to design a state feedback control law to stabilize the origin for all uncertainties in G and δ.

Let $T : D \to R^n$ be a diffeomorphism such that

$$\frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{14.2}$$

where I is the $p \times p$ identity matrix.[4] The change of variables

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x), \qquad \eta \in R^{n-p}, \; \xi \in R^p \tag{14.3}$$

transforms the system into the form

$$\dot{\eta} = f_a(\eta, \xi) \tag{14.4}$$
$$\dot{\xi} = f_b(\eta, \xi) + G(x)E(x)u + \delta(t, x, u) \tag{14.5}$$

The form (14.4)–(14.5) is usually referred to as the *regular form*. To design the sliding mode control, we start by designing the sliding manifold $s = \xi - \phi(\eta) = 0$ such that, when motion is restricted to the manifold, the reduced-order model

$$\dot{\eta} = f_a(\eta, \phi(\eta)) \tag{14.6}$$

has an asymptotically stable equilibrium point at the origin. The design of $\phi(\eta)$ amounts to solving a stabilization problem for the system

$$\dot{\eta} = f_a(\eta, \xi)$$

with ξ viewed as the control input. This stabilization problem may be solved by using the techniques of linearization or feedback linearization presented in the previous two chapters or some nonlinear design tools that will be introduced later in this chapter, such as backstepping or passivity-based control. We assume that we can find a stabilizing continuously differentiable function $\phi(\eta)$ with $\phi(0) = 0$. Next, we design u to bring s to zero in finite time and maintain it there for all future time. Toward that end, let us write the $\dot{s}$-equation:

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)E(x)u + \delta(t, x, u) \tag{14.7}$$

As we saw in the introductory example, we can design u as a pure switching component or it may contain an additional continuous component that cancels the known terms on the right-hand side of (14.7).[5] If $\hat{G}(x)$ is a nominal model of $G(x)$, the continuous component of u will be $-E^{-1}\hat{G}^{-1}[f_b-(\partial\phi/\partial\eta)f_a]$. In the absence of uncertainty; that is, when $\delta=0$ and G is known, taking $u=-E^{-1}G^{-1}[f_b-(\partial\phi/\partial\eta)f_a]$ results in $\dot{s}=0$, which ensures that the condition $s=0$ can be maintained for all future time. To analyze both cases simultaneously, we write the control u as

---

[3] The method can be extended to the case where G is not diagonal by including the off-diagonal elements in δ. Since the dependence of δ on u will be restricted, the method works for cases where G is diagonally dominant.

[4] The existence of T is explored in Exercise 14.9.

$$u=E^{-1}(x)\left\{-L(x)\left[f_b(\eta,\xi)-\frac{\partial\phi}{\partial\eta}f_a(\eta,\xi)\right]+v\right\} \tag{14.8}$$

where $L(x)=\hat{G}^{-1}(x)$, if the known terms are cancelled, and $L=0$, otherwise. Substituting (14.8) into (14.7) yields

$$\dot{s}_i=g_i(x)v_i+\Delta_i(t,x,v), \quad 1\le i\le p \tag{14.9}$$

where Δ_i is the ith component of

$$\begin{aligned}\Delta(t,x,v) &= \delta(t,x,-E^{-1}(x)L(x)(f_b(\eta,\xi)-(\partial\phi/\partial\eta)f_a(\eta,\xi))+E^{-1}(x)v) \\ &\quad +[I-G(x)L(x)]\,[f_b(\eta,\xi)-(\partial\phi/\partial\eta)f_a(\eta,\xi)]\end{aligned}$$

and g_i is the ith diagonal element of G. We assume that the ratio Δ_i/g_i satisfies the inequality

$$\left|\frac{\Delta_i(t,x,v)}{g_i(x)}\right|\le\varrho(x)+\kappa_0\|v\|_\infty, \quad \forall\ (t,x,v)\in[0,\infty)\times D\times R^p, \ \forall\ 1\le i\le p \tag{14.10}$$

where $\varrho(x)\ge0$ (a continuous function) and $\kappa_0\in[0,1)$ are known. Using the estimate (14.10), we proceed to design v to force s toward the manifold $s=0$. Utilizing $V_i=(1/2)s_i^2$ as a Lyapunov function candidate for (14.9), we obtain

$$\dot{V}_i=s_i\dot{s}_i=s_ig_i(x)v_i+s_i\Delta_i(t,x,v)\le g_i(x)\{s_iv_i+|s_i|[\varrho(x)+\kappa_0\|v\|_\infty]\}$$

Take[6]

$$v_i=-\beta(x)\ \mathrm{sgn}(s_i), \quad 1\le i\le p \tag{14.11}$$

where

$$\beta(x)\ge\frac{\varrho(x)}{1-\kappa_0}+\beta_0, \quad \forall\ x\in D \tag{14.12}$$

and $\beta_0>0$. Then,

$$\begin{aligned}\dot{V}_i &\le g_i(x)[-\beta(x)+\varrho(x)+\kappa_0\beta(x)]|s_i| = g_i(x)[-(1-\kappa_0)\beta(x)+\varrho(x)]|s_i| \\ &\le g_i(x)[-\varrho(x)-(1-\kappa_0)\beta_0+\varrho(x)]|s_i| \le -g_0\beta_0(1-\kappa_0)|s_i|\end{aligned}$$

The inequality $\dot{V}_i\le-g_0\beta_0(1-\kappa_0)|s_i|$ ensures that all trajectories starting off the manifold $s=0$ reach it in finite time and those on the manifold cannot leave it.

The procedure for designing a sliding mode stabilizing controller can be summarized by the following steps:

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[5] The continuous component is usually referred to as the *equivalent control.*

[6] For convenience, we take the coefficient of the signum function to be the same for all control components. See Exercise 14.12 for a relaxation of this restriction.

- Design the sliding manifold $\xi = \phi(\eta)$ to stabilize the reduced-order system (14.6).
- Take the control u as $u = E^{-1}\{-\hat{G}^{-1}[f_b - (\partial\phi/\partial\eta)f_a] + v\}$ or $u = E^{-1}v$.
- Estimate $\varrho(x)$ and κ_0 in (14.10), where Δ depends on the choice made in the previous step.
- Choose $\beta(x)$ that satisfies (14.12) and take the switching (discontinuous) control v as given by (14.11).

This procedure exhibits model-order reduction because the main design task is performed on the reduced-order system (14.6). The key feature of sliding mode control is its robustness to matched uncertainties. During the reaching phase, the task of forcing the trajectories toward the sliding manifold and maintaining them there is achieved by the switching control (14.11), provided $\beta(x)$ satisfies the inequality (14.12). From (14.10), we see that $\varrho(x)$ is a measure of the size of the uncertainty. Since we do not require $\varrho(x)$ to be small, the switching controller can handle fairly large uncertainties, limited only by practical constraints on the amplitude of the control signals. During the sliding phase, the motion of the system, as determined by (14.6), is independent of the matched uncertain terms G and δ.

The sliding mode controller contains the discontinuous signum function $\text{sgn}(s_i)$, which raises some theoretical as well as practical issues. Theoretical issues like existence and uniqueness of solutions and validity of Lyapunov analysis will have to be examined in a framework that does not require the state equation to have locally Lipschitz right-hand-side functions.[7] There is also the practical issue of chattering due to imperfections in switching devices and delays, which was illustrated in the introductory example. To eliminate chattering, we use a continuous approximation of the signum function.[8] By using a continuous approximation, we also avoid the theoretical difficulties associated with discontinuous controllers.[9] We approximate the signum function $\text{sgn}(s_i)$ by the high-slope saturation function $\text{sat}(s_i/\varepsilon)$;[10] that is,

$$v_i = -\beta(x)\ \text{sat}\left(\frac{s_i}{\varepsilon}\right), \quad 1 \le i \le p \tag{14.13}$$

where $\beta(x)$ satisfies (14.12). To analyze the performance of the "continuous" sliding mode controller, we examine the reaching phase by using the Lyapunov function

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[7] To read about differential equations with discontinuous right-hand side, consult [58], [147], [173], and [198].

[8] Other approaches to eliminate chattering include the use of observers [197] and extending the dynamics of the system by using integrators [177]. It should be noted that the continuous approximation approach cannot be used in applications where actuators have to be used in an on–off operation mode, like thyristors, for example.

[9] While we do not pursue rigorous analysis of the discontinuous sliding mode controller, the reader is encouraged to use simulations to examine the performance of both the discontinuous controller and its continuous approximation.

[10] Smooth approximations are discussed in Exercises 14.11.

$V_i = (1/2)s_i^2$. The derivative of V_i satisfies the inequality

$$\dot{V}_i \leq g_i(x)\left[-\beta(x)s_i \text{ sat}\left(\frac{s_i}{\varepsilon}\right) + \varrho(x)|s_i| + \kappa_0\beta(x)|s_i|\right]$$

In the region $|s_i| \geq \varepsilon$, we have

$$\dot{V}_i \leq g_i(x)[-(1-\kappa_0)\beta(x) + \varrho(x)]|s_i| \leq -g_0\beta_0(1-\kappa_0)|s_i|$$

which shows that whenever $|s_i(0)| > \varepsilon$, $|s_i(t)|$ will decrease until it reaches the set $\{|s_i| \leq \varepsilon\}$ in finite time and remains inside thereafter. The set $\{|s_i| \leq \varepsilon,\ 1 \leq i \leq p\}$ is called the *boundary layer*. To study the behavior of η, we assume that, together with the sliding-manifold design $\xi = \phi(\eta)$, there is a (continuously differentiable) Lyapunov function $V(\eta)$ that satisfies the inequalities

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \tag{14.14}$$

$$\frac{\partial V}{\partial \eta} f_a(\eta, \phi(\eta) + s) \leq -\alpha_3(\|\eta\|), \quad \forall\ \|\eta\| \geq \gamma(\|s\|) \tag{14.15}$$

for all $(\eta, \xi) \in T(D)$, where α_1, α_2, α_3, and γ are class $\mathcal{K}$ functions.[11] Noting that

$$|s_i| \leq c, \text{ for } 1 \leq i \leq p \ \Rightarrow\ \|s\| \leq k_1 c \Rightarrow \dot{V} \leq -\alpha_3(\|\eta\|), \text{ for } \|\eta\| \geq \gamma(k_1 c)$$

for some positive constant k_1,[12] we define a class $\mathcal{K}$ function α by

$$\alpha(r) = \alpha_2(\gamma(k_1 r))$$

Then,

$$\begin{aligned} V(\eta) \geq \alpha(c) \ &\Rightarrow\ V(\eta) \geq \alpha_2(\gamma(k_1 c)) \Rightarrow \alpha_2(\|\eta\|) \geq \alpha_2(\gamma(k_1 c)) \\ &\Rightarrow\ \|\eta\| \geq \gamma(k_1 c) \ \Rightarrow\ \dot{V} \leq -\alpha_3(\|\eta\|) \leq -\alpha_3(\gamma(k_1 c)) \end{aligned}$$

which shows that the set $\{V(\eta) \leq c_0\}$ with $c_0 \geq \alpha(c)$ is positively invariant because $\dot{V}$ is negative on the boundary $V(\eta) = c_0$. (See Figure 14.12.) It follows that the set

$$\Omega = \{V(\eta) \leq c_0\} \times \{|s_i| \leq c,\ 1 \leq i \leq p\}, \text{ with } c_0 \geq \alpha(c) \tag{14.16}$$

is positively invariant whenever $c > \varepsilon$ and $\Omega \subset T(D)$. Choose ε, $c > \varepsilon$, and $c_0 \geq \alpha(c)$ such that $\Omega \subset T(D)$. The compact set Ω serves as our estimate of the "region of attraction." For all initial states in Ω, the trajectories will be bounded for all $t \geq 0$. After some finite time, we have $|s_i(t)| \leq \varepsilon$. It follows from the foregoing analysis that $\dot{V} \leq -\alpha_3(\gamma(k_1\varepsilon))$ for all $V(\eta) \geq \alpha(\varepsilon)$. Hence, the trajectories will reach the positively invariant set

$$\Omega_\varepsilon = \{V(\eta) \leq \alpha(\varepsilon)\} \times \{|s_i| \leq \varepsilon,\ 1 \leq i \leq p\} \tag{14.17}$$

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[11] Inequality (14.15) implies local input-to-state stability of the system $\dot{\eta} = f_a(\eta, \phi(\eta) + s)$ when s is viewed as the input. (See Exercise 4.60.)

[12] The constant k_1 depends on the type of norm used in the analysis.

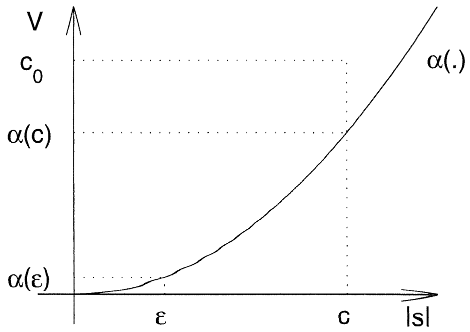


Figure 14.12: A representation of the set Ω for a scalar s. $\dot{V} < 0$ above the $\alpha(\cdot)$-curve.

in finite time. The set Ω_ε can be made arbitrarily small by choosing ε small enough. In the limit, as $\varepsilon \to 0$, Ω_ε shrinks to the origin, which shows that the "continuous" sliding mode controller recovers the performance of its discontinuous counterpart. Finally, we note that if all the assumptions hold globally and $V(\eta)$ is radially unbounded, we can choose Ω arbitrarily large to include any initial state. We summarize our conclusions in the next theorem.

Theorem 14.1 *Consider the system* (14.4)–(14.5). *Suppose there exist* $\phi(\eta)$, $V(\eta)$, $\varrho(x)$, *and* κ_0, *which satisfy* (14.10), (14.14), *and* (14.15). *Let* u *and* v *be given by* (14.8) *and* (14.11), *respectively. Suppose* ε, $c > \varepsilon$, *and* $c_0 \geq \alpha(c)$ *are chosen such that the set* Ω, *defined by* (14.16), *is contained in* $T(D)$. *Then, for all* $(\eta(0), \xi(0)) \in \Omega$, *the trajectory* $(\eta(t), \xi(t))$ *is bounded for all* $t \geq 0$ *and reaches the positively invariant set* Ω_ε, *defined by* (14.17), *in finite time. Moreover, if the assumptions hold globally and* $V(\eta)$ *is radially unbounded, the foregoing conclusion holds for any initial state.* ◇

The theorem shows that the "continuous" sliding mode controller achieves ultimate boundedness with an ultimate bound that can be controlled by the design parameter ε. It also gives conditions for global ultimate boundedness. Since the uncertainty δ could be nonvanishing at $x = 0$, ultimate boundedness is the best we can expect, in general. If, however, δ vanishes at the origin, then we may be able to show asymptotic stability of the origin, as we do in the next theorem.

Theorem 14.2 *Suppose all the assumptions of Theorem* 14.1 *are satisfied with* $\varrho(0) = 0$ *and* $\kappa_0 = 0$. *Suppose further that the origin of* $\dot{\eta} = f_a(\eta, \phi(\eta))$ *is exponentially stale. Then, there exits* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon < \varepsilon^*$, *the origin of the closed-loop system is exponentially stable and* Ω *is a subset of its region of attraction. Moreover, if the assumptions hold globally, the origin will be globally uniformly asymptotically stable.* ◇

Proof: For inequality (14.10) to hold with $\kappa_0 = 0$, Δ_i must be independent of v. Therefore, we write $\Delta_i = \Delta_i(t, x)$. Theorem 14.1 confirms that all trajectories starting in Ω enter Ω_ε in finite time. Inside Ω_ε, the closed-loop system is given by

$$\begin{aligned} \dot{\eta} &= f_a(\eta, \phi(\eta) + s) \\ \dot{s}_i &= \Delta_i(t, x) - \frac{g_i(x)\beta(x)}{\varepsilon} s_i, \quad 1 \le i \le p \end{aligned}$$

By (the converse Lyapunov) Theorem 4.14, there exists a Lyapunov function $V_0(\eta)$ that satisfies

$$c_1\|\eta\|_2^2 \le V_0(\eta) \le c_2\|\eta\|_2^2$$

$$\frac{\partial V_0}{\partial \eta} f_a(\eta, \phi(\eta)) \le -c_3\|\eta\|_2^2$$

$$\left\|\frac{\partial V_0}{\partial \eta}\right\|_2 \le c_4\|\eta\|_2$$

in some neighborhood N_η of $\eta = 0$. By the smoothness of f_a and Δ and the fact that $|\Delta_i(t, x)| \le g_i(x)\varrho(x)$ with $\varrho(0) = 0$, we have

$$\|f_a(\eta, \phi(\eta) + s) - f_a(\eta, \phi(\eta))\|_2 \le k_1\|s\|_2$$

$$\|\Delta\|_2 \le k_2\|\eta\|_2 + k_3\|s\|_2$$

in some neighborhood N of $(\eta, \xi) = (0, 0)$. Choose ε small enough that $\Omega_\varepsilon \subset N_\eta$ and $\Omega_\varepsilon \subset N$. Using the Lyapunov function candidate

$$W = V_0(\eta) + \tfrac{1}{2}\sum_{i=1}^{p} s_i^2$$

it can be shown that

$$\dot{W} \le -c_3\|\eta\|_2^2 + c_4k_1\|\eta\|_2\|s\|_2 + k_2\|\eta\|_2\|s\|_2 + k_3\|s\|_2^2 - \frac{\beta_0 g_0}{\varepsilon}\|s\|_2^2$$

The right-hand side can be made negative definite in Ω_ε by choosing ε small enough. The rest of the proof is straightforward. □

The basic idea of the foregoing proof is that, inside the boundary layer, the control $v_i = -\beta(x)s_i/\varepsilon$ acts as high-gain feedback control for small ε. By choosing ε small enough, the high-gain feedback stabilizes the origin. We could have used the high-gain feedback control throughout the space, but the control amplitude will be too high when s is far from zero.

We have emphasized the robustness of sliding mode control with respect to matched uncertainties. What about unmatched uncertainties? Suppose equation (14.1) is modified to

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(t, x, u)] + \delta_1(x) \qquad (14.18)$$

The change of variables (14.3) transforms the system into

$$\begin{aligned}\dot{\eta} &= f_a(\eta,\xi) + \delta_a(\eta,\xi)\\ \dot{\xi} &= f_b(\eta,\xi) + G(x)E(x)u + \delta(t,x,u) + \delta_b(\eta,\xi)\end{aligned}$$

where

$$\begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix} = \frac{\partial T}{\partial x}\delta_1$$

The term δ_b is added to the matched uncertainty δ. Its only effect is to change the upper bound on Δ_i/g_i. The term δ_a, on the other hand, is unmatched. It changes the reduced-order model on the sliding manifold to

$$\dot{\eta} = f_a(\eta,\phi(\eta)) + \delta_a(\eta,\phi(\eta))$$

The design of ϕ will have to guarantee asymptotic stability of the origin $\eta = 0$ in the presence of the uncertainty δ_a. This is a robust stabilization problem that may be approached by other robust stabilization techniques such as high-gain feedback. The difference between matched and unmatched uncertainties is that sliding mode control guarantees robustness for any matched uncertainty provided an upper bound is known and the needed control effort can be provided. There is no such guarantee for unmatched uncertainties. We may have to restrict its size to robustly stabilize the system on the sliding manifold. The next two examples illustrate these points.

Example 14.1 Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1 x_1 \sin x_2\\ \dot{x}_2 &= \theta_2 x_2^2 + x_1 + u\end{aligned}$$

where θ_1 and θ_2 are unknown parameters that satisfy $|\theta_1| \le a$ and $|\theta_2| \le b$ for some known bounds a and b. The system is already in the regular form with $\eta = x_1$ and $\xi = x_2$. Uncertainty due to θ_2 is matched, while uncertainty due to θ_1 is unmatched. We consider the system

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2$$

and design x_2 to robustly stabilize the origin $x_1 = 0$. This can be achieved with $x_2 = -kx_1$, $k > a$, because

$$x_1\dot{x}_1 = -kx_1^2 + \theta_1 x_1^2 \sin(-kx_1) \le -(k-a)x_1^2$$

The sliding manifold is $s = x_2 + kx_1 = 0$ and

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

To cancel the known term on the right-hand side, we take

$$u = -x_1 - kx_2 + v$$

to obtain

$$\dot{s} = v + \Delta(x)$$

where $\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$. Because

$$|\Delta(x)| \le ak|x_1| + bx_2^2$$

we take

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \qquad \beta_0 > 0$$

and

$$u = -x_1 - kx_2 - \beta(x)\ \mathrm{sgn}(s)$$

This controller, or its continuous approximation with sufficiently small ε, globally stabilizes the origin. △

In the foregoing example, we were able to use high-gain feedback to robustly stabilize the reduced-order model for unmatched uncertainties that satisfy $|\theta_1| \le a$, without having to restrict a. In general, this may not be possible, as illustrated by the next example.

Example 14.2 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + (1-\theta_1)x_2 \\ \dot{x}_2 &= \theta_2 x_2^2 + x_1 + u \end{aligned}$$

where θ_1 and θ_2 are unknown parameters that satisfy $|\theta_1| \le a$ and $|\theta_2| \le b$. We consider the system

$$\dot{x}_1 = x_1 + (1-\theta_1)x_2$$

and design x_2 to robustly stabilize the origin $x_1 = 0$. We note that the system is not stabilizable at $\theta_1 = 1$. Hence, we must limit a to be less than one. Using $x_2 = -kx_1$, we obtain

$$x_1\dot{x}_1 = x_1^2 - k(1-\theta_1)x_1^2 \le -[k(1-a) - 1]x_1^2$$

Hence, the origin $x_1 = 0$ can be stabilized by taking $k > 1/(1-a)$. The sliding manifold is $s = x_2 + kx_1 = 0$. Proceeding as in the previous example, we end up with the sliding mode control

$$u = -(1+k)x_1 - kx_2 - \beta(x)\ \mathrm{sgn}(s)$$

where $\beta(x) = bx_2^2 + ak|x_2| + \beta_0$ with $\beta_0 > 0$. △

14.1.3 Tracking

Consider the single-input–single-output system

$$\begin{aligned} \dot{x} &= f(x) + \delta_1(x) + g(x)[u + \delta(t,x,u)] && (14.19) \\ y &= h(x) && (14.20) \end{aligned}$$

where x, u, and y are the state, control input, and controlled output, respectively. We assume that f, g, h, and δ_1 are sufficiently smooth in a domain $D \subset R^n$ and δ is piecewise continuous in t and sufficiently smooth in (x,u) for $(t,x,u) \in [0,\infty) \times D \times R$. Furthermore, we assume that f and h are known, while g, δ, and δ_1 could be uncertain. For all possible uncertainties in g, we assume that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u && (14.21) \\ y &= h(x) && (14.22) \end{aligned}$$

has relative degree ρ in D; that is,

$$L_g h(x) = \cdots = L_g L_f^{\rho-2} h(x) = 0, \qquad L_g L_f^{\rho-1} h(x) \geq a > 0$$

for all $x \in D$.[13] Our goal is to design a state feedback control law such that the output y asymptotically tracks a reference signal $r(t)$, where

- $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$, and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t;
- the signals $r,\ldots,r^{(\rho)}$ are available on line.

We know from our study of input–output linearization (Section 13.2) that the system (14.21)–(14.22) can be transformed into the normal form by the change of variables

$$\begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} = \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} = T(x) \qquad (14.23)$$

where ϕ_1 to $\phi_{n-\rho}$ satisfy the partial differential equations

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n-\rho, \ \forall \ x \in D$$

---

[13] Without loss of generality, we assume that $L_g L_f^{\rho-1} h$ is positive. If it is negative, we can substitute $u = -\tilde{u}$ and proceed to design $\tilde{u}$. Thus, by solving the problem for a positive $L_g L_f^{\rho-1} h$, we can cover both signs by multiplying the control by $\text{sign}(L_g L_f^{\rho-1} h)$.

We assume that $T(x)$ is a diffeomorphism over D. Since f and h are known, while g could be uncertain, the function ψ is known, while ϕ could be unknown. We would like to restrict the perturbations δ and δ_1 such that when the change of variables (14.23) is applied to the perturbed system (14.19)–(14.20) it preserves the normal form structure. From the relative-degree condition, it is clear that the state equation for η will be independent of u. Let us calculate the state equation for ξ:

$$\dot{\xi}_1 = \frac{\partial h}{\partial x}[f + \delta_1 + g(u + \delta)] = \frac{\partial h}{\partial x}(f + \delta_1)$$

If δ_1 belongs to the null space of $[\partial h/\partial x]$; that is, $[\partial h/\partial x]\delta_1(x) = 0$, for all $x \in D$, we have

$$\dot{\xi}_1 = L_f h(x) = \xi_2$$

Similarly,

$$\dot{\xi}_2 = \frac{\partial(L_f h)}{\partial x}[f + \delta_1 + g(u + \delta)] = \frac{\partial(L_f h)}{\partial x}(f + \delta_1)$$

If δ_1 belongs to the null space of $[\partial(L_f h)/\partial x]$ for all $x \in D$, we have

$$\dot{\xi}_2 = L_f^2 h(x) = \xi_2$$

Continuing in the same manner, it can be seen that if

$$\frac{\partial(L_f^i h)}{\partial x}\delta_1(x) = 0, \quad \text{for } 1 \le i \le \rho - 2, \quad \forall\ x \in D \tag{14.24}$$

then the change of variables (14.23) produces the normal form

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi)\\
\dot{\xi}_1 &= \xi_2\\
\vdots\ &\quad \vdots\\
\dot{\xi}_{\rho-1} &= \xi_\rho\\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)]\\
y &= \xi_1
\end{aligned}$$

Let

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}$$

The change of variables $e = \xi - \mathcal{R}$ yields

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi)\\
\dot{e}_1 &= e_2\\
\vdots\ &\quad \vdots\\
\dot{e}_{\rho-1} &= e_\rho\\
\dot{e}_\rho &= L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)] - r^\rho(t)
\end{aligned}$$

Asymptotic tracking will be achieved if we design a state feedback control law to ensure that $e(t)$ is bounded and converges to zero as t tends to infinity. Boundedness of e will ensure boundedness of ξ because $\mathcal{R}(t)$ is bounded. We need also to ensure boundedness of η. For that, we restrict our analysis to the case where the system

$$\dot{\eta} = f_0(\eta, \xi)$$

is bounded-input–bounded-state stable. This will be the case for any bounded input ξ and any initial state $\eta(0)$ if the system $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable. So, from this point on, we concentrate our attention on showing boundedness and convergence of e. The $\dot{e}$-equation takes the regular form of (14.4)–(14.5) with $\eta = [e_1, \ldots, e_{\rho-1}]^T$ and $\xi = e_\rho$. We start with the system

$$\begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \vdots & & \vdots \\ \dot{e}_{\rho-1} & = & e_\rho \end{array}$$

where e_ρ is viewed as the control input. We want to design e_ρ to stabilize the origin. For this linear system (in the controllable canonical form), we can achieve this task by the linear control

$$e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})$$

where k_1 to $k_{\rho-1}$ are chosen such that the polynomial

$$s^{\rho-1} + k_{\rho-1} s^{\rho-2} + \cdots + k_1$$

is Hurwitz. Then, the sliding manifold is

$$s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho = 0$$

and

$$\dot{s} = k_1 e_2 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)] - r^\rho(t)$$

We can proceed by designing $u = v$ as a pure switching component, or we can take

$$u = -\frac{1}{L_{\hat{g}} L_f^{\rho-1} h(x)} \left[k_1 e_2 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) - r^\rho(t) \right] + v$$

to cancel the known terms on the right-hand side, where $\hat{g}(x)$ is a nominal model of $g(x)$. Note that when g is known; that is, when $\hat{g} = g$, the term

$$-\frac{1}{L_g L_f^{\rho-1} h(x)} \left[L_f^\rho h(x) - r^\rho(t) \right]$$

is the feedback linearizing term we used in Section 13.4.2. In either case, the $\dot{s}$-equation can be written as

$$\dot{s} = L_g L_f^{\rho-1} h(x) v + \Delta(t, x, v)$$

Suppose

$$\left|\frac{\Delta(t,x,v)}{L_gL_f^{\rho-1}h(x)}\right| \leq \varrho(x) + \kappa_0|v|, \qquad 0 \leq \kappa_0 < 1$$

for all $(t,x,v) \in [0,\infty) \times D \times R$, where ϱ and κ_0 are known. Then,

$$v = -\beta(x)\ \mathrm{sgn}(s)$$

where $\beta(x) \geq \varrho(x)/(1-\kappa_0) + \beta_0$ with $\beta_0 > 0$, and its continuous approximation is obtained by replacing $\mathrm{sgn}(s)$ by $\mathrm{sat}(s/\varepsilon)$. We leave it to the reader (Exercise 14.13) to show that with the "continuous" sliding mode controller, there exits a finite time T_1, possibly dependent on ε and the initial states, and a positive constant k, independent of ε and the initial states, such that $|y(t) - r(t)| \leq k\varepsilon$ for all $t \geq T_1$.

14.1.4 Regulation via Integral Control

Consider the single-input–single-output system

$$\begin{aligned} \dot{x} &= f(x) + \delta_1(x,w) + g(x,w)[u + \delta(x,u,w)] && (14.25)\\ y &= h(x) && (14.26) \end{aligned}$$

where $x \in R^n$ is the state, $u \in R$ is the control input, $y \in R$ is the controlled output, and $w \in R^l$ is a vector of unknown constant parameters and disturbances. The functions f, g, h, δ, and δ_1 are sufficiently smooth in (x,u) and continuous in w for $x \in D \subset R^n$, $u \in R$, and $w \in D_w \subset R^l$, where D and D_w are open connected sets. We assume that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x,w)u && (14.27)\\ y &= h(x) && (14.28) \end{aligned}$$

has relative degree ρ in D uniformly in w; that is,

$$L_gh(x,w) = \cdots = L_gL_f^{\rho-2}h(x,w) = 0, \qquad L_gL_f^{\rho-1}h(x,w) \geq a > 0$$

for all $(x,w) \in D \times D_w$. Our goal is to design a state feedback control law such that the output y asymptotically tracks a constant reference $r \in D_r \subset R$, where D_r is an open connected set. This is a special case of the tracking problem of the previous section, where the reference is constant and the uncertainty is parameterized by w. Therefore, we can use the sliding mode controller of the previous section. When the signum function $\mathrm{sgn}(s)$ is approximated by the saturation function $\mathrm{sat}(s/\varepsilon)$, the regulation error will be ultimately bounded by a constant $k\varepsilon$ for some $k > 0$. This is the best we can achieve in a general tracking problem, but in a regulation problem, we can use integral control to achieve zero steady-state error. Proceeding as in Section 12.3, we augment the integral of the regulation error $y - r$ with the system and design a feedback controller that stabilizes the augmented system at an equilibrium point, where $y = r$. Toward that end, we assume that, for each pair $(r, w) \in D_r \times D_w$, there is a unique pair (x_{ss}, u_{ss}) that depends continuously on (r, w) and satisfies the equations

$$\begin{aligned} 0 &= f(x_{ss}) + \delta_1(x_{ss}, w) + g(x_{ss}, w)[u_{ss} + \delta(x_{ss}, u_{ss}, w)] \\ r &= h(x_{ss}) \end{aligned}$$

so that x_{ss} is the desired equilibrium point and u_{ss} is the steady-state control that is needed to maintain equilibrium at x_{ss}. Assuming that

$$\frac{\partial(L_f^i h)}{\partial x}\delta_1(x, w) = 0, \qquad \text{for } 1 \le i \le \rho - 2, \qquad \forall\ (x, w) \in D \times D_w$$

the change of variables (14.23) transforms the system (14.25)–(14.26) into the normal form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi, w) \\ \dot{\xi}_1 &= \xi_2 \\ &\ \ \vdots \\ \dot{\xi}_{\rho-1} &= \xi_\rho \\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x, w) + L_g L_f^{\rho-1} h(x, w)[u + \delta(x, u, w)] \\ y &= \xi_1 \end{aligned}$$

and maps the equilibrium point x_{ss} into (η_{ss}, ξ_{ss}) where $\xi_{ss} = [r, 0, \ldots, 0]^T$. Augmenting the integrator

$$\dot{e}_0 = y - r$$

with the foregoing equation and applying the change of variables

$$z = \eta - \eta_{ss}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\rho \end{bmatrix} = \begin{bmatrix} \xi_1 - r \\ \xi_2 \\ \vdots \\ \xi_\rho \end{bmatrix}$$

we obtain the augmented system

$$\begin{aligned} \dot{z} &= f_0(z + \eta_{ss}, \xi, w) \overset{\text{def}}{=} \tilde{f}_0(z, e, w, r) \\ \dot{e}_0 &= e_1 \\ \dot{e}_1 &= e_2 \\ &\ \ \vdots \\ \dot{e}_{\rho-1} &= e_\rho \\ \dot{e}_\rho &= L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x, w) + L_g L_f^{\rho-1} h(x, w)[u + \delta(x, u, w)] \end{aligned}$$

which preserves the normal form structure with a chain of $\rho+1$ integrators. Therefore, the design of sliding mode control can proceed as in the previous section. In particular, we can take

$$s = k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho$$

where k_0 to $k_{\rho-1}$ are chosen such that the polynomial

$$s^\rho + k_{\rho-1} s^{\rho-1} + \cdots + k_1 s + k_0$$

is Hurwitz. Then,

$$\dot{s} = k_0 e_1 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x,w) + L_g L_f^{\rho-1} h(x,w)[u + \delta(x,u,w)]$$

The control u can be taken as

$$u = v \quad \text{or} \quad u = -\frac{1}{L_{\hat{g}} L_f^{\rho-1} h(x)} \left[k_0 e_1 + \cdots + k_{\rho-1} e_\rho + L_f^\rho(x) \right] + v$$

where $\hat{g}(x)$ is a nominal model of $g(x,w)$, to obtain

$$\dot{s} = L_g L_f^{\rho-1} h(x,w) v + \Delta(x,v,w,r)$$

If

$$\left| \frac{\Delta(x,v,w,r)}{L_g L_f^{\rho-1} h(x,w)} \right| \le \varrho(x) + \kappa_0 |v|, \qquad 0 \le \kappa_0 < 1$$

for all $(x,v,w,r) \in D \times R \times D_w \times D_r$, where ϱ and κ_0 are known, we can take

$$v = -\beta(x) \ \text{sat}\left(\frac{s}{\varepsilon}\right)$$

where $\beta(x) \ge \varrho(x)/(1-\kappa_0) + \beta_0$ with $\beta_0 > 0$. The closed-loop system has an equilibrium point at $(z, e_0, e) = (0, \bar{e}_0, 0)$. Showing convergence to this equilibrium point can be done by repeating the analysis of Section 14.1.2. In particular, if there is a Lyapunov function $V_1(z,w,r)$ for the system $\dot{z} = \tilde{f}_0(z,e,w,r)$ that satisfies the inequalities

$$\tilde{\alpha}_1(\|z\|) \le V_1(z,w,r) \le \tilde{\alpha}_2(\|z\|)$$

$$\frac{\partial V_1}{\partial z} \tilde{f}_0(z,e,w,r) \le -\tilde{\alpha}_3(\|z\|), \qquad \forall \ \|z\| \ge \tilde{\gamma}(\|e\|)$$

uniformly in (w,r) for some class $\mathcal{K}$ functions $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, and $\tilde{\gamma}$, we can show that there are two compact positively invariant sets Ω and Ω_ε such that every trajectory starting in Ω enters Ω_ε in finite time. The construction of the sets Ω and Ω_ε is done in three steps. We write the closed-loop system in the form

$$\begin{aligned} \dot{z} &= \tilde{f}_0(z,e,w,r) \\ \dot{\zeta} &= A\zeta + Bs \\ \dot{s} &= -(L_g L_f^{\rho-1} h)\beta \ \text{sat}\left(\frac{s}{\varepsilon}\right) + \Delta \end{aligned}$$

where $\zeta = [e_0, \dots, e_{\rho-1}]^T$ and A is Hurwitz, and use the inequality

$$s\dot{s} \leq -a\beta_0(1-\kappa_0)|s|, \qquad \text{for } |s| \geq \varepsilon$$

to show that the set $\{|s| \leq c\}$, with $c > \varepsilon$, is positively invariant. In the second step, we use the Lyapunov function $V_2(\zeta) = \zeta^T P \zeta$, where P is the solution of the Lyapunov equation $PA + A^T P = -I$, and the inequality

$$\dot{V}_2 \leq -\zeta^T \zeta + 2\|\zeta\| \; \|PB\| \; |s|$$

to show that the set $\{|s| \leq c\} \cap \{V_2 \leq c^2 \rho_1\}$ is positively invariant, for some $\rho_1 > 0$. Inside this set, we have $\|e\| \leq c\rho_2$, for some $\rho_2 > 0$. Finally, we use the inequality

$$\dot{V}_1 \leq -\tilde{\alpha}_3(\|z\|), \qquad \forall \; \|z\| \geq \tilde{\gamma}(c\rho_2)$$

to show that

$$\Omega = \{|s| \leq c\} \cap \{V_2 \leq c^2\rho_1\} \cap \{V_1 \leq c_0\}$$

is positively invariant for any $c_0 \geq \tilde{\alpha}_2(\tilde{\gamma}(c\rho_2))$. Similarly, it can be shown that

$$\Omega_\varepsilon = \{|s| \leq \varepsilon\} \cap \{V_2 \leq \varepsilon^2\rho_1\} \cap \{V_1 \leq \tilde{\alpha}_2(\tilde{\gamma}(\varepsilon\rho_2))\}$$

is positively invariant and every trajectory starting in Ω enters Ω_ε in finite time.

If $z = 0$ is an exponentially stable equilibrium point of $\dot{z} = \tilde{f}_0(z, e, w, r)$, we can repeat the proof of Theorem 14.2 to show that every trajectory in Ω_ε approaches the desired equilibrium point as $t \to \infty$. In particular, if $V_3(z, w, r)$ is a converse Lyapunov function for the exponentially stable origin $z = 0$, provided by Theorem 4.14, P is the solution of the Lyapunov equation $PA + A^T P = -I$, and $(\tilde{\zeta}, \tilde{s})$ are deviations of (ζ, s) from their equilibrium values, then it can be shown that the derivative of

$$V_0 = V_3 + \lambda \tilde{\zeta}^T P \tilde{\zeta} + \tfrac{1}{2}\tilde{s}^2$$

with $\lambda > 0$, satisfies the inequality

$$\dot{V}_0 \leq - \begin{bmatrix} \|z\| \\ \|\tilde{\zeta}\|_2 \\ |\tilde{s}| \end{bmatrix}^T \begin{bmatrix} k_1 & -k_3 & -k_4 \\ -k_3 & \lambda & -(\lambda k_5 + k_6) \\ -k_4 & -(\lambda k_5 + k_6) & (k_2/\varepsilon) - k_7 \end{bmatrix} \begin{bmatrix} \|z\| \\ \|\tilde{\zeta}\|_2 \\ |\tilde{s}| \end{bmatrix}$$

where k_1 and k_2 are positive constants, while k_3 through k_7 are nonnegative constants. The derivative $\dot{V}_0$ can be made negative definite by choosing $\lambda > k_3^2/k_1$ and then choosing ε small enough to make the 3×3 matrix positive definite.

In the special case when $\beta = k$ (a constant) and $u = v$, the "continuous" sliding mode controller is given by

$$u = -k \; \text{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\varepsilon}\right) \tag{14.29}$$

When $\rho = 1$, the controller (14.29) is a classical PI controller followed by saturation (Figure 14.13), and when $\rho = 2$, it is a classical PID controller followed by saturation (Figure 14.14). This is an interesting connection between the "continuous" sliding mode controller and these classical controllers.

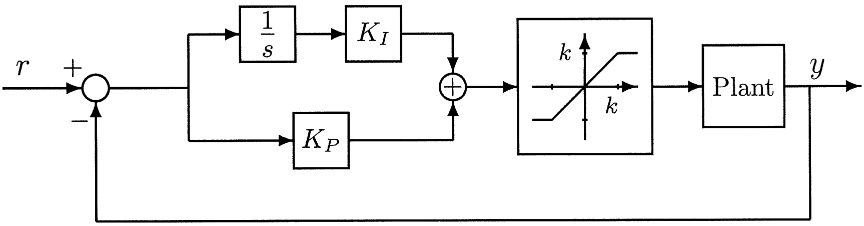


Figure 14.13: The "continuous" sliding mode controller (14.29) for relative-degree-one systems: a PI controller with $K_I = kk_0/\varepsilon$ and $K_P = k/\varepsilon$, followed by saturation.

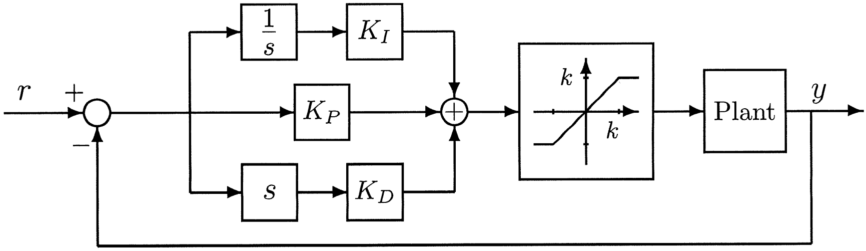


Figure 14.14: The "continuous" sliding mode controller (14.29) for relative-degree-two systems: a PID controller with $K_I = kk_0/\varepsilon$, $K_P = kk_1/\varepsilon$, and $K_D = k/\varepsilon$, followed by saturation.

14.2 Lyapunov Redesign

14.2.1 Stabilization

Consider the system

unknown

$$\dot{x} = f(t,x) + G(t,x)[u + \delta(t,x,u)] \tag{14.30}$$

where $x \in R^n$ is the state and $u \in R^p$ is the control input. The functions f, G, and δ are defined for $(t,x,u) \in [0,\infty) \times D \times R^p$, where $D \subset R^n$ is a domain that contains the origin. We assume that f, G, and δ are piecewise continuous in t and locally Lipschitz in x and u. The functions f and G are known precisely, while the function δ is an unknown function that lumps together various uncertain terms due to model simplification, parameter uncertainty, and so on. The uncertain term δ satisfies the matching condition. A nominal model of the system can be taken as

$$\dot{x} = f(t,x) + G(t,x)u \tag{14.31}$$

We proceed to design a stabilizing state feedback controller by using this nominal model. Suppose we have succeeded to design a feedback control law $u = \psi(t,x)$ such that the origin of the nominal closed-loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x) \tag{14.32}$$

is uniformly asymptotically stable. Suppose further that we know a Lyapunov function for (14.32); that is, we have a continuously differentiable function $V(t,x)$ that satisfies the inequalities

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|) \tag{14.33}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t,x) + G(t,x)\psi(t,x)] \le -\alpha_3(\|x\|) \tag{14.34}$$

for all $(t,x) \in [0,\infty) \times D$, where α_1, α_2, and α_3 are class $\mathcal{K}$ functions. Assume that, with $u = \psi(t,x) + v$, the uncertain term δ satisfies the inequality

$$\|\delta(t,x,\psi(t,x)+v)\| \le \rho(t,x) + \kappa_0\|v\|, \qquad 0 \le \kappa_0 < 1 \tag{14.35}$$

where $\rho : [0,\infty) \times D \to R$ is a nonnegative continuous function. The estimate (14.35) is the only information we need to know about the uncertain term δ. The function ρ is a measure of the size of the uncertainty. It is important to emphasize that we will not require ρ to be small. We will only require it to be known. Our goal in this section is to show that with the knowledge of the Lyapunov function V, the function ρ, and the constant κ_0 in (14.35), we can design an additional feedback control v such that the overall control $u = \psi(t,x) + v$ stabilizes the actual system (14.30) in the presence of the uncertainty. The design of v is called *Lyapunov redesign.*

Before we carry on with the Lyapunov redesign technique, let us illustrate that the feedback linearization problem of the previous chapter fits into the framework of the current problem.

Example 14.3 Consider the feedback linearizable system

$$\dot{x} = f(x) + G(x)u$$

where $f : D \to R^n$ and $G : D \to R^{n\times p}$ are smooth functions on a domain $D \subset R^n$ and there is a diffeomorphism $T : D \to R^n$ such that $D_z = T(D)$ contains the origin and $T(x)$ satisfies the partial differential equations

$$\begin{aligned}\frac{\partial T}{\partial x}f(x) &= AT(x) - B\gamma(x)\alpha(x)\\ \frac{\partial T}{\partial x}G(x) &= B\gamma(x)\end{aligned}$$

where (A,B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$. The change of variables $z = T(x)$ transforms the system into the form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)]$$

Consider also the perturbed system

$$\dot{x} = f(x) + \Delta_f(x) + [G(x) + \Delta_G(x)]u$$

with smooth perturbations that satisfy the conditions

$$\frac{\partial T}{\partial x}\Delta_f(x) = B\gamma(x)\Delta_1(x), \qquad \Delta_G(x) = G(x)\Delta_2(x)$$

on D.[14] The perturbed system can be represented in the form (14.30); that is,

$$\dot{z} = Az - B\gamma(x)\alpha(x) + B\gamma(x)[u + \delta(x,u)]$$

where $\delta(x,u) = \Delta_1(x) + \Delta_2(x)u$. Since the nominal system is feedback linearizable, we can take the nominal stabilizing feedback control as

$$\psi(x) = \alpha(x) - \gamma^{-1}(x)Kz = \alpha(x) - \gamma^{-1}(x)KT(x)$$

where K is chosen such that $(A - BK)$ is Hurwitz. A Lyapunov function for the nominal closed-loop system

$$\dot{z} = (A - BK)z$$

can be taken as $V(z) = z^T Pz$, where P is the solution of the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -I$$

With $u = \psi(x) + v$, the uncertain term $\delta(x,u)$ satisfies the inequality

$$\|\delta(x, \psi(x) + v)\| \le \|\Delta_1(x) + \Delta_2(x)\alpha(x) - \Delta_2(x)\gamma^{-1}(x)Kz\| + \|\Delta_2(x)\|\,\|v\|$$

Thus, to satisfy (14.35), we need the inequalities

$$\|\Delta_2(x)\| \le \kappa_0 < 1 \tag{14.36}$$

and

$$\|\Delta_1(x) + \Delta_2(x)\alpha(x) - \Delta_2(x)\gamma^{-1}(x)KT(x)\| \le \rho(x) \tag{14.37}$$

to hold over a domain that contains the origin for some continuous function $\rho(x)$. Inequality (14.36) is restrictive because it puts a definite limit on the perturbation Δ_2. Inequality (14.37), on the other hand, is not restrictive, because we do not require ρ to be small. It is basically a choice of a function ρ to estimate the growth of the left-hand side of (14.37). △

Consider now the system (14.30) and apply the control $u = \psi(t,x) + v$. The closed-loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x) + G(t,x)[v + \delta(t,x,\psi(t,x) + v)] \tag{14.38}$$

---

[14] It can be easily seen that the perturbed system is still feedback linearizable with the same diffeomorphism $T(x)$, provided $I + \Delta_2$ is nonsingular. Condition (14.36) implies that $I + \Delta_2$ is nonsingular.

is a perturbation of the nominal closed-loop system (14.32). Let us calculate the derivative of $V(t,x)$ along the trajectories of (14.38). For convenience, we will not write the argument of the various functions. We have

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + G\psi) + \frac{\partial V}{\partial x}G(v+\delta) \leq -\alpha_3(\|x\|) + \frac{\partial V}{\partial x}G(v+\delta)$$

Set $w^T = [\partial V/\partial x]G$ and rewrite the last inequality as

$$\dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \delta$$

The first term on the right-hand side is due to the nominal closed-loop system. The second and third terms represent, respectively, the effect of the control v and the uncertain term δ on $\dot{V}$. Due to the matching condition, the uncertain term δ appears on the right-hand side exactly at the same point where v appears. Consequently, it is possible to choose v to cancel the (destabilizing) effect of δ on $\dot{V}$. We will now explore two different methods for choosing v so that $w^T v + w^T \delta \leq 0$. Suppose inequality (14.35) is satisfied with $\|\cdot\|_2$; that is,

$$\|\delta(t,x,\psi(t,x)+v)\|_2 \leq \rho(t,x) + \kappa_0\|v\|_2, \qquad 0 \leq \kappa_0 < 1$$

We have

$$w^T v + w^T \delta \leq w^T v + \|w\|_2 \; \|\delta\|_2 \leq w^T v + \|w\|_2[\rho(t,x) + \kappa_0\|v\|_2]$$

Taking

$$v = -\eta(t,x)\cdot\frac{w}{\|w\|_2} \tag{14.39}$$

with a nonnegative function η, we obtain

$$w^T v + w^T \delta \leq -\eta\|w\|_2 + \rho\|w\|_2 + \kappa_0\eta\|w\|_2 = -\eta(1-\kappa_0)\|w\|_2 + \rho\|w\|_2$$

Choosing $\eta(t,x) \geq \rho(t,x)/(1-\kappa_0)$ for all $(t,x) \in [0,\infty)\times D$ yields

$$w^T v + w^T \delta \leq -\rho\|w\|_2 + \rho\|w\|_2 = 0$$

Hence, with the control (14.39), the derivative of $V(t,x)$ along the trajectories of the closed-loop system (14.38) is negative definite.

As an alternative idea, suppose (14.35) is satisfied with $\|\cdot\|_\infty$; that is,

$$\|\delta(t,x,\psi(t,x)+v)\|_\infty \leq \rho(t,x) + \kappa_0\|v\|_\infty, \qquad 0 \leq \kappa_0 < 1$$

We have

$$w^T v + w^T \delta \leq w^T v + \|w\|_1 \; \|\delta\|_\infty \leq w^T v + \|w\|_1[\rho(t,x) + \kappa_0\|v\|_\infty]$$

Choose

$$v = -\eta(t,x)\ \mathrm{sgn}(w) \tag{14.40}$$

where $\eta(t,x) \geq \rho(t,x)/(1-\kappa_0)$ for all $(t,x) \in [0,\infty) \times D$ and $\mathrm{sgn}(w)$ is a p-dimensional vector whose ith component is $\mathrm{sgn}(w_i)$. Then,

$$\begin{aligned} w^T v + w^T \delta &\leq -\eta\|w\|_1 + \rho\|w\|_1 + \kappa_0\eta\|w\|_1 \\ &= -\eta(1-\kappa_0)\|w\|_1 + \rho\|w\|_1 \\ &\leq -\rho\|w\|_1 + \rho\|w\|_1 \;=\; 0 \end{aligned}$$

Hence, with the control (14.40), the derivative of $V(t,x)$ along the trajectories of the closed-loop system (14.38) is negative definite. Notice that the control laws given by (14.39) and (14.40) coincide for single-input systems ($p = 1$).

The control laws given by (14.39) and (14.40) are discontinuous functions of the state x. This discontinuity causes some theoretical as well as practical problems. Theoretically, we have to change the definition of the control law to avoid division by zero. We also have to examine the question of existence and uniqueness of solutions more carefully, since the feedback functions are not locally Lipschitz in x. Practically, the implementation of such discontinuous controllers is characterized by the phenomenon of *chattering*, where, due to imperfections in switching devices or computational delays, the control has fast switching fluctuations across the switching surface.[15] Instead of trying to work out all these problems, we will choose the easy and more practical route of approximating the discontinuous control law by a continuous one. The development of such approximation is similar for both control laws. Therefore, we continue with the control (14.39) and leave the parallel development of a continuous approximation of (14.40) to Exercises 14.21 and 14.22.

Consider the feedback control law

$$v = \begin{cases} -\eta(t,x)(w/\|w\|_2), & \text{if } \eta(t,x)\|w\|_2 \geq \varepsilon \\ \\ -\eta^2(t,x)(w/\varepsilon), & \text{if } \eta(t,x)\|w\|_2 < \varepsilon \end{cases} \tag{14.41}$$

With (14.41), the derivative of V along the trajectories of the closed-loop system (14.38) will be negative definite whenever $\eta(t,x)\|w\|_2 \geq \varepsilon$. We only need to check $\dot{V}$ when $\eta(t,x)\|w\|_2 < \varepsilon$. In this case,

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x\|_2) + w^T\left[-\eta^2 \cdot \frac{w}{\varepsilon} + \delta\right] \\ &\leq -\alpha_3(\|x\|_2) - \frac{\eta^2}{\varepsilon}\|w\|_2^2 + \rho\|w\|_2 + \kappa_0\|w\|_2\|v\|_2 \\ &= -\alpha_3(\|x\|_2) - \frac{\eta^2}{\varepsilon}\|w\|_2^2 + \rho\|w\|_2 + \frac{\kappa_0\eta^2}{\varepsilon}\|w\|_2^2 \\ &\leq -\alpha_3(\|x\|_2) + (1-\kappa_0)\left(-\frac{\eta^2}{\varepsilon}\|w\|_2^2 + \eta\|w\|_2\right) \end{aligned}$$

---

[15] See Section 14.1 for further discussion of chattering.

The term

$$-\frac{\eta^2}{\varepsilon}\|w\|_2^2 + \eta\|w\|_2$$

attains a maximum value $\varepsilon/4$ at $\eta\|w\|_2 = \varepsilon/2$. Therefore,

$$\dot{V} \le -\alpha_3(\|x\|_2) + \frac{\varepsilon(1-\kappa_0)}{4}$$

whenever $\eta(t,x)\|w\|_2 < \varepsilon$. On the other hand, when $\eta(t,x)\|w\|_2 \ge \varepsilon$, $\dot{V}$ satisfies

$$\dot{V} \le -\alpha_3(\|x\|_2) \le -\alpha_3(\|x\|_2) + \frac{\varepsilon(1-\kappa_0)}{4}$$

Thus, the inequality

$$\dot{V} \le -\alpha_3(\|x\|_2) + \frac{\varepsilon(1-\kappa_0)}{4}$$

is satisfied irrespective of the value of $\eta(t,x)\|w\|_2$. Take $r > 0$ such that $B_r \subset D$, choose $\varepsilon < 2\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$, and set $\mu = \alpha_3^{-1}(\varepsilon(1-\kappa_0)/2) < \alpha_2^{-1}(\alpha_1(r))$. Then,

$$\dot{V} \le -\tfrac{1}{2}\alpha_3(\|x\|_2), \quad \forall\ \mu \le \|x\|_2 < r$$

Application of Theorem 4.18 results in the following theorem, which shows that the solutions of the closed-loop system are uniformly ultimately bounded by a class $\mathcal{K}$ function of ε.

Theorem 14.3 *Consider the system* (14.30). *Let $D \subset R^n$ be a domain that contains the origin and $B_r = \{\|x\|_2 \le r\} \subset D$. Let $\psi(t,x)$ be a stabilizing feedback control law for the nominal system* (14.31) *with a Lyapunov function $V(t,x)$ that satisfies* (14.33) *and* (14.34) *in 2-norm for all $t \ge 0$ and all $x \in D$, with some class $\mathcal{K}$ functions α_1, α_2, and α_3. Suppose the uncertain term δ satisfies* (14.35) *in 2-norm for all $t \ge 0$ and all $x \in D$. Let v be given by* (14.41) *and choose $\varepsilon < 2\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$. Then, for any $\|x(t_0)\|_2 < \alpha_2^{-1}(\alpha_1(r))$, there exists a finite time t_1 such that the solution of the closed-loop system* (14.38) *satisfies*

$$\|x(t)\|_2 \le \beta(\|x(t_0)\|_2, t-t_0), \quad \forall\ t_0 \le t < t_1 \tag{14.42}$$

$$\|x(t)\|_2 \le b(\varepsilon), \quad \forall\ t \ge t_1 \tag{14.43}$$

where β is a class $\mathcal{KL}$ function and b is a class $\mathcal{K}$ function defined by

$$b(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu)) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\varepsilon(1-\kappa_0)/2)))$$

If all the assumptions hold globally and α_1 belongs to class $\mathcal{K}_\infty$, then (14.42) *and* (14.43) *hold for any initial state $x(t_0)$.* ◇

Thus, in general, the continuous Lyapunov redesign given by (14.41) does not stabilize the origin as its discontinuous counterpart (14.39) does. Nevertheless, it guarantees uniform ultimate boundedness of the solutions. Since the ultimate bound $b(\varepsilon)$ is a class $\mathcal{K}$ function of ε, it can be made arbitrarily small by choosing ε small enough. In the limit, as $\varepsilon \to 0$, we recover the performance of the discontinuous controller. Notice that there is no analytical reason to require ε to be very small. The only analytical restriction on ε is the requirement $\varepsilon < 2\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$. This requirement is satisfied for any ε when the assumptions hold globally and α_i $(i = 1, 2, 3)$ are class $\mathcal{K}_\infty$ functions. Of course, from a practical viewpoint, we would like to make ε as small as feasible, because we would like to drive the state of the system to a neighborhood of the origin that is as small as it could be. Exploiting the smallness of ε in the analysis, we can arrive at a sharper result when the uncertainty δ vanishes at the origin. Suppose there is a ball $B_a = \{\|x\|_2 \leq a\}$, $a \leq r$, such that the following inequalities are satisfied for all $x \in B_a$:

$$\begin{aligned} \alpha_3(\|x\|_2) &\geq \phi^2(x) && (14.44)\\ \eta(t,x) &\geq \eta_0 > 0 && (14.45)\\ \rho(t,x) &\leq \rho_1\phi(x) && (14.46) \end{aligned}$$

Here, $\phi : R^n \to R$ is a positive definite function of x. Choosing $\varepsilon < b^{-1}(a)$ ensures that the trajectories of the closed-loop systems will be confined to B_a after a finite time. When $\eta(t,x)\|w\|_2 < \varepsilon$, the derivative $\dot{V}$ satisfies

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x\|_2) - \frac{\eta^2(1-\kappa_0)}{\varepsilon}\|w\|_2^2 + \rho\|w\|_2 \\ &\leq -\frac{1}{2}\alpha_3(\|x\|_2) - \frac{1}{2}\phi^2(x) - \frac{\eta_0^2(1-\kappa_0)}{\varepsilon}\|w\|_2^2 + \rho_1\phi(x)\|w\|_2 \\ &\leq -\frac{1}{2}\alpha_3(\|x\|_2) - \frac{1}{2}\begin{bmatrix} \phi(x) \\ \|w\|_2 \end{bmatrix}^T \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 2\eta_0^2(1-\kappa_0)/\varepsilon \end{bmatrix} \begin{bmatrix} \phi(x) \\ \|w\|_2 \end{bmatrix} \end{aligned}$$

The matrix of the quadratic form will be positive definite if $\varepsilon < 2\eta_0^2(1-\kappa_0)/\rho_1^2$. Thus, choosing $\varepsilon < 2\eta_0^2(1-\kappa_0)/\rho_1^2$, we have $\dot{V} \leq -\alpha_3(\|x\|_2)/2$. Since $\dot{V} \leq -\alpha_3(\|x\|_2) \leq -\alpha_3(\|x\|_2)/2$ when $\eta(t,x)\|w\|_2 \geq \varepsilon$, we conclude that

$$\dot{V} \leq -\frac{1}{2}\alpha_3(\|x\|_2)$$

which shows that the origin is uniformly asymptotically stable.

Corollary 14.1 *Assume the inequalities* (14.44) *through* (14.46) *are satisfied, in addition to all the assumptions of Theorem* 14.3. *Then, for all* $\varepsilon < \min\{2\eta_0^2(1-\kappa_0)/\rho_1^2,\ b^{-1}(a)\}$, *the origin of the closed-loop system* (14.38) *is uniformly asymptotically stable. If* $\alpha_i(r) = k_i r^c$, *then the origin is exponentially stable.* ◇

Corollary 14.1 is particularly useful when the origin of the nominal closed-loop system (14.32) is exponentially stable and the perturbation $\delta(t, x, u)$ is Lipschitz in

x and u and vanishes at $(x = 0,\ u = 0)$. In this case, $\phi(x)$ is proportional to $\|x\|_2$ and the uncertain term satisfies (14.35) with $\rho(x)$ that satisfies (14.46). In general, the condition (14.46) may require more than just a vanishing perturbation at the origin. For example if, in a scalar case, $\phi(x) = |x|^3$, then a perturbation term x cannot be bounded by $\rho_1\phi(x)$.

The stabilization result of Corollary 14.1 is dependent on the choice of η to satisfy (14.45). It can be shown (Exercise 14.20) that if η does not satisfy (14.45), the feedback control may fail to stabilize the origin. When η satisfies (14.45), the feedback control law (14.41) acts in the region $\eta\|w\|_2 < \varepsilon$ as a high-gain feedback control $v = -kw$ with $k \geq \eta_0^2/\varepsilon$. Such high-gain feedback control law can stabilize the origin when (14.44) through (14.46) are satisfied (Exercise 14.24).

Example 14.4 Let us continue Example 14.3 on feedback linearizable systems. Suppose inequality (14.36) is satisfied in $\|\cdot\|_2$. Suppose further that

$$\|\Delta_1(x) + \Delta_2(x)\alpha(x) + \Delta_2(x)\gamma^{-1}(x)Kz\|_2 \leq \rho_1\|z\|_2$$

for all $z \in B_r \subset D_z$. Then, (14.37) is satisfied with $\rho = \rho_1\|z\|_2$. We take the control v as in (14.41) with

$$\eta = 1 + \frac{\rho_1\|z\|_2}{(1-\kappa_0)}, \qquad w^T = 2z^TPB, \qquad \varepsilon < \min\left\{\frac{2(1-\kappa_0)}{\rho_1^2},\ \frac{2r^2\lambda_{\min}(P)}{(1-\kappa_0)\lambda_{\max}(P)}\right\}$$

It can be verified that all the assumptions of Theorem 14.3 and Corollary 14.1 are satisfied with $\alpha_1(r) = \lambda_{\min}(P)r^2$, $\alpha_2(r) = \lambda_{\max}(P)r^2$, $\alpha_3(r) = r^2$, $\phi(z) = \|z\|_2$, and $a = r$. Thus, with the overall feedback control $u = \psi(x) + v$, the origin of the perturbed closed-loop system is exponentially stable. If all the assumptions hold globally and $T(x)$ is a global diffeomorphism, the origin $x = 0$ will be globally asymptotically stable.[16] △

Example 14.5 Reconsider the pendulum equation of Example 13.18, with $\delta_1 = \pi$,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= a\sin x_1 - bx_2 + cu\end{aligned}$$

We want to stabilize the pendulum at the open-loop equilibrium point $x = 0$. This system is feedback linearizable with $T(x) = x$. A nominal stabilizing feedback control can be taken as

$$\psi(x) = -\left(\frac{\hat{a}}{\hat{c}}\right)\sin x_1 - \left(\frac{1}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

---

[16] The origin $z = 0$ will be globally exponentially stable, but we cannot conclude that $x = 0$ will be globally exponentially stable, unless the linear growth conditions $\|T(x)\| \leq L_1\|x\|$ and $\|T^{-1}(z)\| \leq L_2\|z\|$ hold globally.

where $\hat{a}$ and $\hat{c}$ are the nominal values of a and c, respectively, and k_1 and k_2 are chosen such that

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix}$$

is Hurwitz. With $u = \psi(x) + v$, the uncertainty term δ is given by

$$\delta = \frac{1}{\hat{c}} \left[\left(\frac{a\hat{c} - \hat{a}c}{\hat{c}} \right) \sin x_1 - \left(\frac{c - \hat{c}}{\hat{c}} \right) (k_1 x_1 + k_2 x_2) \right] + \left(\frac{c - \hat{c}}{\hat{c}} \right) v$$

Hence,

$$|\delta| \le \rho_1 \|x\|_2 + \kappa_0 |v|, \qquad \forall\ x \in R^2\ \forall\ v \in R$$

where

$$\kappa_0 \ge \left| \frac{c - \hat{c}}{\hat{c}} \right|, \quad \rho_1 = \frac{k}{\hat{c}}, \quad \text{and} \quad k \ge \left| \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right| + \left| \frac{c - \hat{c}}{\hat{c}} \right| \sqrt{k_1^2 + k_2^2}$$

Assuming $\kappa_0 < 1$ and taking v as in the previous example, we find that the control law $u = \psi(x) + v$ makes the origin globally exponentially stable. In Example 13.18, we analyzed the same system under the control $u = \psi(x)$. Comparing the results shows exactly the contribution of the additional control component v. In Example 13.18, we were able to show that the control $u = \psi(x)$ stabilizes the system when k is restricted to satisfy

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

This restriction has now been completely removed, provided we know k. $\triangle$

Example 14.6 Once again, consider the pendulum equation of the previous example. This time, suppose the suspension point of the pendulum is subjected to a time-varying, bounded, horizontal acceleration. For simplicity, neglect friction ($b = 0$). The state equation is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a \sin x_1 + cu + h(t) \cos x_1 \end{aligned}$$

where $h(t)$ is the (normalized) horizontal acceleration of the suspension point. We have $|h(t)| \le H$ for all $t \ge 0$. The nominal model and the nominal stabilizing control can be taken as in the previous example (with $b = 0$). The uncertain term δ satisfies

$$|\delta| \le \rho_1\ \|x\|_2 + \kappa_0\ |v| + H/\hat{c}$$

where ρ_1 and κ_0 are the same as in that example. This time, we have $\rho(x) = \rho_1 \|x\|_2 + H/\hat{c}$, which does not vanish at $x = 0$. The choice of η in the control law (14.41) must satisfy $\eta \ge (\rho_1 \|x\|_2 + H/\hat{c})/(1 - \kappa_0)$. We take $\eta(x) = \eta_0 + \rho_1 \|x\|_2/(1 -$

κ_0) with $\eta_0 \geq H/\hat{c}(1-\kappa_0)$. In the previous example, we arbitrarily set $\eta(0) = 1$. This is the only modification we have to make to accommodate the nonvanishing disturbance term $h(t)\cos x_1$. Since $\rho(0) \neq 0$, Corollary 14.1 does not apply. We can only conclude, by Theorem 14.3, that the solutions of the closed-loop system are uniformly ultimately bounded and the ultimate bound is proportional to $\sqrt{\varepsilon}$. △

14.2.2 Nonlinear Damping

Reconsider the system (14.30) with $\delta(t,x,u) = \Gamma(t,x)\delta_0(t,x,u)$; that is,

$$\dot{x} = f(t,x) + G(t,x)[u + \Gamma(t,x)\delta_0(t,x,u)] \tag{14.47}$$

As before, we assume that f and G are known, while $\delta_0(t,x,u)$ is an uncertain term. The function $\Gamma(t,x)$ is known. We assume that f, G, Γ, and δ_0 are piecewise continuous in t and locally Lipschitz in x and u for all $(t,x,u) \in [0,\infty) \times R^n \times R^p$. We assume also that δ_0 is uniformly bounded for all (t,x,u). Let $\psi(t,x)$ be a nominal stabilizing feedback control law such that the origin of the nominal closed-loop system (14.32) is globally uniformly asymptotically stable and there is a known Lyapunov function $V(t,x)$ that satisfies (14.33) and (14.34) for all $(t,x) \in [0,\infty) \times R^n$ with class $\mathcal{K}_\infty$ functions α_1, α_2, and α_3. If an upper bound on $\|\delta_0(t,x,u)\|$ is known, we can design the control component v, as before, to ensure robust global stabilization. In this section, we show that even when no upper bound on δ_0 is known, we can design the control component v to ensure boundedness of the trajectories of the closed-loop system. Toward that end, let $u = \psi(t,x) + v$ and recall that the derivative of V along the trajectories of the closed-loop system satisfies

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + G\psi) + \frac{\partial V}{\partial x}G(v + \Gamma\delta_0) \leq -\alpha_3(\|x\|) + w^T(v + \Gamma\delta_0)$$

where $w^T = [\partial V/\partial x]G$. Taking

$$v = -kw\|\Gamma(t,x)\|_2^2, \qquad k > 0 \tag{14.48}$$

we obtain

$$\dot{V} \leq -\alpha_3(\|x\|) - k\|w\|_2^2\|\Gamma\|_2^2 + \|w\|_2\|\Gamma\|_2 k_0$$

in which k_0 is an (unknown) upper bound on $\|\delta_0\|$. The term

$$-k\|w\|_2^2\|\Gamma\|_2^2 + \|w\|_2\|\Gamma\|_2 k_0$$

attains a maximum value $k_0^2/4k$ at $\|w\|_2\|\Gamma\|_2 = k_0/2k$. Therefore,

$$\dot{V} \leq -\alpha_3(\|x\|_2) + \frac{k_0^2}{4k}$$

Since α_3 is class $\mathcal{K}_\infty$, it is always true that $\dot{V}$ is negative outside some ball. It follows from Theorem 4.18 that for any initial state $x(t_0)$, the solution of the closed-loop system is uniformly bounded. The Lyapunov redesign (14.48) is called *nonlinear damping*. We summarize our conclusion in the next lemma.

Lemma 14.1 *Consider the system* (14.47) *and let* $\psi(t,x)$ *be a stabilizing feedback control for the nominal system* (14.31) *with a Lyapunov function* $V(t,x)$ *that satisfies* (14.33) *and* (14.34) *for all* $t \geq 0$ *and all* $x \in R^n$, *with some class* $\mathcal{K}_\infty$ *functions* α_1, α_2, *and* α_3. *Suppose the uncertain term* δ_0 *is uniformly bounded for* $(t,x,u) \in [0,\infty) \times R^n \times R^p$. *Let* v *be given by* (14.48) *and take* $u = \psi(t,x) + v$. *Then, for any* $x(t_0) \in R^n$, *the solution of the closed-loop system is uniformly bounded.* ◇

Example 14.7 Consider the scalar system

$$\dot{x} = x^2 + u + x\delta_0(t)$$

where $\delta_0(t)$ is a bounded function of t. With the nominal stabilizing control $\psi(x) = -x^2 - x$, the Lyapunov function $V(x) = x^2$ satisfies (14.33) and (14.34) globally with $\alpha_1(r) = \alpha_2(r) = \alpha_3(r) = r^2$. The nonlinear damping component (14.48), with $k = 1$, is given by $v = -2x^3$. The closed-loop system

$$\dot{x} = -x - 2x^3 + x\delta_0(t)$$

has a bounded solution no matter how large the bounded disturbance δ_0 is, thanks to the nonlinear damping term $-2x^3$. △

14.3 Backstepping

We start with the special case of *integrator backstepping*. Consider the system

$$\dot{\eta} = f(\eta) + g(\eta)\xi \qquad (14.49)$$

$$\dot{\xi} = u \qquad (14.50)$$

where $[\eta^T, \xi]^T \in R^{n+1}$ is the state and $u \in R$ is the control input. The functions $f : D \to R^n$ and $g : D \to R^n$ are smooth[17] in a domain $D \subset R^n$ that contains $\eta = 0$ and $f(0) = 0$. We want to design a state feedback control law to stabilize the origin ($\eta = 0$, $\xi = 0$). We assume that both f and g are known. This system can be viewed as a cascade connection of two components, as shown in Figure 14.15(a); the first component is (14.49), with ξ as input, and the second component is the integrator (14.50). Suppose the component (14.49) can be stabilized by a smooth state feedback control law $\xi = \phi(\eta)$, with $\phi(0) = 0$; that is, the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

is asymptotically stable. Suppose further that we know a (smooth, positive definite) Lyapunov function $V(\eta)$ that satisfies the inequality

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad \forall\ \eta \in D \qquad (14.51)$$

---

[17] We require smoothness of all functions for convenience. It will become clear, however, that in a particular problem, we only need existence of derivatives up to a certain order.

where $W(\eta)$ is positive definite. By adding and subtracting $g(\eta)\phi(\eta)$ on the right-hand side of (14.49), we obtain the equivalent representation

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)] \\ \dot{\xi} &= u \end{aligned}$$

which is shown in Figure 14.15(b). The change of variables

$$z = \xi - \phi(\eta)$$

results in the system

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= u - \dot{\phi} \end{aligned}$$

which is shown in Figure 14.15(c). Going from Figure 14.15(b) to Figure 14.15(c) can be viewed as "backstepping" $-\phi(\eta)$ through the integrator. Since f, g, and ϕ are known, the derivative $\dot{\phi}$ can be computed by using the expression

$$\dot{\phi} = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$$

Taking $v = u - \dot{\phi}$ reduces the system to the cascade connection

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v \end{aligned}$$

which is similar to the system we started from, except that now the first component has an asymptotically stable origin when the input is zero. This feature will be exploited in the design of v to stabilize the overall system. Using

$$V_c(\eta, \xi) = V(\eta) + \tfrac{1}{2}z^2$$

as a Lyapunov function candidate, we obtain

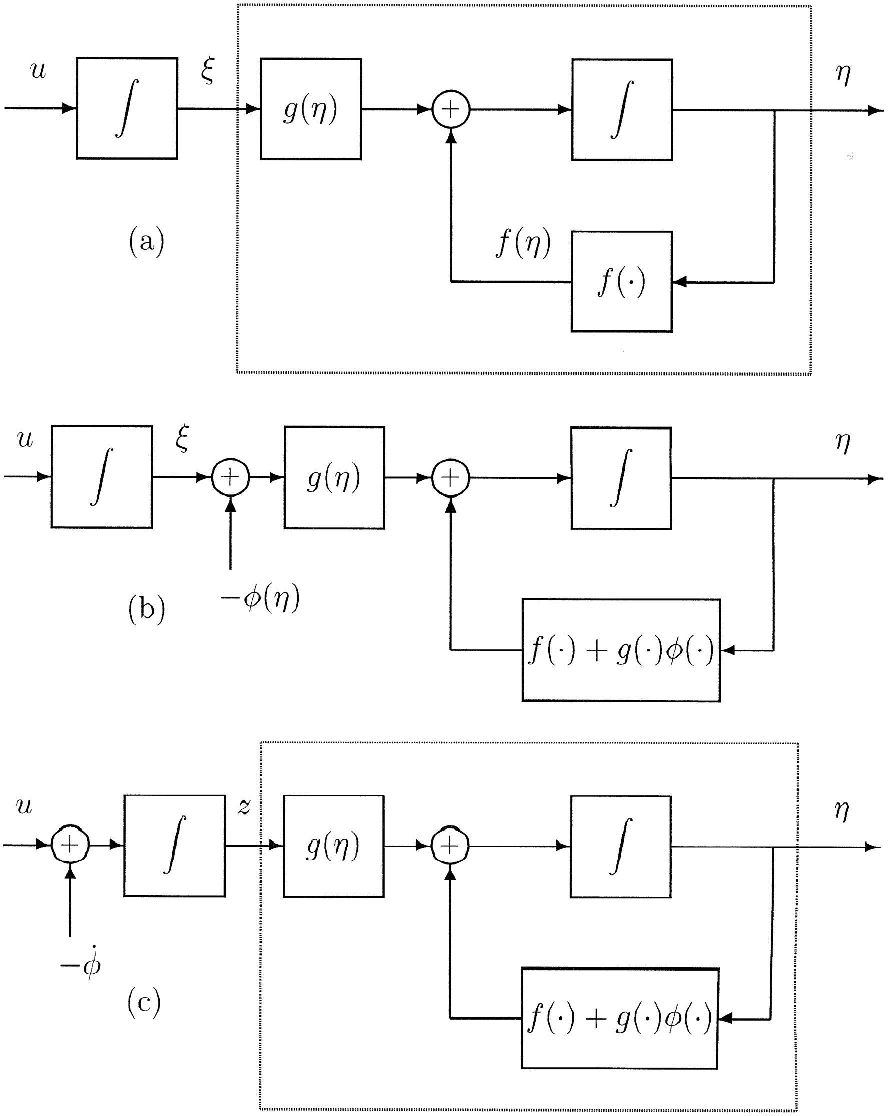
$$\begin{aligned} \dot{V}_c &= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + zv \\ &\le -W(\eta) + \frac{\partial V}{\partial \eta}g(\eta)z + zv \end{aligned}$$

Choosing

$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz, \quad k > 0$$

yields

$$\dot{V}_c \le -W(\eta) - kz^2$$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$f(\cdot) + g(\cdot)\phi(\cdot)$

Figure 14.15: (a) The block diagram of system (14.49)–(14.50); (b) introducing $\phi(\eta)$; (c) "backstepping" $-\phi(\eta)$ through the integrator.

which shows that the origin ($\eta = 0,\ z = 0$) is asymptotically stable. Since $\phi(0) = 0$, we conclude that the origin ($\eta = 0,\ \xi = 0$) is asymptotically stable. Substituting for v, z, and $\dot{\phi}$, we obtain the state feedback control law

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta} g(\eta) - k[\xi - \phi(\eta)] \tag{14.52}$$

If all the assumptions hold globally and $V(\eta)$ is radially unbounded, we can conclude that the origin is globally asymptotically stable. We summarize our conclusions in the next lemma.

Lemma 14.2 *Consider the system* (14.49)–(14.50). *Let* $\phi(\eta)$ *be a stabilizing state feedback control law for* (14.49) *with* $\phi(0) = 0$, *and* $V(\eta)$ *be a Lyapunov function that satisfies* (14.51) *with some positive definite function* $W(\eta)$. *Then, the state feedback control law* (14.52) *stabilizes the origin of* (14.49)–(14.50), *with* $V(\eta) + [\xi - \phi(\eta)]^2/2$ *as a Lyapunov function. Moreover, if all the assumptions hold globally and* $V(\eta)$ *is radially unbounded, the origin will be globally asymptotically stable.* ◇

Example 14.8 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

which takes the form (14.49)–(14.50) with $\eta = x_1$ and $\xi = x_2$. We start with the scalar system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

with x_2 viewed as the input and proceed to design the feedback control $x_2 = \phi(x_1)$ to stabilize the origin $x_1 = 0$. With

$$x_2 = \phi(x_1) = -x_1^2 - x_1$$

we cancel the nonlinear term x_1^2 to obtain[18]

$$\dot{x}_1 = -x_1 - x_1^3$$

and $V(x_1) = x_1^2/2$ satisfies

$$\dot{V} = -x_1^2 - x_1^4 \le -x_1^2, \quad \forall\, x_1 \in R$$

Hence, the origin of $\dot{x}_1 = -x_1 - x_1^3$ is globally exponentially stable. To backstep, we use the change of variables

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

---

[18] We do not cancel $-x_1^3$ because it provides nonlinear damping. (See Example13.19.)

to transform the system into the form

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_1^3 + z_2\\ \dot{z}_2 &= u + (1+2x_1)(-x_1 - x_1^3 + z_2)\end{aligned}$$

Taking

$$V_c(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

as a composite Lyapunov function, we obtain

$$\begin{aligned}\dot{V}_c &= x_1(-x_1 - x_1^3 + z_2) + z_2[u + (1+2x_1)(-x_1 - x_1^3 + z_2)]\\ &= -x_1^2 - x_1^4 + z_2[x_1 + (1+2x_1)(-x_1 - x_1^3 + z_2) + u]\end{aligned}$$

Taking

$$u = -x_1 - (1+2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

yields

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$$

Hence, the origin is globally asymptotically stable. △

Application of integrator backstepping in the preceding example is straightforward due to the simplicity of scalar designs. For higher order systems, we may retain this simplicity via recursive application of integrator backstepping, as illustrated by the next example.

Example 14.9 The third-order system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}$$

is composed of the second-order system of the previous example with an additional integrator at the input side. We proceed to apply integrator backstepping as in the previous example. After one step of backstepping, we know that the second-order system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\end{aligned}$$

with x_3 as input, can be globally stabilized by the control

$$x_3 = -x_1 - (1+2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2) \overset{\text{def}}{=} \phi(x_1, x_2)$$

and

$$V(x_1, x_2) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_2 + x_1 + x_1^2)^2$$

is the corresponding Lyapunov function. To backstep, we apply the change of variables

$$z_3 = x_3 - \phi(x_1, x_2)$$

to obtain

$$\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{aligned}$$

Using $V_c = V + z_3^2/2$ as a composite Lyapunov function, we obtain

$$\begin{aligned}
\dot{V}_c &= \frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}(z_3 + \phi) \\
&\quad + z_3 \left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) \right] \\
&= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\
&\quad + z_3 \left[\frac{\partial V}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u \right]
\end{aligned}$$

Taking

$$u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

yields

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 - z_3^2$$

Hence, the origin is globally asymptotically stable. △

Let us move now from (14.49)–(14.50) to the more general system

$$\dot{\eta} = f(\eta) + g(\eta)\xi \tag{14.53}$$

$$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u \tag{14.54}$$

where f_a and g_a are smooth. If $g_a(\eta, \xi) \neq 0$ over the domain of interest, the input transformation

$$u = \frac{1}{g_a(\eta, \xi)}[u_a - f_a(\eta, \xi)] \tag{14.55}$$

will reduce (14.54) to the integrator $\dot{\xi} = u_a$. Therefore, if a stabilizing state feedback control law $\phi(\eta)$ and a Lyapunov function $V(\eta)$ exist such that the conditions of Lemma 14.2 are satisfied for (14.53), then the lemma and (14.55) yield

$$\begin{aligned}
u &= \phi_c(\eta, \xi) \\
&= \frac{1}{g_a(\eta, \xi)} \left\{ \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)] - f_a(\eta, \xi) \right\}
\end{aligned} \tag{14.56}$$

for some $k > 0$ and

$$V_c(\eta, \xi) = V(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2 \tag{14.57}$$

as a stabilizing state feedback control law and a Lyapunov function, respectively, for the overall system (14.53)–(14.54). By recursive application of backstepping, we can stabilize *strict-feedback* systems of the form

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\;\;\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned}$$

where $x \in R^n$, z_1 to z_k are scalars, and f_0 to f_k vanish at the origin. The reason for referring to such systems as "strict feedback" is that the nonlinearities f_i and g_i in the $\dot{z}_i$-equation ($i = 1, \ldots, k$) depend only on $x, z_1, \ldots, z_i$; that is, on the state variables that are "fed back." We assume that

$$g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k$$

over the domain of interest. The recursive procedure starts with the system

$$\dot{x} = f_0(x) + g_0(x)z_1$$

where z_1 is viewed as the control input. We assume that it is possible to determine a stabilizing state feedback control law $z_1 = \phi_0(x)$, with $\phi_0(0) = 0$, and a Lyapunov function $V_0(x)$ such that

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \leq -W(x)$$

over the domain of interest for some positive definite function $W(x)$. In many applications of backstepping, the variable x is scalar, which simplifies this stabilization problem. With $\phi_0(x)$ and $V_0(x)$ in hand, we proceed to apply backstepping in a systematic way. First, we consider the system

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2
\end{aligned}$$

as a special case of (14.53)–(14.54) with

$$\eta = x, \quad \xi = z_1, \quad u = z_2, \quad f = f_0, \quad g = g_0, \quad f_a = f_1, \quad g_a = g_1$$

We use (14.56) and (14.57) to obtain the stabilizing state feedback control law and the Lyapunov function as

$$\phi_1(x, z_1) = \frac{1}{g_1}\left[\frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi) - f_1\right], \quad k_1 > 0$$

$$V_1(x, z_1) = V_0(x) + \tfrac{1}{2}[z_1 - \phi(x)]^2$$

Next, we consider the system

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x) z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\ \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \end{aligned}$$

as a special case of (14.53)–(14.54) with

$$\eta = \begin{bmatrix} x \\ z_1 \end{bmatrix}, \ \xi = z_2, \ u = z_3, \ f = \begin{bmatrix} f_0 + g_0 z_1 \\ f_1 \end{bmatrix}, \ g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \ f_a = f_2, \ g_a = g_2$$

Using (14.56) and (14.57), we obtain the stabilizing state feedback control law and the Lyapunov function as

$$\phi_2(x, z_1, z_2) = \frac{1}{g_2}\left[\frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi_1) - f_2\right]$$

for some $k_2 > 0$ and

$$V_2(x, z_1, z_2) = V_1(x, z_1) + \tfrac{1}{2}[z_2 - \phi_2(x, z_1)]^2$$

This process is repeated k times to obtain the overall stabilizing state feedback control law $u = \phi_k(x, z_1, \ldots, z_k)$ and the Lyapunov function $V_k(x, z_1, \ldots, z_k)$.

Example 14.10 Consider a single-input–single output system in the special normal form

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x) z_1 \\ \dot{z}_1 &= z_2 \\ &\ \vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= \gamma(x, z)[u - \alpha(x, z)] \\ y &= z_1 \end{aligned}$$

where $x \in R^{n-r}$, z_1 to z_r are scalars, and $\gamma(x, z) \neq 0$ for all (x, z). This representation is a special case of the normal form (13.16)–(13.18) because the $\dot{x}$-equation takes the form $f_0(x) + g_0(x) z_1$, instead of the more general form $f_0(x, z)$. The system is in the strict feedback form. The origin can be globally stabilized by recursive backstepping if we can find a smooth function $\phi_0(x)$ and a smooth radially unbounded Lyapunov function $V_0(x)$ such that

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \leq -W(x), \quad \forall\, x \in R^n$$

for some positive definite function $W(x)$. If the system is minimum phase, the origin of $\dot{x} = f_0(x)$ is globally asymptotically stable, and we know a Lyapunov function $V_0(x)$ that satisfies

$$\frac{\partial V_0}{\partial x} f_0(x) \leq -W(x), \quad \forall\, x \in R^n$$

for some positive definite function $W(x)$, we can simply take $\phi_0(x) = 0$. Otherwise, we have to search for $\phi_0(x)$ and $V_0(x)$. This shows that backstepping allows us to stabilize nonminimum phase systems, provided the stabilization problem for the zero dynamics is solvable. △

Example 14.11 The second-order system

$$\begin{aligned} \dot{\eta} &= -\eta + \eta^2 \xi \\ \dot{\xi} &= u \end{aligned}$$

was considered in Example 13.16, where it was shown that $u = -k\xi$, with sufficiently large $k > 0$, can achieve semiglobal stabilization. In this example, we achieve global stabilization via backstepping. Starting with the system

$$\dot{\eta} = -\eta + \eta^2 \xi$$

it is clear that $\xi = 0$ and $V_0(\eta) = \eta^2/2$ result in

$$\frac{\partial V_0}{\partial \eta}(-\eta) = -\eta^2, \quad \forall\, \eta \in R$$

Using $V = V_0 + \xi^2/2 = (\eta^2 + \xi^2)/2$, we obtain

$$\dot{V} = \eta(-\eta + \eta^2\xi) + \xi u = -\eta^2 + \xi(\eta^3 + u)$$

Thus,

$$u = -\eta^3 - k\xi, \quad k > 0$$

globally stabilizes the origin. △

Example 14.12 As a variation from the previous example, consider the system[19]

$$\begin{aligned} \dot{\eta} &= \eta^2 - \eta\xi \\ \dot{\xi} &= u \end{aligned}$$

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[19] With the output $y = \xi$, the system is nonminimum phase, because the origin of the zero-dynamics equation $\dot{\eta} = \eta^2$ is unstable.

This time, $\dot{\eta} = \eta^2 - \eta\xi$ cannot be stabilized by $\xi = 0$. It is easy, however, to see that $\xi = \eta + \eta^2$ and $V_0(\eta) = \eta^2/2$ result in

$$\frac{\partial V_0}{\partial \eta}[\eta^2 - \eta(\eta + \eta^2)] = -\eta^4, \qquad \forall\ \eta \in R$$

Using $V = V_0 + (\xi - \eta - \eta^2)^2/2$, we obtain

$$\begin{aligned} \dot{V} &= \eta(\eta^2 - \eta\xi) + (\xi - \eta - \eta^2)[u - (1 + 2\eta)(\eta^2 - \eta\xi)] \\ &= -\eta^4 + (\xi - \eta - \eta^2)[-\eta^2 + u - (1 + 2\eta)(\eta^2 - \eta\xi)] \end{aligned}$$

The control

$$u = (1 + 2\eta)(\eta^2 - \eta\xi) + \eta^2 - k(\xi - \eta - \eta^2), \qquad k > 0$$

yields

$$\dot{V} = -\eta^4 - k(\xi - \eta - \eta^2)^2$$

which shows that the origin is globally asymptotically stable. △

In the previous two sections, we saw how to use sliding mode control and Lyapunov redesign to robustly stabilize an uncertain system when the uncertainty satisfies the matching condition. Backstepping can be used to relax the matching condition. To illustrate the idea, let us consider the single-input system

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\xi + \delta_\eta(\eta, \xi) && (14.58) \\ \dot{\xi} &= f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta_\xi(\eta, \xi) && (14.59) \end{aligned}$$

defined on a domain $D \subset R^{n+1}$ that contains the origin ($\eta = 0,\ \xi = 0$), where $\eta \in R^n$ and $\xi \in R$. Suppose $g_a(\eta, \xi) \neq 0$ and all functions are smooth for all $(\eta, \xi) \in D$. Let f, g, f_a, and g_a be known and δ_η and δ_ξ be uncertain terms. We assume that f and f_a vanish at the origin and the uncertain terms satisfy the inequalities

$$\begin{aligned} \|\delta_\eta(\eta, \xi)\|_2 &\leq a_1\|\eta\|_2 && (14.60) \\ |\delta_\xi(\eta, \xi)| &\leq a_2\|\eta\|_2 + a_3|\xi| && (14.61) \end{aligned}$$

for all $(\eta, \xi) \in D$. Inequality (14.60) restricts the class of uncertainties, because it restricts the upper bound on $\delta_\eta(\eta, \xi)$ to depend only on η. Nevertheless, it is less restrictive than the matching condition that would have required $\delta_\eta = 0$. Starting with the system (14.58), suppose we can find a stabilizing state feedback control law $\xi = \phi(\eta)$ with $\phi(0) = 0$ and a (smooth, positive definite) Lyapunov function $V(\eta)$ such that

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \xi)] \leq -b\|\eta\|_2^2 \qquad (14.62)$$

for all $(\eta, \xi) \in D$ for some positive constant b. Inequality (14.62) shows that $\eta = 0$ is an asymptotically stable equilibrium point of the system

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \xi)$$

Suppose further that $\phi(\eta)$ satisfies the inequalities

$$|\phi(\eta)| \le a_4 \|\eta\|_2, \qquad \left\| \frac{\partial \phi}{\partial \eta} \right\|_2 \le a_5 \tag{14.63}$$

over D. Consider now the Lyapunov function candidate

$$V_c(\eta, \xi) = V(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2 \tag{14.64}$$

The derivative of V_c along the trajectories of (14.58)–(14.59) is given by

$$\begin{aligned} \dot{V}_c &= \frac{\partial V}{\partial \eta}(f + g\phi + \delta_\eta) + \frac{\partial V}{\partial \eta} g \; (\xi - \phi) \\ &\quad + (\xi - \phi) \left[f_a + g_a u + \delta_\xi - \frac{\partial \phi}{\partial \eta}(f + g\xi + \delta_\eta) \right] \end{aligned}$$

Taking

$$u = \frac{1}{g_a} \left[\frac{\partial \phi}{\partial \eta}(f + g\xi) - \frac{\partial V}{\partial \eta} g - f_a - k(\xi - \phi) \right], \qquad k > 0 \tag{14.65}$$

and using (14.62), we obtain

$$\dot{V}_c \le -b\|\eta\|_2^2 + (\xi - \phi) \left[\delta_\xi - \frac{\partial \phi}{\partial \eta} \delta_\eta \right] - k(\xi - \phi)^2$$

By using (14.60), (14.61), and (14.63), it can be shown that

$$\begin{aligned} \dot{V}_c &\le -b\|\eta\|_2^2 + 2a_6 \|\eta\|_2 |\xi - \phi| - (k - a_3)(\xi - \phi)^2 \\ &= -\begin{bmatrix} \|\eta\|_2 \\ |\xi - \phi| \end{bmatrix}^T \begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix} \begin{bmatrix} \|\eta\|_2 \\ |\xi - \phi| \end{bmatrix} \end{aligned}$$

for some $a_6 \ge 0$. Choosing

$$k > a_3 + \frac{a_6^2}{b}$$

yields

$$\dot{V}_c \le -\sigma[\|\eta\|_2^2 + |\xi - \phi|^2]$$

for some $\sigma > 0$. Thus, we have completed the proof of the next lemma.

Lemma 14.3 *Consider the system* (14.58)–(14.59), *where the uncertainty satisfies inequalities* (14.60) *and* (14.61). *Let* $\phi(\eta)$ *be a stabilizing state feedback control law for* (14.58) *that satisfies* (14.63), *and* $V(\eta)$ *be a Lyapunov function that satisfies* (14.62). *Then, the state feedback control law* (14.65), *with* k *sufficiently large, stabilizes the origin of* (14.58)–(14.59). *Moreover, if all the assumptions hold globally and* $V(\eta)$ *is radially unbounded, the origin will be globally asymptotically stable.* ◇

As an application of backstepping, consider the robust stabilization of the system

$$\left.\begin{array}{rcl} \dot{x}_i & = & x_{i+1} + \delta_i(x), \quad 1 \le i \le n-1 \\ \dot{x}_n & = & \gamma(x)[u - \alpha(x)] + \delta_n(x) \end{array}\right\} \tag{14.66}$$

defined on a domain $D \subset R^n$ that contains the origin $x = 0$, where $x = [x_1, \ldots, x_n]^T$. Suppose $\gamma(x) \neq 0$ and all functions are smooth for all $x \in D$. Let α and γ be known, and δ_i for $1 \le i \le n$ be uncertain terms. The nominal system is feedback linearizable. We assume that the uncertain terms satisfy the inequalities

$$|\delta_i(x)| \le a_i \sum_{k=1}^{i} |x_k|, \qquad \text{for } 1 \le i \le n \tag{14.67}$$

for all $x \in D$, where the nonnegative constants a_1 to a_n are known. Inequalities (14.67) restrict the class of uncertainties for $1 \le i \le n-1$ because they restrict the upper bound on $\delta_i(x)$ to depend only on x_1 to x_i; that is, the upper bounds appear in a strict feedback form. Nevertheless, they are less restrictive than the matching condition that would have required $\delta_i = 0$ for $1 \le i \le n-1$. To apply the backstepping recursive design procedure, we start with the scalar system

$$\dot{x}_1 = x_2 + \delta_1(x)$$

where x_2 is viewed as the control input and $\delta_1(x)$ satisfies the inequality $|\delta_1(x)| \le a_1|x_1|$. In this scalar system, the uncertainty satisfies the matching condition. The origin $x_1 = 0$ can be robustly stabilized by the high-gain feedback control $x_2 = -k_1 x_1$ where $k_1 > 0$ is chosen sufficiently large. In particular, let $V_1(x_1) = x_1^2/2$ be a Lyapunov function candidate. Then,

$$\dot{V}_1 = x_1[-k_1 x_1 + \delta_1(x)] \le -(k_1 - a_1)x_1^2$$

and the origin is stabilized for all $k_1 > a_1$. From this point on, backstepping and Lemma 14.3 are applied recursively to derive the stabilizing state feedback control. The procedure is illustrated by the following example.

Example 14.13 We want to design a state feedback control law to stabilize the second-order system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + \theta_1 x_1 \sin x_2 \\ \dot{x}_2 & = & \theta_2 x_2^2 + x_1 + u \end{array}$$

where θ_1 and θ_2 are unknown parameters that satisfy $|\theta_1| \le a$ and $|\theta_2| \le b$ for some known bounds a and b. The system takes the form (14.66) with $\delta_1 = \theta_1 x_1 \sin x_2$ and $\delta_2 = \theta_2 x_2^2$. The function δ_1 satisfies the inequality $|\delta_1| \le a|x_1|$ globally. The function δ_2 satisfies the inequality $|\delta_2| \le b\rho|x_2|$ on the set $|x_2| \le \rho$. Starting with the system

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2$$

we take $x_2 = \phi_1(x_1) = -k_1 x_1$ and $V_1(x_1) = x_1^2/2$ to obtain

$$\dot{V}_1 = x_1\phi_1(x_1) + \theta_1 x_1^2 \sin x_2 \le -(k_1 - a)x_1^2$$

We choose $k_1 = 1 + a$. To backstep, we apply the change of variables $z_2 = x_2 + (1+a)x_1$ and rewrite the system as

$$\begin{aligned} \dot{x}_1 &= -(1+a)x_1 + \theta_1 x_1 \sin x_2 + z_2 \\ \dot{z}_2 &= \psi_1(x) + \psi_2(x,\theta) + u \end{aligned}$$

where

$$\psi_1 = x_1 + (1+a)x_2, \qquad \psi_2 = (1+a)\theta_1 x_1 \sin x_2 + \theta_2 x_2^2$$

Using $V_c = (x_1^2 + z_2^2)/2$ as a composite Lyapunov function, we obtain

$$\dot{V}_c \le -x_1^2 + z_2[x_1 + \psi_1(x) + \psi_2(x,\theta) + u]$$

Taking

$$u = -x_1 - \psi_1(x) - kz_2$$

yields

$$\begin{aligned} \dot{V}_c &\le -x_1^2 + z_2\psi_2(x,\theta) - kz_2^2 \\ &\le -x_1^2 + a(1+a)|x_1|\ |z_2| + bx_2^2|z_2| - kz_2^2 \end{aligned}$$

On the set

$$\Omega_c = \{x \in R^2 \mid V_c(x) \le c\}$$

we have $|x_2| \le \rho$ for some ρ dependent on c.[20] Restricting our analysis to Ω_c, we obtain

$$\begin{aligned} \dot{V}_c &\le -x_1^2 + a(1+a)|x_1|\ |z_2| + b\rho|z_2 - (1+a)x_1|\ |z_2| - kz_2^2 \\ &\le -x_1^2 + (1+a)(a+b\rho)|x_1|\ |z_2| - (k - b\rho)z_2^2 \end{aligned}$$

Choosing

$$k > b\rho + (1+a)^2(a+b\rho)^2/4$$

ensures that the origin is exponentially stable[21] with Ω_c contained in the region of attraction. Since the preceding inequality can be satisfied for any $c > 0$ by choosing k sufficiently large, the feedback control can achieve semiglobal stabilization. △

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[20] ρ can be estimated by $\sqrt{2c(1+k_1^2)}$.

[21] Note that we conclude exponential stability, rather than asymptotic stability as guaranteed by Lemma 14.3. Why?

Example 14.14 Consider again the system

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1 x_1 \sin x_2\\ \dot{x}_2 &= \theta_2 x_2^2 + x_1 + u\end{aligned}$$

from the previous example, where $|\theta_1| \le a$ and $|\theta_2| \le b$. In that example, we limited the analysis to a compact set in order to satisfy the linear growth inequalities (14.67). Consequently, we could only achieve semiglobal stabilization. In this example, we achieve global stabilization by combining backstepping with Lyapunov redesign. We proceed as in the previous example until we reach the point

$$\dot{V}_c \le -x_1^2 + z_2[x_1 + \psi_1(x) + \psi_2(x,\theta) + u]$$

In Example 14.13, we used $u = -x_1 - \psi_1(x) - kz_2$ and relied on the high gain k to deal with the uncertainty. This required the nonlinear term x_2^2 to be bounded by the linear term $\rho|x_2|$. Here, we take

$$u = -x_1 - \psi_1(x) - kz_2 + v, \qquad k > 0$$

where the control component v is to be designed. Then,

$$\dot{V}_c \le -x_1^2 - kz_2^2 + z_2[\psi_2(x,\theta) + v]$$

Noting that

$$|\psi_2(x,\theta)| \le a(1+a)|x_1| + bx_2^2$$

we take

$$v = \begin{cases} -\eta(x)\ \mathrm{sgn}(z_2), & \text{if } \eta(x)|z_2| \ge \varepsilon \\ -\eta^2(x)z_2/\varepsilon, & \text{if } \eta(x)|z_2| < \varepsilon \end{cases}$$

where $\eta(x) = \eta_0 + a(1+a)|x_1| + bx_2^2$ for some $\eta_0 > 0$ and $\varepsilon > 0$. When $\eta(x)|z_2| \ge \varepsilon$,

$$z_2[\psi_2(x,\theta) + v] \le |\psi_2||z_2| - \eta|z_2| \le 0$$

and when $\eta(x)|z_2| < \varepsilon$,

$$z_2[\psi_2(x,\theta) + v] \le |\eta||z_2| - \frac{\eta^2 z_2^2}{\varepsilon} \le \frac{\varepsilon}{4}$$

Thus,

$$\dot{V}_c \le -x_1^2 - kz_2^2 + \frac{\varepsilon}{4}$$

This inequality shows that within a finite time interval, the state x enters a ball B_r of radius $r = k_0\sqrt{\varepsilon}$ for some $k_0 > 0$. Inside B_r, we have

$$|\psi_2(x,\theta)| \le a(1+a)|x_1| + br|x_2| \le \rho_1(|x_1| + |z_2|)$$

for some $\rho_1 > 0$ independent of ε. In the intersection of B_r and the boundary layer $\{\eta(x)|z_2| < \varepsilon\}$, we have

$$\begin{aligned}\dot{V}_c &\leq -x_1^2 - kz_2^2 + \rho_1|x_1||z_2| + \rho_1|z_2|^2 - \frac{\eta_0^2 z_2^2}{\varepsilon} \\ &= -\tfrac{1}{2}x_1^2 - kz_2^2 - \left[\tfrac{1}{2}x_1^2 - \rho_1|x_1||z_2| + \left(\frac{\eta_0^2}{\varepsilon} - \rho_1\right) z_2^2\right]\end{aligned}$$

The bracketed term can be made nonnegative by choosing ε small enough. Hence,

$$\dot{V}_c \leq -\tfrac{1}{2}x_1^2 - kz_2^2$$

and the origin is globally asymptotically stable. △

We conclude the section by noting that backstepping can be applied to multiinput systems in what is known as *block backstepping*, provided certain nonsingularity conditions are satisfied. Consider the system

$$\dot{\eta} = f(\eta) + G(\eta)\xi \tag{14.68}$$

$$\dot{\xi} = f_a(\eta,\xi) + G_a(\eta,\xi)u \tag{14.69}$$

where $\eta \in R^n$, $\xi \in R^m$, and $u \in R^m$, in which m could be greater than one. Suppose f, f_a, G, and G_a are (known) smooth functions over the domain of interest, f and f_a vanish at the origin, and the $m \times m$ matrix G_a is nonsingular. Suppose further that the component (14.68) can be stabilized by a smooth state feedback control law $\xi = \phi(\eta)$ with $\phi(0) = 0$, and we know a (smooth, positive definite) Lyapunov function $V(\eta)$ that satisfies the inequality

$$\frac{\partial V}{\partial \eta}[f(\eta) + G(\eta)\phi(\eta)] \leq -W(\eta)$$

for some positive definite function $W(\eta)$. Using

$$V_c = V(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^T[\xi - \phi(\eta)]$$

as a Lyapunov function candidate for the overall system, we obtain

$$\dot{V}_c = \frac{\partial V}{\partial \eta}(f + G\phi) + \frac{\partial V}{\partial \eta}G\ (\xi - \phi) + [\xi - \phi]^T\left[f_a + G_a u - \frac{\partial \phi}{\partial \eta}(f + G\xi)\right]$$

Taking

$$u = G_a^{-1}\left[\frac{\partial \phi}{\partial \eta}(f + G\xi) - \left(\frac{\partial V}{\partial \eta}G\right)^T - f_a - k(\xi - \phi)\right], \quad k > 0$$

results in

$$\dot{V}_c = \frac{\partial V}{\partial \eta}(f + G\phi) - k[\xi - \phi(\eta)]^T[\xi - \phi(\eta)] \leq -W(\eta) - k[\xi - \phi(\eta)]^T[\xi - \phi(\eta)]$$

which shows that the origin $(\eta = 0,\ \xi = 0)$ is asymptotically stable.

14.4 Passivity-Based Control

In Chapter 6, we introduced the notion of passivity and studied its role in analyzing the stability of feedback connections. The ideas of passivity-based control that we are going to introduce here are straightforward applications of the results of Chapter 6. However, we do not need the elaborate details of Chapter 6 to understand these ideas. It is enough to recall the definitions of passivity and zero-state observability.

We consider the p-input–p-output system

$$\begin{aligned} \dot{x} &= f(x,u) && (14.70)\\ y &= h(x) && (14.71) \end{aligned}$$

where f is locally Lipschitz in (x, u) and h is continuous in x, for all $x \in R^n$ and $u \in R^m$. We assume that $f(0,0) = 0$, so that the origin $x = 0$ is an open-loop equilibrium point, and $h(0) = 0$. We recall that the system (14.70)–(14.71) is passive if there exists a continuously differentiable positive semidefinite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall\ (x,u) \in R^n \times R^m \tag{14.72}$$

The system is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in the set $\{h(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$. Throughout this section we will require the storage function to be positive definite. The basic idea of passivity-based control is illustrated in the next theorem.

Theorem 14.4 *If the system* (14.70)–(14.71) *is*

(1) *passive with a radially unbounded positive definite storage function and*

(2) *zero-state observable,*

then the origin $x = 0$ can be globally stabilized by $u = -\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$.

Proof: Use the storage function $V(x)$ as a Lyapunov function candidate for the closed-loop system

$$\dot{x} = f(x, -\phi(y))$$

The derivative of V is given by

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T\phi(y) \leq 0$$

Hence, $\dot{V}$ is negative semidefinite and $\dot{V} = 0$ if and only if $y = 0$. By zero-state observability,

$$y(t) \equiv 0 \ \Rightarrow\ u(t) \equiv 0 \ \Rightarrow\ x(t) \equiv 0$$

Therefore, by the invariance principle, the origin is globally asymptotically stable. □

The intuition behind the theorem becomes clear when we think of the storage function as the energy of the system. A passive system has a stable origin. All that is needed to stabilize the origin is the injection of damping so that energy will dissipate whenever $x(t)$ is not identically zero. The required damping is injected by the function ϕ. There is great freedom in the choice of ϕ. We can choose it to meet any constraint on the magnitude of u. For example, if u is constrained to $|u_i| \leq k_i$ for $1 \leq i \leq p$, we can choose $\phi_i(y) = k_i \operatorname{sat}(y_i)$ or $\phi_i(y) = (2k_i/\pi)\tan^{-1}(y_i)$.

The utility of Theorem 14.4 can be increased by transforming nonpassive systems into passive ones. Consider, for example, a special case of (14.70), where

$$\dot{x} = f(x) + G(x)u \tag{14.73}$$

Suppose a radially unbounded, positive definite, continuously differentiable function $V(x)$ exists such that

$$\frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall\, x$$

Take

$$y = h(x) \overset{\text{def}}{=} \left[\frac{\partial V}{\partial x} G(x)\right]^T$$

Then the system with input u and output y is passive. If it is also zero-state observable, we can apply Theorem 14.4.

Example 14.15 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u \end{aligned}$$

Let $V(x) = x_1^4/4 + x_2^2/2$. Then

$$\dot{V} = x_1^3 x_2 - x_2 x_1^3 + x_2 u = x_2 u$$

Set $y = x_2$ and note that, with $u = 0$, $y(t) \equiv 0$ implies that $x(t) \equiv 0$. Thus, all the conditions of Theorem 14.4 are satisfied and a globally stabilizing state feedback control can be taken as $u = -kx_2$ or $u = -(2k/\pi)\tan^{-1}(x_2)$ with any $k > 0$. △

Allowing ourselves the freedom to choose the output function is useful, but we are still limited to state equations for which the origin is open-loop stable. We can cover a wider class of systems if we use feedback to achieve passivity. Consider again the system (14.73). If a feedback control

$$u = \alpha(x) + \beta(x)v \tag{14.74}$$

and an output function $h(x)$ exist such that the system

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v \tag{14.75}$$

$$y = h(x) \tag{14.76}$$

with input v and output y, satisfies the conditions of Theorem 14.4, we can globally stabilize the origin by using $v = -\phi(y)$. The use of feedback to convert a nonpassive system into a passive one is known as *feedback passivation*.[22]

Example 14.16 The nonlinear dynamic equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

where q is an m-dimensional vector of generalized coordinates representing joint positions, u is an m-dimensional control input, and $M(q)$ is a symmetric inertia matrix that is positive definite for all $q \in R^m$. The term $C(q,\dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces. The matrix C has the property that $\dot{M} - 2C$ is a skew-symmetric matrix for all $q, \dot{q} \in R^m$, where $\dot{M}$ is the total derivative of $M(q)$ with respect to t. The term $D\dot{q}$, where D is a positive semidefinite symmetric matrix, accounts for viscous damping. The term $g(q)$, which accounts for gravity forces, is given by $g(q) = [\partial P(q)/\partial q]^T$, where $P(q)$ is the total potential energy of the links due to gravity. Consider the regulation problem of designing a state feedback control law so that q asymptotically tracks a constant reference q_r. Let $e = q - q_r$. Then, e satisfies the differential equation

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

Our goal is to stabilize the system at $(e = 0, \dot{e} = 0)$, but this point is not an open-loop equilibrium point. Let

$$u = g(q) - K_p e + v$$

where K_p is a positive definite symmetric matrix and v is an additional control component to be chosen. Substituting u into the $\ddot{e}$-equation yields

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

Take

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

as a storage function candidate. The function V is positive definite and its derivative satisfies

$$\begin{aligned} \dot{V} &= \dot{e}^T M\ddot{e} + \tfrac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e} \\ &= \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \\ &\le \dot{e}^T v \end{aligned}$$

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[22] For the system (14.73) with output $y = h(x)$, it is shown in [31] that the system is locally feedback equivalent to a passive system with a positive definite storage function if $\text{rank}\{[\partial h/\partial x](0)G(0)\} = p$ and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function.

Defining the output as $y = \dot{e}$, we see that the system with input v and output y is passive with V as the storage function. It is interesting to note that the role of the passifying feedback component $g(q) - K_p e$ is to reshape the potential energy to $(1/2)e^T K_p e$, which has a unique minimum at $e = 0$. The sum of the kinetic energy and the reshaped potential energy is the storage function. With $v = 0$,

$$y(t) \equiv 0 \;\Leftrightarrow\; \dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; K_p e(t) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

which shows that the system is zero-state observable. Hence, it can be globally stabilized by the control $v = -\phi(\dot{e})$ with any function ϕ such that $\phi(0) = 0$ and $y^T \phi(y) > 0$ for all $y \neq 0$. The choice $v = -K_d \dot{e}$ with a positive definite symmetric matrix K_d results in the control

$$u = g(q) - K_p(q - q_r) - K_d \dot{q}$$

which takes the form of a classical PD controller plus a gravity compensation term. △

One class of systems that is amenable to feedback passivation is the cascade connection of a passive system with a system whose unforced dynamics have a stable equilibrium point at the origin. Consider the system

$$\begin{aligned} \dot{z} &= f_a(z) + F(z, y)y && (14.77) \\ \dot{x} &= f(x) + G(x)u && (14.78) \\ y &= h(x) && (14.79) \end{aligned}$$

where $f_a(0) = 0$, $f(0) = 0$, and $h(0) = 0$. The functions f_a, F, f, and G are locally Lipschitz and h is continuous. We view the system as a cascade connection of the driving system (14.78)–(14.79) and the driven system (14.77).[23] We assume that the representation (14.77)–(14.79) is valid globally, the driving system is passive with a radially unbounded positive definite storage function $V(x)$, the origin of $\dot{z} = f_a(z)$ is stable, and we know a radially unbounded Lyapunov function $W(z)$ for $\dot{z} = f_a(z)$, which satisfies

$$\frac{\partial W}{\partial z} f_a(z) \leq 0, \qquad \forall\ z$$

Using $U(z, x) = W(z) + V(x)$ as a storage function candidate for the full system (14.77)–(14.79), we obtain

$$\dot{U} \;=\; \frac{\partial W}{\partial z} f_a(z) + \frac{\partial W}{\partial z} F(z, y)y + \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x)u$$

---

[23] A driven system of the form $\dot{z} = f_0(z, y)$ with sufficiently smooth f_0 can be represented in the form (14.77) by taking

$$f_a(z) = f_0(z, 0) \quad \text{and} \quad F(z, y) = \int_0^1 \frac{\partial f_0}{\partial y}(z, sy)\ ds$$

$$\leq \frac{\partial W}{\partial z} F(z,y)y + y^T u = y^T \left[u + \left(\frac{\partial W}{\partial z} F(z,y) \right)^T \right]$$

The feedback control

$$u = -\left(\frac{\partial W}{\partial z} F(z,y) \right)^T + v$$

results in

$$\dot{U} \leq y^T v$$

Hence, the system

$$\begin{aligned} \dot{z} &= f_a(z) + F(z,y)y && (14.80) \\ \dot{x} &= f(x) - G(x) \left(\frac{\partial W}{\partial z} F(z,y) \right)^T + G(x)v && (14.81) \\ y &= h(x) && (14.82) \end{aligned}$$

with input v and output y is passive with U as the storage function. If the system (14.80)–(14.82) is zero-state observable, we can apply Theorem 14.4 to globally stabilize the origin.

Example 14.17 The rotational motion of a rigid body subject to three independent scalar control torques can be modeled by[24]

$$\begin{aligned} \dot{\rho} &= \tfrac{1}{2}[I_3 + S(\rho) + \rho\rho^T]\omega \\ M\dot{\omega} &= -S(\omega)M\omega + u \end{aligned}$$

where $\omega \in R^3$ is the velocity vector and $\rho \in R^3$ is a particular choice of the kinematic parameters, which leads to a three-dimensional representation of the rotation group. The matrix $S(x)$ is a skew-symmetric matrix, defined by

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

M is a positive definite symmetric inertia matrix, and I_3 is the 3×3 identity matrix. Taking $y = \omega$, it can be seen that the system takes the form of the cascade connection (14.77)–(14.79) with

$$M\dot{\omega} = -S(\omega)M\omega + u, \qquad y = \omega$$

as the driving system and

$$\dot{\rho} = \tfrac{1}{2}[I_3 + S(\rho) + \rho\rho^T]\omega$$

---

[24] See [97] and [151] for the derivation of the model. If $\varepsilon \in R^3$ and $\eta \in R$ are the Euler parameters, then $\rho = \varepsilon/\eta$.

as the driven system. Taking $V(\omega) = (1/2)\omega^T M\omega$, it can be seen that

$$\dot{V} = \omega^T M\dot{\omega} = -\omega^T S(\omega)M\omega + \omega^T u = y^T u$$

where we used the property $\omega^T S(\omega) = 0$. Hence, the driving system is passive. The unforced driven system $\dot{\rho} = 0$ has a stable equilibrium point at $\rho = 0$ and any radially unbounded, positive definite, continuously differentiable function $W(\rho)$ will qualify as a Lyapunov function. Thus, all our assumptions are satisfied and the system can be made passive by the control

$$u = -\left\{\frac{\partial W}{\partial \rho}\tfrac{1}{2}[I_3 + S(\rho) + \rho\rho^T]\right\}^T + v$$

Taking $W(\rho) = k\ln(1 + \rho^T\rho)$, with $k > 0$, yields

$$u = -\left\{\frac{k\rho^T}{1+\rho^T\rho}[I_3 + S(\rho) + \rho\rho^T]\right\}^T + v = -k\rho + v$$

where we used the property $\rho^T S(\rho) = 0$. We need to check zero-state observability of the passive system

$$\begin{aligned} \dot{\rho} &= \tfrac{1}{2}[I_3 + S(\rho) + \rho\rho^T]\omega \\ M\dot{\omega} &= -S(\omega)M\omega - k\rho + v \\ y &= \omega \end{aligned}$$

With $v = 0$,

$$y(t) \equiv 0 \;\Leftrightarrow\; \omega(t) \equiv 0 \;\Rightarrow\; \dot{\omega}(t) \equiv 0 \;\Rightarrow\; \rho(t) \equiv 0$$

Hence, the system is zero-state observable and can be globally stabilized by

$$u = -k\rho - \phi(\omega)$$

with any locally Lipschitz function ϕ such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$. △

Checking zero-state observability of the full system (14.80)–(14.82) can be avoided if we strengthen the assumption on $W(z)$ to

$$\frac{\partial W}{\partial z} f_a(z) < 0, \quad \forall\ z \neq 0, \quad \text{and} \quad \frac{\partial W}{\partial z}(0) = 0 \tag{14.83}$$

which implies that the origin of $\dot{z} = f_a(z)$ is globally asymptotically stable. Taking

$$u = -\left(\frac{\partial W}{\partial z}F(z,y)\right)^T - \phi(y) \tag{14.84}$$

where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$, and using U as a Lyapunov function candidate for the closed-loop system, we obtain

$$\dot{U} \leq \frac{\partial W}{\partial z} f_a(z) - y^T\phi(y) \leq 0$$

Moreover, $\dot{U} = 0$ implies that $z = 0$ and $y = 0$, which in turn imply that $u = 0$. If the driving system (14.78)–(14.79) is zero-state observable, the conditions $u(t) \equiv 0$ and $y(t) \equiv 0$ imply that $x(t) \equiv 0$. Hence, by the invariance principle, the origin $(z = 0, x = 0)$ is globally asymptotically stable. We summarize this conclusion in the next theorem.

Theorem 14.5 *Suppose the system* (14.78)–(14.79) *is zero-state observable and passive with a radially unbounded, positive definite storage function. Suppose the origin of* $\dot{z} = f_a(z)$ *is globally asymptotically stable and let* $W(z)$ *be a radially unbounded, positive definite Lyapunov function that satisfies* (14.83). *Then, the control* (14.84) *globally stabilizes the origin* $(z = 0, x = 0)$. ◇

Example 14.18 Consider the system

$$\begin{aligned} \dot{\eta} &= -\eta + \eta^2\xi \\ \dot{\xi} &= u \end{aligned}$$

which was studied in Examples 13.16 and 14.11. With $y = \xi$ as the output, the system takes the form (14.77)–(14.79). The system $\dot{\xi} = u$, $y = \xi$ is passive with the storage function $V(\xi) = \xi^2/2$. It is clearly zero-state observable, since $y = \xi$. The origin of $\dot{\eta} = -\eta$ is globally exponentially stable with the Lyapunov function $W(\eta) = \eta^2/2$. Thus, all the conditions of Theorem 14.5 are satisfied and a globally stabilizing state feedback control can be taken as

$$u = -\eta^3 - k\zeta, \qquad k > 0$$

which is the same control we derived by using backstepping △

14.5 High-Gain Observers

The nonlinear design techniques discussed in this chapter, and the previous one, assume state feedback; that is, measurements of all state variables are available. In many practical problems we cannot measure all state variables, or we may choose not to measure some of them due to technical or economic reasons. Therefore, it is important to extend these techniques to output feedback. In some special cases, we can modify the technique to produce an output feedback controller. Examples are explored in Exercises 14.47 and 14.48. The first exercise shows that for minimum-phase, relative-degree-one systems, we can design sliding mode control as output feedback. The second exercise shows passivity-based control for a system that has a passive map from the input to the derivative of the output. In more general cases, we have to use dynamic compensation to extend state feedback designs to output feedback. One form of dynamic compensation is to use observers that asymptotically estimate the state from output measurements. For some nonlinear systems, the design of such observers could be as easy as in linear systems. Suppose, for example, that a nonlinear system can be transformed into the form[25]

$$\dot{x} = Ax + g(y, u) \tag{14.85}$$

$$y = Cx \tag{14.86}$$

where (A, C) is observable. This form is special because the nonlinear function g depends only on the output y and the control input u. Taking the observer as

$$\dot{\hat{x}} = A\hat{x} + g(y, u) + H(y - C\hat{x}) \tag{14.87}$$

it can be easily seen that the estimation error $\tilde{x} = x - \hat{x}$ satisfies the linear equation

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Hence, designing C such that $A - HC$ is Hurwitz guarantees asymptotic error convergence, that is, $\lim_{t\to\infty} \tilde{x}(t) = 0$. Exercise 14.49 explores the use of the observer (14.87) in output feedback control. Aside from the fact that the observer (14.87) works only for a special class of nonlinear systems, its main drawback is the assumption that the nonlinear function g is perfectly known. Any error in modeling g will be reflected in the estimation error equation. In particular, if the observer is implemented as

$$\dot{\hat{x}} = A\hat{x} + g_0(y, u) + H(y - C\hat{x})$$

where g_0 is a nominal model of g, the $\dot{\tilde{x}}$-equation becomes

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + g(y, u) - g_0(y, u)$$

It is no longer obvious that a Hurwitz $A - HC$ can handle the perturbation term $g - g_0$. In this section, we give a special design of the observer gain that makes the observer robust to uncertainties in modeling the nonlinear functions. The technique, known as high-gain observers, works for a wide class of nonlinear systems and guarantees that the output feedback controller recovers the performance of the state feedback controller when the observer gain is sufficiently high. In Section 14.5.1, we use a second-order example to motivate the idea of high-gain observers. In Section 14.5.2, we use the observer in output feedback stabilization. The main result of that section is a *separation principle* that allows us to separate the design into two tasks. First, we design a state feedback controller that stabilizes the system and meets other design specifications. Then, an output feedback controller is obtained by replacing the state x by its estimate $\hat{x}$ provided by the high-gain observer. A key property that makes this separation possible is the design of the state feedback controller to be globally bounded in x. High-gain observers can be used in a wide range of control problems.[26] As an example, we show in Section 14.5.3 the use of high-gain observers in output feedback integral regulators.

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[25] Necessary and sufficient conditions for a nonlinear system to be equivalent to the form (14.85)–(14.86) are given in [124, Chapter 5]. The same reference gives a different approach for introducing dynamic compensation through filtered transformations.

14.5.1 Motivating Example

Consider the second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 &\quad (14.88)\\ \dot{x}_2 &= \phi(x,u) &\quad (14.89)\\ y &= x_1 &\quad (14.90) \end{aligned}$$

where $x = [x_1, x_2]^T$. Suppose $u = \gamma(x)$ is a locally Lipschitz state feedback control law that stabilizes the origin $x = 0$ of the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 &\quad (14.91)\\ \dot{x}_2 &= \phi(x,\gamma(x)) &\quad (14.92) \end{aligned}$$

To implement this feedback control using only measurements of the output y, we use the observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + h_1(y - \hat{x}_1) &\quad (14.93)\\ \dot{\hat{x}}_2 &= \phi_0(\hat{x},u) + h_2(y - \hat{x}_1) &\quad (14.94) \end{aligned}$$

where $\phi_0(x,u)$ is a nominal model of the nonlinear function $\phi(x,u)$. The estimation error

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix}$$

satisfies the equation

$$\begin{aligned} \dot{\tilde{x}}_1 &= -h_1\tilde{x}_1 + \tilde{x}_2 &\quad (14.95)\\ \dot{\tilde{x}}_2 &= -h_2\tilde{x}_1 + \delta(x,\tilde{x}) &\quad (14.96) \end{aligned}$$

where $\delta(x,\tilde{x}) = \phi(x,\gamma(\hat{x})) - \phi_0(\hat{x},\gamma(\hat{x}))$. We want to design the observer gain $H = [h_1, h_2]^T$ such that $\lim_{t\to\infty} \tilde{x}(t) = 0$. In the absence of the disturbance term δ, asymptotic error convergence is achieved by designing H such that

$$A_o = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}$$

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[26] See [100] for a survey of the use of high-gain observers in various control problem formulations.

is Hurwitz. For this second-order system, A_o is Hurwitz for any positive constants h_1 and h_2. In the presence of δ, we need to design H with the additional goal of rejecting the effect of δ on $\tilde{x}$. This is ideally achieved, for any δ, if the transfer function

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2} \left[\begin{array}{c} 1 \\ s + h_1 \end{array} \right]$$

from δ to $\tilde{x}$ is identically zero. While this is not possible, we can make $\sup_{\omega \in R} |G_o(j\omega)|$ arbitrarily small by choosing $h_2 \gg h_1 \gg 1$. In particular, taking

$$h_1 = \frac{\alpha_1}{\varepsilon}, \quad h_2 = \frac{\alpha_2}{\varepsilon^2} \tag{14.97}$$

for some positive constants α_1, α_2, and ε, with $\varepsilon \ll 1$, it can be shown that

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \left[\begin{array}{c} \varepsilon \\ \varepsilon s + \alpha_1 \end{array} \right]$$

Hence, $\lim_{\varepsilon \to 0} G_o(s) = 0$. The disturbance rejection property of the high-gain observer can be also seen in the time domain by representing the error equation (14.95)–(14.96) in the singularly perturbed form. Towards that end, define the scaled estimation errors

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}, \quad \eta_2 = \tilde{x}_2 \tag{14.98}$$

The newly defined variables satisfy the singularly perturbed equation

$$\varepsilon \dot{\eta}_1 = -\alpha_1 \eta_1 + \eta_2 \tag{14.99}$$

$$\varepsilon \dot{\eta}_2 = -\alpha_2 \eta_1 + \varepsilon \delta(x, \tilde{x}) \tag{14.100}$$

This equation shows clearly that reducing ε diminishes the effect δ. It shows also that, for small ε, the scaled-estimation error η will be much faster than x. Notice, however, that $\eta_1(0)$ will be $O(1/\varepsilon)$ whenever $x_1(0) \neq \hat{x}_1(0)$. Consequently, the solution of (14.99)–(14.100) will contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$ for some $a > 0$. While this exponential mode decays rapidly, it exhibits an impulsive-like behavior where the transient peaks to $O(1/\varepsilon)$ values before it decays rapidly towards zero. In fact, the function $(a/\varepsilon)e^{-at/\varepsilon}$ approaches an impulse function as ε tends to zero. This behavior is known as the *peaking phenomenon*. It is important to realize that the peaking phenomenon is not a consequence of using the change of variables (14.98) to represent the error dynamics in the singularly perturbed form. It is an intrinsic feature of any high-gain-observer with $h_2 \gg h_1 \gg 1$. This point can be seen by calculating the transition matrix $\exp(A_o t)$ and noting that its (2,1) element is given by

$$\frac{-2h_2}{\sqrt{4h_2 - h_1^2}} e^{-h_1 t/2} \sin \left(\frac{t\sqrt{4h_2 - h_1^2}}{2} \right)$$

when $4h_2 > h_1^2$ and

$$\frac{-h_2}{\sqrt{h_1^2 - 4h_2}} \left\{ \exp\left[-\left(\frac{h_1 - \sqrt{h_1^2 - 4h_2}}{2} \right) t \right] - \exp\left[-\left(\frac{h_1 + \sqrt{h_1^2 - 4h_2}}{2} \right) t \right] \right\}$$

when $4h_2 < h_1^2$. The amplitude of the exponential mode is greater than $\sqrt{h_2}$ in the first case and h_2/h_1 in the second one. Thus, as we increase h_1 and h_2/h_1, we drive the amplitude toward infinity.

To get a better feel for the peaking phenomenon, let us simulate the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2^3 + u \\ y &= x_1 \end{aligned}$$

which can be globally stabilized by the state feedback controller

$$u = -x_2^3 - x_1 - x_2$$

The output feedback controller is taken as

$$\begin{aligned} u &= -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2 \\ \dot{\hat{x}}_1 &= \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= (1/\varepsilon^2)(y - \hat{x}_1) \end{aligned}$$

where the observer gain assigns the eigenvalues of A_o at $-1/\varepsilon$ and $-1/\varepsilon$. Figure 14.16 shows the performance of the closed-loop system under state and output feedback. Output feedback is simulated for three different values of ε. The initial conditions are $x_1(0) = 0.1$, $x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. Peaking is induced by $[x_1(0) - \hat{x}_1(0)]/\varepsilon = 0.1/\varepsilon$ when ε is sufficiently small. Figure 14.16 shows a counter intuitive behavior as ε decreases. Since decreasing ε causes the estimation error to decay faster toward zero, one would expect the response under output feedback to approach the response under state feedback as ε decreases. Figure 14.16 shows the opposite behavior, where the response under output feedback deviates from the response under state feedback as ε decreases. This is the impact of the peaking phenomenon. The same figure shows the control u on a much shorter time interval to exhibit peaking. This control peaking is transmitted to the plant causing its state to peak. If peaking of the state takes it outside the region of attraction, it could destabilize the system. Figure 14.17 shows that this is exactly what happens in the current case as we decrease ε to 0.004 where the system has a finite escape time shortly after $t = 0.07$.

Fortunately, we can overcome the peaking phenomenon by saturating the control outside a compact region of interest in order to create a buffer that protects the plant from peaking. Suppose the control is saturated as

$$u = \text{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$$

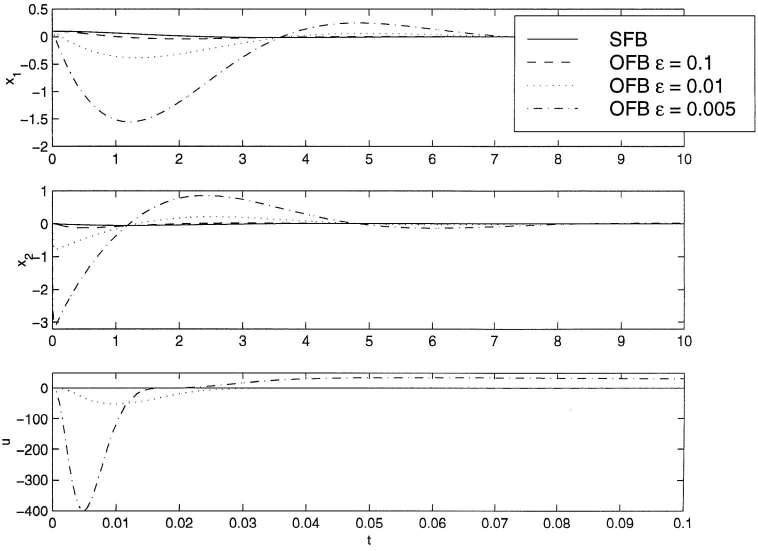

Figure 14.16: Performance under state (SFB) and output (OFB) feedback.

Figure 14.18 shows the performance of the closed-loop system under saturated state and output feedback. The control u is shown on a shorter time interval that exhibits control saturation during peaking. The peaking period decreases with ε. The states x_1 and x_2 exhibit the intuitive behavior we expected earlier; namely, the response under output feedback approaches the response under state feedback as ε decreases. Note that we decrease ε to 0.001, beyond the value 0.004 where instability was detected in the unsaturated case. Not only does the system remain stable, but the response under output feedback is almost indistinguishable from the response under state feedback. What is more interesting is that the region of attraction under output feedback approaches the region of attraction under state feedback as ε tends to zero. This is shown in Figures 14.19 and 14.20. The first figure shows the phase portrait of the closed-loop system under $u = \text{sat}(-x_2^3 - x_1 - x_2)$. It has a bounded region of attraction enclosed by a limit cycle. The second figure shows that the intersection of the boundary of the region of attraction under $u = \text{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$ with the x_1–x_2 plane approaches the limit cycle as ε tends to zero.

The behavior we saw in Figures 14.18 and 14.20 will be realized with any globally bounded stabilizing function $\gamma(x)$. During the peaking period, the control $\gamma(\hat{x})$ saturates. Since the peaking period shrinks to zero as ε tends to zero, for sufficiently small ε, the peaking period becomes so small that the state of the plant x remains close to its initial value. After the peaking period, the estimation error becomes $O(\varepsilon)$ and the feedback control $\gamma(\hat{x})$ becomes close to $\gamma(x)$. Consequently, the trajectories of the closed-loop system under output feedback asymptotically approach

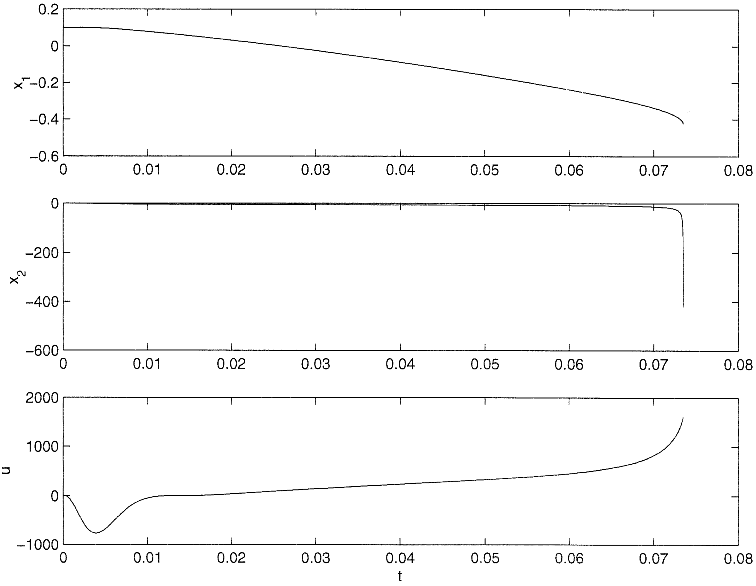

Figure 14.17: Instability induced by peaking at $\varepsilon = 0.004$.

its trajectories under state feedback as ε tends to zero. This leads to recovery of the performance achieved under state feedback. The global boundedness of $\gamma(x)$ can be always achieved by saturating the state feedback control, or the state estimates, outside a compact region of interest.

The analysis of the closed-loop system under output feedback proceeds as follows: The system is represented in the singularly perturbed form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \phi(x, \gamma(\hat{x})) \\
\varepsilon\dot{\eta}_1 &= -\alpha_1\eta_1 + \eta_2 \\
\varepsilon\dot{\eta}_2 &= -\alpha_2\eta_1 + \varepsilon\delta(x, \tilde{x})
\end{aligned}$$

where $\hat{x}_1 = x_1 - \varepsilon\eta_1$ and $\hat{x}_2 = x_2 - \eta_2$. The motion of the slow variables (x_1, x_2) can be approximated by a slow model obtained by setting $\varepsilon = 0$. Since $\varepsilon = 0$ yields $\eta = 0$, the slow model is the closed-loop system under state feedback, given by (14.91)–(14.92). The fast motion of (η_1, η_2) can be approximated by the fast model

$$\varepsilon\dot{\eta} = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \eta \overset{\text{def}}{=} A_0\eta$$

obtained by neglecting $\varepsilon\delta$. Let $V(x)$ be a Lyapunov function for the slow model and $W(\eta) = \eta^T P_0 \eta$ be a Lyapunov function for the fast model, where P_0 is the

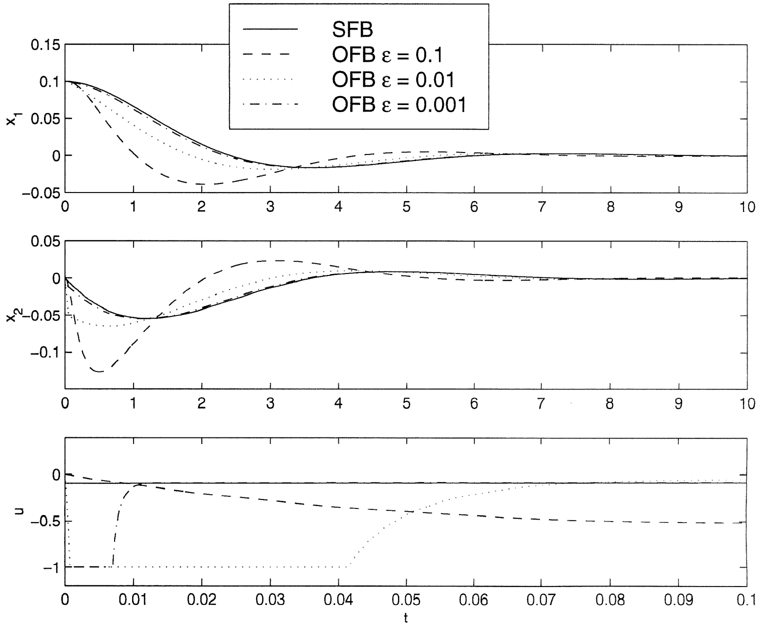

Figure 14.18: Performance under state (SFB) and output (OFB) feedback with saturation.

solution of the Lyapunov equation $P_0A_0 + A_0^TP_0^T = -I$. Define the sets Ω_c and Σ by $\Omega_c = \{V(x) \leq c\}$ and $\Sigma = \{W(\eta) \leq \rho\varepsilon^2\}$, where $c > 0$ is chosen such that Ω_c is in the interior of the region of attraction of (14.91)–(14.92). The analysis can be divided in two basic steps. In the first step we show that for sufficiently large ρ, there is $\varepsilon_1^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the origin of the closed-loop system is asymptotically stable and the set $\Omega_c \times \Sigma$ is a positively invariant subset of the region of attraction. The proof makes use of the fact that in $\Omega_c \times \Sigma$, η is $O(\varepsilon)$. In the second step, we show that for any bounded $\hat{x}(0)$ and any $x(0) \in \Omega_b$, where $0 < b < c$, there exists $\varepsilon_2^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, the trajectory enters the set $\Omega_c \times \Sigma$ in finite time. The proof makes use of the fact that Ω_b is in the interior of Ω_c and $\gamma(\hat{x})$ is globally bounded. Hence, there exits a time $T_1 > 0$, independent of ε, such that any trajectory starting in Ω_b will remain in Ω_c for all $t \in [0, T_1]$. Using the fact that η decays faster than an exponential mode of the form $(1/\varepsilon)e^{-at/\varepsilon}$, we can show that the trajectory enters the set $\Omega_c \times \Sigma$ within a time interval $[0, T(\varepsilon)]$, where $\lim_{\varepsilon\to 0} T(\varepsilon) = 0$. Thus, by choosing ε small enough, we can ensure that $T(\varepsilon) < T_1$. Figure 14.21 gives a sketch that illustrates this behavior.

The full-order observer (14.93)–(14.94) provides estimates $(\hat{x}_1, \hat{x}_2)$ that are used to replace (x_1, x_2) in the feedback control law. Since $y = x_1$ is measured, we can use x_1 in the control law and only replace x_2 by $\hat{x}_2$. This approach does not change

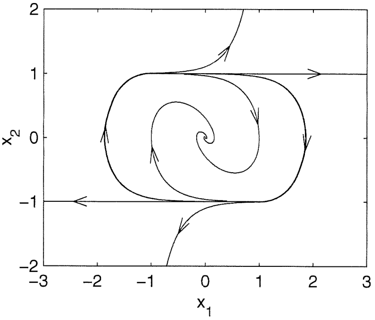

Figure 14.19: Phase portrait of the closed-loop system under $u = \text{sat}(-x_2^3 - x_1 - x_2)$.

the analysis of the closed-loop system, and we obtain the same results as before. On the other hand, we can use the reduced-order observer

$$\dot{w} = -h(w + hy) + \phi_o(\hat{x}, u) \tag{14.101}$$

$$\hat{x}_2 = w + hy \tag{14.102}$$

where $h = \alpha/\varepsilon$ for some positive constants α and ε with $\varepsilon \ll 1$, to estimate $\hat{x}_2$. It is not difficult to see that the high-gain reduced-order observer (14.101)–(14.102) exhibits the peaking phenomenon, and global boundedness of the state feedback control plays the same role as in the full-order observer case.

The high-gain observer is basically an approximate differentiator. This point can be easily seen in the special case when the nominal function ϕ_0 is chosen to be zero, for then the observer is linear. For the full-order observer (14.93)–(14.94), the transfer function from y to $\hat{x}$ is

$$\frac{\alpha_2}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} 1 + (\varepsilon\alpha_1/\alpha_2)s \\ s \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ s \end{bmatrix} \quad \text{as } \varepsilon \rightarrow 0$$

and for the reduced-order observer (14.101)–(14.102), the transfer function from y to $\hat{x}_2$ is

$$\frac{s}{(\varepsilon/\alpha)s + 1} \rightarrow s \quad \text{as } \varepsilon \rightarrow 0$$

Thus, on a compact frequency interval, the high-gain observer approximates $\dot{y}$ for sufficiently small ε.

Realizing that the high-gain observer is basically an approximate differentiator, we can see that measurement noise and unmodeled high-frequency sensor dynamics will put a practical limit on how small ε could be. Despite this limitation, there are interesting applications where the range of permissible values of ε allows for successful application of high-gain observers.[27]

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[27] Examples of application to induction motors and mechanical systems are given in [3], [47], and [186].

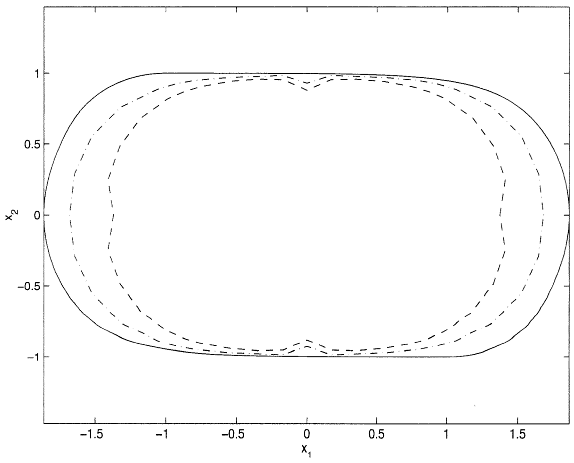

Figure 14.20: The region of attraction under state feedback (solid) and intersection of the region of attraction under output feedback with the x_1-x_2 plane for $\varepsilon = 0.1$ (dashed) and $\varepsilon = 0.05$ (dash-dot).

14.5.2 Stabilization

Consider the multi-input–multi-output nonlinear system

$$
\begin{aligned}
\dot{x} &= Ax + B\phi(x, z, u) && (14.103)\\
\dot{z} &= \psi(x, z, u) && (14.104)\\
y &= Cx && (14.105)\\
\zeta &= q(x, z) && (14.106)
\end{aligned}
$$

where $u \in R^p$ is the control input, $y \in R^m$ and $\zeta \in R^s$ are measured outputs, and $x \in R^\rho$ and $z \in R^\ell$ constitute the state vector. The $\rho \times \rho$ matrix A, the $\rho \times m$ matrix B, and the $m \times \rho$ matrix C, given by

$$
A = \text{block diag}[A_1, \ldots, A_m], \quad A_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{\rho_i \times \rho_i}
$$

$$
B = \text{block diag}[B_1, \ldots, B_m], \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho_i \times 1}
$$

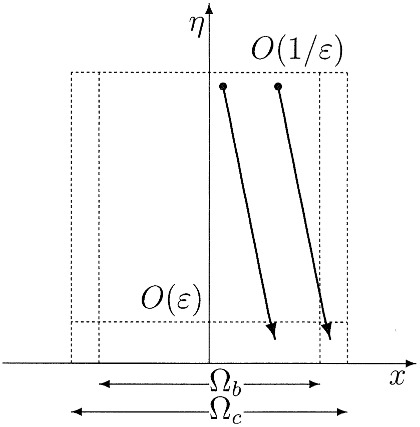

Figure 14.21: Illustration of fast convergence to the set $\Omega_c \times \Sigma$.

$$C = \text{block diag}[C_1, \ldots, C_m], \quad C_i = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{1\times\rho_i}$$

where $1 \leq i \leq m$ and $\rho = \rho_1 + \cdots + \rho_m$, represent m chains of integrators. The functions ϕ, ψ, and q are locally Lipschitz in their arguments for $(x, z, u) \in D_x \times D_z \times R^p$, where $D_x \subset R^\rho$ and $D_z \subset R^s$ are domains that contain their respective origins. Moreover, $\phi(0,0,0) = 0$, $\psi(0,0,0) = 0$, and $q(0,0) = 0$. Our goal is to design an output feedback controller to stabilize the origin.

The two main sources for the model (14.103)–(14.106) are the normal form of input–output linearizable systems and models of mechanical and electromechanical systems, where displacement variables are measured while their derivatives (velocities, accelerations, etc.) are not measured. The normal form for a single-input–single-output system is given by (13.16)–(13.18). It is easy to see that the equation takes the form (14.103)–(14.105) with $x = \xi$ and $z = \eta$.[28] If y is the only measured variable, we can drop equation (14.106). However, in many problems, we can measure some state variables in addition to those at the end of the chains of integrators. For example, the magnetic suspension system of Exercise 1.18 is modeled by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a+x_1)^2} + u\right]
\end{aligned}$$

where x_1 is the ball position, x_2 is its velocity, and x_3 is the electromagnet current. Typically, we measure the ball position x_1 and the current x_3. The model fits the form (14.103)–(14.106) with (x_1, x_2) as the x component and x_3 as the z

---

[28] See [88, Section 5.1] for the multivariable normal form.

component. The measured outputs are $y = x_1$ and $\zeta = x_3$. Another source for the model (14.103)–(14.106) where (14.106) is significant arises in system in which the dynamics are extended by adding integrators. In Example 13.8, we saw a system represented by an nth order differential equation. The dynamics were extended by adding m integrators at the input side. Inspection of the resulting state model shows that it fits the form (14.103)–(14.106) with z as the states of the m integrators and x as the output and its derivatives up to $y^{(n-1)}$. In this case, the whole vector z is measured, and equation (14.106) takes the form $\zeta = z$.

We use a two-step approach to design the output feedback controller. First, a partial state feedback controller that uses measurements of x and ζ is designed to asymptotically stabilize the origin. Then, a high-gain observer is used to estimate x from y. The state feedback controller is allowed to be a dynamical system of the form

$$\dot{\vartheta} = \Gamma(\vartheta, x, \zeta) \tag{14.107}$$

$$u = \gamma(\vartheta, x, \zeta) \tag{14.108}$$

where γ and Γ are locally Lipschitz functions in their arguments over the domain of interest and globally bounded functions of x. Moreover, $\gamma(0,0,0) = 0$ and $\Gamma(0,0,0) = 0$. A static state feedback controller $u = \gamma(x, \zeta)$ will be viewed as a special case of the foregoing equation by dropping the $\dot{\vartheta}$-equation. For convenience, we write the closed-loop system under state feedback as

$$\dot{\mathcal{X}} = f(\mathcal{X}) \tag{14.109}$$

where $\mathcal{X} = (x, z, \vartheta)$. The output feedback controller is taken as

$$\dot{\vartheta} = \Gamma(\vartheta, \hat{x}, \zeta) \tag{14.110}$$

$$u = \gamma(\vartheta, \hat{x}, \zeta) \tag{14.111}$$

where $\hat{x}$ is generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, u) + H(y - C\hat{x}) \tag{14.112}$$

The observer gain H is chosen as

$$H = \text{block diag}[H_1, \ldots, H_m], \quad H_i = \begin{bmatrix} \alpha_1^i/\varepsilon \\ \alpha_2^i/\varepsilon^2 \\ \vdots \\ \alpha_{\rho_i-1}^i/\varepsilon^{\rho_i-1} \\ \alpha_{\rho_i}^i/\varepsilon^{\rho_i} \end{bmatrix}_{\rho_i \times 1} \tag{14.113}$$

where ε is a positive constant to be specified and the positive constants α_j^i are chosen such that the roots of

$$s^{\rho_i} + \alpha_1^i s^{\rho_i - 1} + \cdots + \alpha_{\rho_i-1}^i s + \alpha_{\rho_i}^i = 0$$

are in the open left-half plane, for all $i = 1, \ldots, m$. The function $\phi_0(x, \zeta, u)$ is a nominal model of $\phi(x, z, u)$, which is required to be locally Lipschitz in its arguments over the domain of interest and globally bounded in x. Moreover, $\phi_0(0, 0, 0) = 0$.

Theorem 14.6 *Consider the closed-loop system of the plant* (14.103)–(14.106) *and the output feedback controller* (14.110)–(14.112). *Suppose the origin of* (14.109) *is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of R^ρ. Then,*

- *there exists $\varepsilon_1^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the solutions $(\mathcal{X}(t), \hat{x}(t))$ of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \geq 0$.*
- *given any $\mu > 0$, there exist $\varepsilon_2^* > 0$ and $T_2 > 0$, both dependent on μ, such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, the solutions of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, satisfy*

$$\|\mathcal{X}(t)\| \leq \mu \quad \text{and} \quad \|\hat{x}(t)\| \leq \mu, \quad \forall\, t \geq T_2 \tag{14.114}$$

- *given any $\mu > 0$, there exists $\varepsilon_3^* > 0$, dependent on μ, such that, for every $0 < \varepsilon \leq \varepsilon_3^*$, the solutions of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, satisfy*

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \geq 0 \tag{14.115}$$

  where $\mathcal{X}_r$ is the solution of (14.109), *starting at $\mathcal{X}(0)$.*

- *if the origin of* (14.109) *is exponentially stable and that $f(\mathcal{X})$ is continuously differentiable in some neighborhood of $\mathcal{X} = 0$, then there exists $\varepsilon_4^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_4^*$, the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction.* ◇

Proof: See Appendix C.23.

The theorem shows that the output feedback controller recovers the performance of the state feedback controller for sufficiently small ε. The performance recovery manifests itself in three points. First, recovery of exponential stability. Second, recovery of the region of attraction in the sense that we can recover any compact set in its interior. Third, the solution $\mathcal{X}(t)$ under output feedback approaches the solution under state feedback as ε tends to zero. For convenience, recovery of asymptotic stability is shown only for the exponentially stable case.[29] Notice, however, that the first three bullets of the theorem, which show boundedness, ultimate boundedness, and trajectory convergence, are valid without the exponential stability assumption.

As a corollary of the theorem, it is clear that if the state feedback controller achieves global or semiglobal stabilization with local exponential stability, then for sufficiently small ε, the output feedback controller achieves semiglobal stabilization with local exponential stability.

---

[29] See [16] for the more general case when the origin is asymptotically, but not exponentially, stable.

Example 14.19 In Section 14.1, we designed the continuous sliding mode state feedback controller

$$u = -k \text{ sat}\left(\frac{a_1(\theta - \pi) + \dot{\theta}}{\mu}\right)$$

with $a_1 = 1$, $k = 4$, and $\mu = 1$, to stabilize the pendulum equation

$$m\ell^2\ddot{\theta} + mg_0\ell \sin\theta + k_0\ell^2\dot{\theta} = u$$

at $(\theta = \pi, \dot{\theta} = 0)$. Suppose now that we only measure θ. An output feedback controller can be taken as

$$u = -k \text{ sat}\left(\frac{a_1(\hat{\theta} - \pi) + \hat{\omega}}{\mu}\right)$$

where $\hat{\theta}$ and $\hat{\omega}$ are the estimates of θ and $\omega = \dot{\theta}$ provided by the high-gain observer

$$\begin{aligned} \dot{\hat{\theta}} &= \hat{\omega} + (2/\varepsilon)(\theta - \hat{\theta}) \\ \dot{\hat{\omega}} &= \phi_0(\hat{\theta}, u) + (1/\varepsilon^2)(\theta - \hat{\theta}) \end{aligned}$$

where $\phi_0 = -\hat{a}\sin\hat{\theta} + \hat{c}u$ is a nominal model of $\phi = -(g_0/\ell)\sin\theta - (k_0/m)\dot{\theta} + (1/m\ell^2)u$ in which $\hat{a}$ and $\hat{c}$ are nominal values of (g_0/ℓ) and $(1/m\ell^2)$, respectively, while the nominal value of the coefficient of friction k_0 is taken to be zero. The observer is designed to have multiple poles at $-1/\varepsilon$. In Figure 14.22, we compare the performance of the state and output feedback controllers for $\varepsilon = 0.05$ and $\varepsilon = 0.01$. The pendulum parameters are $m = 0.15$, $\ell = 1.05$, and $k_0 = 0.02$ and the initial conditions are $\theta(0) = \pi/4$ and $\omega(0) = \hat{\theta}(0) = \hat{\omega}(0) = 0$. We consider three cases for the observer. In the first case, the observer uses the nominal values $\hat{a} = 9.81$ and $\hat{c} = 10$, which correspond to the nominal parameters $\hat{m} = 0.1$ and $\hat{\ell} = 1$. In the second case, we use $\hat{a} = 9.3429$ and $\hat{c} = 6.0469$, which correspond to the actual parameters; that is, $\hat{m} = m = 0.15$ and $\hat{\ell} = \ell = 1.05$. In the third case, we use a linear observer by setting $\hat{a} = \hat{c} = 0$. In all cases, we see that the response under output feedback approaches the response under state feedback as ε decreases. When ε is relatively large, we see an advantage for including ϕ_0 in the observer when it is a good model of ϕ. However, if the model is not that good, a linear observer may perform better. The important thing to notice here is that the differences between the three observers diminish as ε decreases. This is expected, because decreasing ε rejects the effect of the uncertainty in modeling ϕ. △

14.5.3 Regulation via Integral Control

Consider the single-input–single-output system

$$\begin{aligned} \dot{x} &= f(x, w) + g(x, w)[u + \delta(x, u, w)] \\ y &= h(x, w) \end{aligned}$$

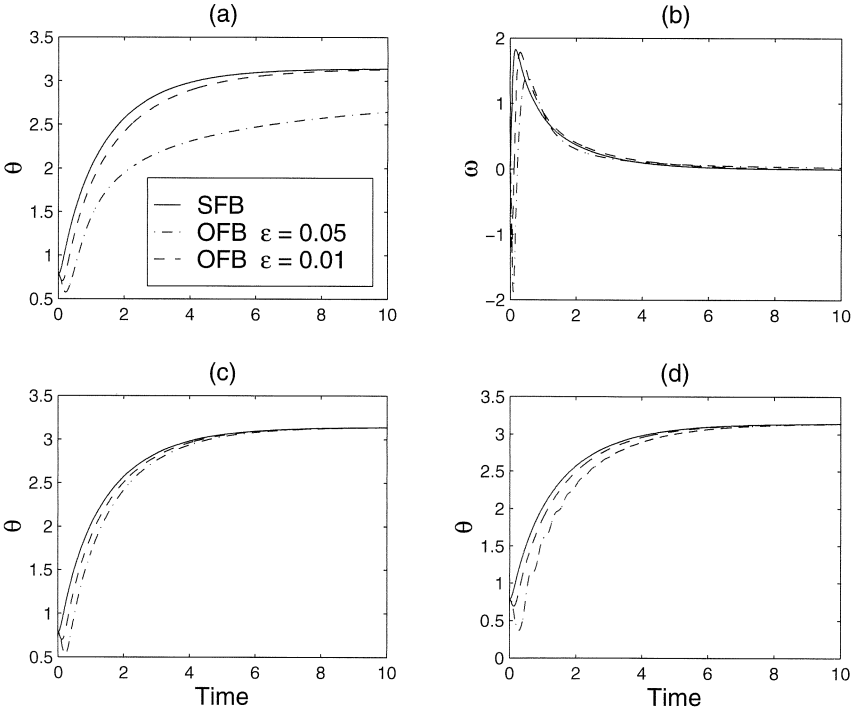

Figure 14.22: Comparison of state feedback (SFB) and output feedback (OFB) for Example 14.19. Figures (a) and (b) show θ and $\omega = \dot{\theta}$ for a nonlinear high-gain observer with nominal m and ℓ. Figure (c) shows θ for a nonlinear high-gain observer with actual m and ℓ. Figure (d) shows θ for a linear high-gain observer.

where $x \in R^n$ is the state, $u \in R$ is the control input, $y \in R$ is the controlled as well as measured output, and $w \in R^l$ is a vector of unknown constant parameters and disturbances. The functions f, g, h, and δ are sufficiently smooth in (x, u) and continuous in w for $x \in D \subset R^n$, $u \in R$, and $w \in D_w \subset R^l$, where D and D_w are open connected sets. We assume that the system

$$\begin{aligned} \dot{x} &= f(x, w) + g(x, w)u \\ y &= h(x, w) \end{aligned}$$

has relative degree ρ in D uniformly in w; that is,

$$L_g h(x, w) = \cdots = L_g L_f^{\rho-2} h(x, w) = 0, \qquad L_g L_f^{\rho-1} h(x, w) \geq a > 0$$

for all $(x, w) \in D \times D_w$. Our goal is to design an output feedback controller such that the output y asymptotically tracks a constant reference $r \in D_r \subset R$, where D_r

is an open connected set.

This is the output feedback version of the problem we studied in Section 14.1.4. The only difference is that we allow f and h to depend on w, while in Section 14.1.4, they were restricted to be independent of w. The restriction was needed there because the variables $h, L_f h, \ldots, L_f^{\rho-1} h$ were used to compute the state feedback control. In the output feedback case, these variables are calculated from the measured signal y by using a high-gain observer. Because we allow f to depend on w, the term δ_1 has been absorbed in f. We are not going to repeat the assumptions and derivations of Section 14.1.4. Let us only recall the sliding mode state feedback controller (14.29), namely,

$$\begin{aligned}
\dot{e}_0 &= e_1 \\
u &= -k \text{ sat}\left(\frac{k_0 e_0 + k_1 e_1 + k_2 e_2 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right)
\end{aligned}$$

where $e_1 = y - r$ is the regulation error and e_2 to e_ρ are the derivatives of e_1.[30] The control is globally bounded and the signal e_1 is available on-line. To implement this controller using output feedback, we estimate e_2 to e_ρ by utilizing a linear high-gain observer. Thus, the output feedback controller can be taken as

$$\begin{aligned}
\dot{e}_0 &= e_1 \\
u &= -k \text{ sat}\left(\frac{k_0 e_0 + k_1 e_1 + k_2 \hat{e}_2 + \cdots + k_{\rho-1} \hat{e}_{\rho-1} + e_\rho}{\mu}\right) \\
\dot{\hat{e}}_i &= \hat{e}_{i+1} + \left(\frac{\alpha_i}{\varepsilon^i}\right)(e_1 - \hat{e}_1), \quad 1 \le i \le \rho - 1 \\
\dot{\hat{e}}_\rho &= \left(\frac{\alpha_\rho}{\varepsilon^\rho}\right)(e_1 - \hat{e}_1)
\end{aligned}$$

where the positive constants α_1 to α_ρ are chosen such the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0$$

have negative real parts. For relative-degree-one systems ($\rho = 1$), the high-gain observer is not used. Under the assumptions of Section 14.1.4, the closed-loop system under state feedback has an exponentially stable equilibrium point at $(z, e_0, e) = (0, \bar{e}_0, 0)$. We leave it to the reader (Exercise 14.50) to verify that, for sufficiently small ε, the output feedback controller recovers the performance of the state feedback controller.

14.6 Exercises

14.1 Consider the system

$$\dot{x}_1 = x_2 + \sin x_1, \qquad \dot{x}_2 = \theta_1 x_1^2 + (1 + \theta_2) u, \qquad y = x_1$$

---

[30] We have written the sliding mode boundary-layer parameter as μ to reserve ε for the high-gain observer parameter.

where $|\theta_1| \leq 2$ and $|\theta_2| \leq 1/2$. Using sliding mode control,

(a) design a continuous state feedback controller to stabilize the origin.

(b) design a continuous state feedback controller so that the output $y(t)$ asymptotically tracks a reference signal $r(t)$. Assume that r, $\dot{r}$, and $\ddot{r}$ are continuous and bounded.

14.2 A simplified model of an underwater vehicle in yaw is given by [60]

$$\ddot{\psi} + a\dot{\psi}|\dot{\psi}| = \tau$$

where ψ is the heading angle, τ is a normalized torque input, and a is a positive parameter. It is desired that ψ tracks a desired trajectory $\psi_r(t)$, where $\psi_r(t)$, $\dot{\psi}_r(t)$, and $\ddot{\psi}_r(t)$ are bounded functions of t. Let $\hat{a} = 1$ be the nominal value of a.

(a) Using $x_1 = \psi$ and $x_2 = \dot{\psi}$ as state variables, $u = \tau$ as control input and $y = \psi$ as output, find the state model.

(b) Show that the system is input–output linearizable.

(c) Assuming that $a = \hat{a} = 1$, use feedback linearization to design a state feedback controller that achieves global asymptotic tracking.

(d) Assuming that $|\hat{a} - a| \leq 0.01$ and $\psi_r(t) = \sin 2t$, show that the controller designed in part (c) will achieve asymptotic tracking with tolerance $|\psi(t) - \psi_r(t)| \leq \delta_1$ and estimate δ_1. Is this tolerance achievable for all initial states?

(e) Assuming that $|\hat{a} - a| \leq k$, where k is known, design a state feedback controller to achieve global asymptotic tracking with tolerance $|\psi(t) - \psi_r(t)| \leq 0.01$.

14.3 (**[176]**) Consider the controlled van der Pol equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega^2 x_1 + \varepsilon\omega(1 - \mu^2 x_1^2) x_2 u$$

where ω, ε, and μ are positive constants and u is the control input.

(a) Show that for $u = 1$, there is a stable limit cycle outside the surface $x_1^2 + x_2^2/\omega^2 = 1/\mu^2$ and for $u = -1$, there is an unstable limit cycle outside the same surface.

(b) Let $s = x_1^2 + x_2^2/\omega^2 - r^2$, with $r < 1/\mu$. Show that restricting the motion of the system to the surface $s = 0$ (that is, $s(t) \equiv 0$) results in a harmonic oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega^2 x_1$$

which produces a sinusoidal oscillation of frequency ω and amplitude r.

(c) Design a state feedback sliding mode controller to drive all trajectories in the band $|x_1| < 1/\mu$ to the manifold $s = 0$ and have them slide on that manifold.

(d) Simulate the response of this system for the ideal sliding mode controller and for a continuous approximation. Use $\omega = \mu = \varepsilon = 1$.

14.4 Consider the system

$$\dot{x}_1 = x_2 + ax_1 \sin x_1, \qquad \dot{x}_2 = bx_1x_2 + u$$

where a and b are unknown constants, but we know the bounds $|a-1| \le 1$ and $|b-1| \le 2$. Using sliding mode control, design a continuous globally stabilizing state feedback controller.

14.5 The equation of motion for a pendulum whose suspension point is subjected to a time-varying, bounded, horizontal acceleration is given by

$$m\ell\ddot{\theta} + mg\sin\theta + k\ell\dot{\theta} = T/\ell + mh(t)\cos\theta$$

where h is the horizontal acceleration, T is the torque input, and the other variables are defined in Section 1.2.1. Assume that

$$0.9 \le \ell \le 1.1, \qquad 0.5 \le m \le 1.5, \qquad 0 \le k \le 0.2, \qquad |h(t)| \le 1$$

and $g = 9.81$. It is desired to stabilize the pendulum at $\theta = 0$ for arbitrary initial conditions $\theta(0)$ and $\dot{\theta}(0)$. Design a continuous sliding mode state feedback controller to achieve ultimate boundedness with $|\theta| \le 0.01$ and $|\dot{\theta}| \le 0.01$.

14.6 **([108])** Consider the system

$$\dot{x}_1 = x_1x_2, \qquad \dot{x}_2 = x_1 + u$$

(a) Using sliding mode control, design a continuous globally stabilizing state feedback controller.

(b) Can you globally stabilize the origin via feedback linearization?

14.7 Consider the system

$$\dot{x}_1 = -x_1 + \tanh(x_2), \qquad \dot{x}_2 = x_2 + x_3, \qquad \dot{x}_3 = u + \delta(x)$$

where $\delta(x)$ is an uncertain function that satisfies $|\delta(x)| \le \varrho(x)$ for all x, for some known function ϱ. Design a continuous sliding mode state feedback controller such that, for all $\|x(0)\|_\infty \le k$, $x(t)$ is bounded and $|x_1(t)|$ is ultimately bounded by 0.01.

14.8 The tank system of Example 12.5 (with integrator included) is given by

$$\dot{y} = \frac{1}{A(y)}(u - c\sqrt{y}), \qquad \dot{\sigma} = y - r$$

where r is a desired set point. Let $\hat{c}$ and $\hat{A}(y)$ be nominal models of c and $A(y)$, respectively, and suppose that you know positive constants $\varrho_1 > 0$, $\varrho_2 > 0$, $\varrho_3 \geq 0$, and $0 \leq \varrho_4 < 1$ such that

$$\varrho_1 \leq A(y) \leq \varrho_2, \quad |\hat{c} - c| \leq \varrho_3, \quad \text{and} \quad \left|\frac{A(y) - \hat{A}(y)}{A(y)}\right| \leq \varrho_4$$

Using sliding mode control, design a continuous state feedback controller such that all state variables are bounded and $|y(t) - r|$ converges to zero as $t \to \infty$.

14.9 Consider the system (14.1).

(a) Let B be a constant matrix of rank p. Show that there is an $n \times n$ nonsingular matrix M such that $MB = [0, \ I_p]^T$, where I_p is the $p \times p$ identity matrix. Verify that $T(x) = Mx$ satisfies (14.2).

(b) Let $B(x)$ be a smooth function of x and assume that B has rank p for all x in a domain $D \subset R^n$. Let $\Delta = \text{span}\{b_1, \ldots, b_p\}$, where $b_1, \ldots, b_p$ are the columns of B. Suppose Δ is involutive. Show that for every $x_0 \in D$, there exist smooth functions $\phi_1(x), \ldots, \phi_{n-p}(x)$, with linearly independent differentials $\partial\phi_1/\partial x, \ldots, \partial\phi_{n-p}/\partial x$ at x_0, such that $[\partial\phi_i/\partial x]B(x) = 0$ for $1 \leq i \leq n-p$. Show that we can find smooth functions $\phi_{n-p+1}(x), \ldots, \phi_n(x)$ such that $T(x) = [\phi_1(x), \ldots, \phi_n(x)]^T$ is a diffeomorphism in a neighborhood of x_0 and satisfies (14.2).
Hint: Apply Frobenius theorem.

14.10 Consider the nonautonomous regular form

$$\begin{aligned} \dot{\eta} &= f_a(t, \eta, \xi) + \delta_a(t, \eta, \xi) \\ \dot{\xi} &= f_b(t, \eta, \xi) + G(t, x)E(t, x)u + \delta(t, x, u) \end{aligned}$$

where, for all $(t, x) \in [0, \infty) \times D$, E is a known nonsingular matrix with a bounded inverse and G is a positive diagonal matrix whose elements are bounded away from zero. Suppose there is a continuously differentiable function $\phi(t, \eta)$, with $\phi(t, 0) = 0$, such that the origin of $\dot{\eta} = f_a(t, \eta, \phi(t, \eta)) + \delta_\eta(t, \eta, \phi(t, \eta))$ is uniformly asymptotically stable. Let

$$u = E^{-1}\left[-L\left(f_b - \frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial\eta}f_a\right) + v\right]$$

where $L = \hat{G}^{-1}$ or $L = 0$ and $\hat{G}(t, x)$ is a nominal model of $G(t, x)$. Let

$$\Delta = (I - GL)\left(f_b - \frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial\eta}f_a\right) + \delta - \frac{\partial\phi}{\partial\eta}\delta_a$$

Suppose Δ_i satisfies the inequality (14.10) with $\varrho = \varrho(t, x)$. Taking $s = \xi - \phi(t, \eta)$, design a sliding mode controller to stabilize the origin. State and prove a theorem similar to Theorem 14.1.

14.11 Suppose (14.13) is replaced by

$$v_i = -\,\beta(x)\,\sigma\left(\frac{s_i}{\varepsilon}\right)$$

where $\sigma : R \rightarrow R$ is a continuously differentiable, odd, monotonically increasing function with the properties

$$\sigma(0) = 0, \qquad \lim_{y\rightarrow\infty} \sigma(y) = 1, \qquad \text{and} \qquad y\sigma(y) \geq \sigma(1)y^2, \ \forall\ |y| \leq 1$$

(a) Verify that $\sigma(y) = \tanh(y)$, $\sigma(y) = (2/\pi)\tan^{-1}(\pi y/2)$, and $\sigma(y) = y/(1 + |y|)$ satisfy the foregoing properties.

(b) Show that if (14.10) is satisfied with $\kappa_0 < \sigma(1)$ and β is chosen to satisfy

$$\beta(x) \geq \frac{\varrho(x)}{\sigma(1) - \kappa_0} + \beta_0, \quad \beta_0 > 0$$

then

$$s_i\dot{s}_i \leq \ -g_0\beta_0[\sigma(1) - \kappa_0]|s_i|, \qquad \text{for } |s_i| \geq \varepsilon$$

(c) Prove Theorems 14.1 and 14.2 for this sliding mode control.

14.12 Replace inequality (14.10) by

$$\frac{\Delta_i}{g_i} \leq \varrho_i(x) + \sum_{j=1}^{p} \kappa_{ij}|v_j|, \qquad \forall\ 1 \leq i \leq p$$

Let $\mathcal{K}$ be the $p\times p$ matrix whose elements are κ_{ij} and suppose $I - K$ is an M-matrix.[31] Recall from the properties of M-matrices that the following three conditions are equivalent:[32]

(i) $I - \mathcal{K}$ is an M-matrix.

(ii) $I - \mathcal{K}$ is nonsingular and all elements of $(I - \mathcal{K})^{-1}$ are nonnegative.

(iii) There is a vector w whose elements are all positive such that the elements of $b = (I - \mathcal{K})w$ are all positive.

Let $\bar{\varrho}_i(x) \geq \varrho_i(x)$ for all $1 \leq i \leq p$ and $\sigma(x) = (I - \mathcal{K})^{-1}[\bar{\varrho}_1(x), \ldots, \bar{\varrho}_p(x)]^T$.

(a) Show that $v_i = -\beta_i(x)\ \mathrm{sgn}(s_i)$ with $\beta_i(x) = \sigma_i(x) + w_i$ yields $s_i\dot{s}_i \leq -b_i g_0|s_i|$ for $1 \leq i \leq p$.

(b) Suppose $\sum_{j=1}^{p} \kappa_{ij} \leq \kappa_0 < 1$ and $\varrho_i(x) = \varrho(x)$ for $1 \leq i \leq p$. Show that $I - \mathcal{K}$ is an M-matrix and $\bar{\varrho}_i(x)$ and w can be chosen to produce the control (14.11).

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[31] See Lemma 9.7 for the definition of M-matrices.

[32] See [57].

14.13 With reference to the "continuous" sliding mode controller of Section 14.1.3, show that there exists a finite time T, possibly dependent on ε and the initial states, and a positive constant k, independent of ε and the initial states, such that $|y(t) - r(t)| \le k\varepsilon$ for all $t \ge T$.

14.14 Repeat Exercise 14.5 using Lyapunov redesign.

14.15 Repeat Exercise 14.8 using Lyapunov redesign.

14.16 Use the Numerical data of the pendulum equation in Section 14.1.1 to simulate the Lyapunov redesign of Example 14.5. Choose the design parameters of the Lyapunov redesign to obtain the same control level as in the sliding mode design. Compare the performance of the two controllers.

14.17 For each of the scalar systems that follow, use the nonlinear damping tool to design a state feedback controller that guarantees boundedness of the state $x(t)$ as well as uniform ultimate boundedness by an ultimate bound μ. The function $\delta(t)$ is bounded for all $t \ge 0$, but we do not know an upper bound on $|\delta(t)|$.

$$\textbf{(a)}\ \ \dot{x} = -x + x^2[u + \delta(t)], \qquad \textbf{(b)}\ \ \dot{x} = x^2[1 + \delta(t)] - xu$$

14.18 Consider a single-input–single-output system in the normal form (13.16)–(13.18) and suppose (13.16) is input-to-state stable. Let $\hat{\alpha}(x)$ and $\hat{\gamma}(x)$ be nominal models of the functions $\alpha(x)$ and $\gamma(x)$, respectively, and suppose the modeling errors satisfy the inequalities

$$|\gamma(x)[\hat{\alpha}(x) - \alpha(x)]| \le \rho_0(x), \qquad \left| \frac{\gamma(x) - \hat{\gamma}(x)}{\hat{\gamma}(x)} \right| \le k < 1$$

where the function $\rho_0(x)$ and the constant k are known. Let $r(t)$ be a reference signal and suppose r and its derivatives up to $r^{(\rho)}$ are continuous and bounded. Using Lyapunov redesign, design a continuous state feedback controller such that the output y asymptotically tracks r with prespecified tolerance μ; that is, $|y(t)-r(t)| \le \mu$ for all $t \ge T$ for some finite time T.

14.19 Repeat the previous exercise for the case of constant reference, using integral control. Verify that the regulation error converges to zero.

14.20 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u + \delta(x)$$

where δ is unknown, but we know an estimate ρ_1 such that $|\delta(x)| \le \rho_1 \|x\|_2$. Let $u = \psi(x) = -x_1 - x_2$ be the nominal stabilizing control, and

$$v = \begin{cases} -\rho_1 \|x\|_2 (w/\|w\|_2), & \text{if } \rho_1 \|x\|_2 \|w\|_2 \ge \varepsilon \\[2ex] -\rho_1^2 \|x\|_2^2 (w/\varepsilon), & \text{if } \rho_1 \|x\|_2 \|w\|_2 < \varepsilon \end{cases}$$

where $w^T = 2x^TPB$ and $V(x) = x^TPx$ is a Lyapunov function for the nominal closed-loop system. Apply the control $u = -x_1 - x_2 + v$.

(a) Verify that all the assumptions of Corollary 14.1 are satisfied, except (14.45), which holds with $\eta_0 = 0$.

(b) Show that when $\delta(x) = 2(x_1 + x_2)$ and $\rho_1 = 2\sqrt{2}$, the origin is unstable.

14.21 Assume that (14.33) through (14.35) are satisfied with $\|\cdot\|_\infty$. Consider the following continuous approximation of the discontinuous control (14.40):

$$v_i = \begin{cases} -\eta(t,x)\,\mathrm{sgn}(w_i), & \text{if } \eta(t,x)|w_i| \geq \varepsilon \\ -\eta^2(t,x)(w_i/\varepsilon), & \text{if } \eta(t,x)|w_i| < \varepsilon \end{cases}$$

for $i = 1, 2, \ldots, p$, where $w^T = [\partial V/\partial x]G(t,x)$.

(a) Show that

$$\dot{V} \leq -\alpha_3(\|x\|_\infty) + \sum_{i \in I} \left[\eta(t,x)|w_i| - \frac{\eta^2(t,x)|w_i|^2}{\varepsilon} \right]$$

where $i \in I$ if $\eta(t,x)|w_i| < \varepsilon$.

(b) State and prove a theorem similar to Theorem 14.3 for the current controller.

14.22 Consider the controller of Exercise 14.21. We want to prove a result similar to Corollary 14.1. Assume that $\alpha_3(\|x\|_\infty) \geq \phi^2(x)$, $\eta(t,x) \geq \eta_0 > 0$, and

$$|\delta_i| \leq \rho_1\phi(x) + \kappa_0|v_i|, \quad 0 \leq \kappa_0 < 1$$

for $i = 1, 2, \ldots, p$. These inequalities imply that (14.35) holds with $\|\cdot\|_\infty$, but they are more restrictive because the upper bound on $|\delta_i|$ depends only on $|v_i|$.

(a) Show that

$$\dot{V} \leq -\phi^2(x) + \sum_{i \in I} \left\{ -(1 - \kappa_0)\eta_0^2 \frac{|w_i|^2}{\varepsilon} + \rho_1\phi(x)|w_i| \right\}$$

(b) State and prove a result similar to Corollary 14.1.

14.23 Suppose the inequality (14.35) takes the form

$$\|\delta(t, x, \psi(t,x) + v)\|_2 \leq \rho_0 + \rho_1\phi(x) + \kappa_0\|v\|_2, \qquad 0 \leq \kappa_0 < 1$$

where $\phi(x) = \sqrt{\alpha_3(\|x\|_2)}$. Let $\eta(x) = \eta_0 + \eta_1\phi(x)$, where $\eta_0 \geq \rho_0/(1 - \kappa_0)$ and $\eta_1 \geq \rho_1/(1 - \kappa_0)$. Consider the feedback control

$$v = \begin{cases} -[\eta_0 + \eta_1\phi(x)](w/\|w\|_2), & \text{if } \|w\|_2 \geq \varepsilon \\ -[\eta_0 + \eta_1\phi(x)](w/\varepsilon), & \text{if } \|w\|_2 < \varepsilon \end{cases}$$

(a) Show that the derivative of V along the trajectories of the closed-loop system (14.38) satisfies

$$\dot{V} \leq -\phi^2(x) + \frac{\varepsilon}{4}[\rho_0 + \rho_1\phi(x)] \leq -\frac{1}{2}\alpha_3(\|x\|_2) + \frac{\varepsilon^2\rho_1^2}{32} + \frac{\varepsilon\rho_0}{4}$$

(b) Apply Theorem 4.18 to arrive at a conclusion similar to Theorem 14.3.

(c) Compare this controller with (14.41).

14.24 Consider the problem treated in Section 14.2 and assume that (14.35) is satisfied with 2-norm. Suppose further that (14.44) and (14.46) are satisfied. Show that the control law

$$u = \psi(t,x) - \gamma w; \quad w^T = \frac{\partial V}{\partial x}G(t,x)$$

with sufficiently large γ will stabilize the origin.

14.25 Consider the problem treated in Section 14.2. Show that, instead of using the control law $u = \psi(t,x) + v$, we can simply use $u = v$, where $\rho(t,x)$ is taken from the inequality

$$\|\delta(t,x,u) - \psi(t,x)\|_2 \leq \rho(t,x) + \kappa_0\|u\|_2, \quad 0 \leq \kappa_0 < 1$$

14.26 Suppose that, in addition to the matched uncertainty δ, the system (14.30) also has unmatched uncertainty Δ; that is,

$$\dot{x} = f(t,x) + \Delta(t,x) + G(t,x)[u + \delta(t,x,u)]$$

Suppose that, over a domain $D \subset R^n$, all the assumptions of Theorem 14.3 are satisfied, the inequalities (14.44) through (14.46) are satisfied, and the unmatched uncertainty satisfies $\|[\partial V/\partial x]\Delta(t,x)\|_2 \leq \mu\phi^2(x)$, for some $\mu \geq 0$. Let $u = \psi(t,x) + v$, where v is determined by (14.41). Show that if $\mu < 1$, then the feedback control law will stabilize the origin of the closed-loop system provided that the ε chosen is small enough to satisfy $\varepsilon < 4(1-\mu)(1-\kappa_0)\eta_0^2/\rho_1^2$.

14.27 Consider the system $\dot{x} = f(x) + G(x)[u + \delta(x,u)]$, and suppose there are known smooth functions $\psi(x)$, $V(x)$, and $\rho(x)$, all vanishing at $x = 0$, and a known constant k such that

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \quad \frac{\partial V}{\partial x}[f(x) + G(x)\psi(x)] \leq -c_3\|x\|^2$$

$$\|\delta(x,\psi(x)+v)\| \leq \rho(x) + \kappa_0\|v\|, \quad 0 \leq \kappa_0 < 1, \quad \forall\, x \in R^n,\ \forall\, v \in R^p$$

where c_1 to c_3 are positive constants.

(a) Show that it is possible to design a continuous state feedback controller $u = \gamma(x)$ such that the origin of

$$\dot{x} = f(x) + G(x)[\gamma(x) + \delta(x, \gamma(x))]$$

is globally exponentially stable.

(b) Apply the result of part (a) to the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = (1 + a_1)(x_1^3 + x_2^3) + (1 + a_2)u$$

where a_1 and a_2 are unknown constants that satisfy $|a_1| \leq 1$ and $|a_2| \leq 1/2$.

14.28 Repeat exercise 14.1 using backstepping.

14.29 Repeat exercise 14.5 using backstepping.

14.30 Repeat exercise 14.6 using backstepping.

14.31 Using backstepping, design a state feedback controller to globally stabilize the system

$$\dot{x}_1 = x_2 + a + (x_1 - a^{1/3})^3, \qquad \dot{x}_2 = x_1 + u$$

where a is a known constant.

14.32 (**[108]**) Consider the system

$$\dot{x}_1 = -x_2 - \tfrac{3}{2}x_1^2 - \tfrac{1}{2}x_1^3, \qquad \dot{x}_2 = u$$

(a) Using backstepping, design a **linear** state feedback controller to globally stabilize the origin.
Hint: Avoid cancellation of nonlinear terms.

(b) Design a globally stabilizing state feedback controller by using feedback linearization.

(c) Compare the two designs. Using computer simulations, compare their performance and the control effort used in each case.

14.33 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_1^3 + u$$

(a) Find a smooth state feedback controller $u = \psi(x)$ such that the origin is globally exponentially stable.

(b) Extend the dynamics of the system by connecting an integrator in series with the input, to obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_1^3 + z, \qquad \dot{z} = v$$

Using backstepping, find a smooth state feedback controller $v = \phi(x, z)$ such that the origin is globally asymptotically stable.

14.34 Consider the system of Exercise 13.17.

(a) Starting with the $\dot{x}_1$-equation and stepping back to the $\dot{x}_2$-equation, design a state feedback controller $u = \psi(x)$ such that the origin $(x_1 = 0,\ x_2 = 0)$ of the first two equations is globally exponentially stable.

(b) Show that, under the feedback controller of part (a), the origin $x = 0$ of the full system is globally asymptotically stable.
Hint: Use input-to-state properties of the third equation.

14.35 Consider the system

$$\dot{x}_1 = x_2 + \theta x_1^2, \qquad \dot{x}_2 = x_3 + u, \qquad \dot{x}_3 = x_1 - x_3, \qquad y = x_1$$

where $\theta \in [0, 2]$. Using backstepping, design a state feedback controller such that $|y - a \sin t|$ is ultimately bounded by μ, where μ is a design parameter that can be chosen arbitrarily small. Assume that $|a| \leq 1$ and $\|x(0)\|_\infty \leq 1$.

14.36 Repeat exercise 14.4 using a combination of backstepping and Lyapunov redesign.

14.37 Consider the system

$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = x_2 + x_3, \qquad \dot{x}_3 = x_1^2 + \delta(x) + u$$

where $\delta(x)$ is an unknown (locally Lipschitz) function of x that satisfies $|\delta(x)| \leq k\|x\|_2$ for all x, with a known constant k. Design a globally stabilizing state feedback controller.

14.38 Consider again Exercise 14.7.

(a) With $\delta = 0$, use backstepping to design a globally stabilizing state feedback controller.

(b) Use the stabilizing controller of part (a) and Lyapunov redesign to design a state feedback controller such that, for all $\|x(0)\|_\infty \leq k$, $x(t)$ is bounded and $|x_1(t)|$ is ultimately bounded by 0.01.

14.39 Consider the magnetic suspension system of Exercise 1.18 and suppose the ball is subject to a vertical disturbance force $d(t)$; that is,

$$m\ddot{y} = -k\dot{y} + mg + F(y, i) + d(t)$$

Suppose further that $|d(t)| \leq d_0$ for all $t \geq 0$, where the upper bound d_0 is known.

(a) Viewing the force F as the control input, use Lyapunov redesign to design a state feedback control $F = \gamma(y, \dot{y})$ such that $|y - r|$ is ultimately bounded by μ, where μ is a design parameter that can be chosen arbitrarily small. Design your control such that γ is a continuously differentiable function of its arguments.

(b) Using backstepping, design a state feedback control law for the voltage input u that will ensure that $|y - r|$ is ultimately bounded by μ.

14.40 Consider the system

$$\dot{x}_1 = -x_1 + x_1^2[x_2 + \delta(t)], \qquad \dot{x}_2 = u$$

where $\delta(t)$ is a bounded function of t for all $t \geq 0$, but we do not know an upper bound on $|\delta(t)|$. By combining backstepping and the nonlinear damping tools, design a state feedback controller that ensures global boundedness of the state x for all initial states $x(0) \in R^2$.

14.41 Repeat Exercise 14.40 for the system

$$\dot{x}_1 = -x_1 x_2 + x_1^2[1 + \delta(t)], \qquad \dot{x}_2 = u$$

14.42 Consider the linear system $\dot{x} = Ax + Bu$ and suppose there exists a positive definite symmetric matrix P such that $PA + A^T P \leq 0$ and the pair $(A, B^T P)$ is observable. Design a globally stabilizing state feedback control law $u = -\psi(x)$ such that $\|\psi(x)\| \leq k$ for all x, where k is a given positive constant.

14.43 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + \psi(u)$$

where ψ is a locally Lipschitz function that satisfies $\psi(0) = 0$ and $u\psi(u) > 0$ for all $u \neq 0$. Design a globally stabilizing state feedback control.

14.44 Consider a relative-degree-one, single-input–single-output system with a globally defined normal form

$$\dot{\eta} = f_0(\eta, y), \qquad \dot{y} = b(\eta, y) + a(\eta, y)u$$

where $f_0(0,0) = 0$ and $a(\eta, y) \geq a_0 > 0$. Suppose there is a (known) radially unbounded Lyapunov function $W(\eta)$, with $[\partial W/\partial \eta](0) = 0$, such that $[\partial W/\partial \eta] f_0(\eta, 0) < 0$ for all $\eta \neq 0$. Design a globally stabilizing state feedback controller.

14.45 Design a passivity-based globally stabilizing state feedback control for the system of Example 13.17.

14.46 Design a globally stabilizing state feedback control law for the system

$$\dot{x}_1 = -(1 + x_3)x_1^3, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_2^2 - 1 + u$$

14.47 Consider a relative-degree-one, single-input–single-output system that is represented, in some domain that contains the origin, by the normal form

$$\dot{\eta} = f_0(\eta, y), \qquad \dot{y} = b(\eta, y) + a(\eta, y)u$$

where f_0, a, and b are sufficiently smooth, $a(\eta, y) \geq a_0 > 0$, $f_0(0,0) = 0$, and $b(0,0) = 0$. Suppose the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable and there is a Lyapunov function $V(\eta)$ such that

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \quad \text{and} \quad \frac{\partial V}{\partial \eta} f_0(\eta, y) \leq -\alpha_3(\|\eta\|), \qquad \forall \ \|\eta\| \geq \gamma(|y|)$$

for some class $\mathcal{K}$ functions α_1, α_2, α_3, and γ. Let $\hat{a}(y)$ and $\hat{b}(y)$ be smooth nominal models of $a(\eta, y)$ and $b(\eta, y)$, respectively, such that $\hat{a}(y) \geq \hat{a}_0 > 0$ and

$$\left| \frac{b(\eta, y)}{a(\eta, y)} - \frac{\hat{b}(y)}{\hat{a}(y)} \right| \leq \varrho(y) \tag{14.116}$$

over the domain of interest, where $\varrho(y)$ is known. The choice $\hat{a} = 1$, $\hat{b} = 0$ is possible.

(a) Show that a continuous stabilizing sliding mode controller can be taken as

$$u = -\frac{\hat{b}(y)}{\hat{a}(y)} - \beta(y) \ \text{sat}\left(\frac{y}{\varepsilon}\right)$$

where $\beta(y) \geq \varrho(y) + \beta_0$, for some positive constants ε and β_0. In particular, show that there are compact positively invariant sets Ω and $\Omega_\varepsilon = \{V(\eta) \leq \alpha(\varepsilon),\ |y| \leq \varepsilon\} \subset \Omega$, for some class $\mathcal{K}$ function α, such that every trajectory starting in Ω enters Ω_ε in finite time.

(b) Show that if the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable, then for sufficiently small ε the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction.

(c) Show that (14.116) can be satisfied on any compact set with a constant ϱ.

(d) Under what conditions will this controller achieve semiglobal stabilization?

(e) Design a stabilizing output feedback controller for the system

$$\dot{x}_1 = -x_1 - x_2^3, \quad \dot{x}_2 = -x_2 - x_3^3 - u, \quad \dot{x}_3 = x_1^2 - x_3 + u, \quad y = x_2$$

14.48 Consider the p-input–p-output system

$$\dot{x} = f(x, u), \quad y = h(x)$$

where f is locally Lipschitz, h is continuously differentiable, $f(0,0) = 0$, and $h(0) = 0$. Suppose the system

$$\dot{x} = f(x, u), \quad \dot{y} = \frac{\partial h}{\partial x} f(x, u) \overset{\text{def}}{=} \tilde{h}(x, u)$$

with output $\dot{y}$, is passive with a radially unbounded, positive definite storage function $V(x)$; that is, $\dot{V} \leq u^T \dot{y}$, and is zero-state observable. Let z_i be the output of the linear transfer function $b_i s/(s + a_i)$ whose input is y_i, where a_i and b_i are positive constants.

(a) Using $V(x) + \sum_{i=1}^{p} (k_i/2b_i) z_i^2$ as a Lyapunov function candidate, show that the output feedback control $u_i = -k_i z_i$, $1 \leq i \leq p$, $k_i > 0$, globally stabilizes the origin.

(b) Using $V(x) + \sum_{i=1}^{p} (1/b_i) \int_0^{z_i} \phi_i(\sigma) \, d\sigma$ as a Lyapunov function candidate, where ϕ_i is a locally Lipschitz function such that $\phi_i(0) = 0$ and $\sigma\phi_i(\sigma) > 0$ for all $\sigma \neq 0$, show that the output feedback control $u_i = -\phi_i(z_i)$, $1 \leq i \leq p$, stabilizes the origin. Under what conditions on ϕ_i will this control law achieve global stabilization?

(c) Apply the result of part (a) to globally stabilize the pendulum equation

$$m\ell\ddot{\theta} + mg \sin\theta = u$$

at the angle $\theta = \delta_1$ by using feedback from θ, but not from $\dot{\theta}$.

14.49 Consider the system (14.85)–(14.86) and suppose $u = \gamma(x)$ is a locally Lipschitz state feedback control that globally stabilizes the origin. Let $\hat{x}$ be the state estimate provided by the observer (14.87). Show that the output feedback control $u = \gamma(\hat{x})$ globally stabilizes the origin $(x = 0, \hat{x} = 0)$ of the closed-loop system if the system

$$\dot{x} = Ax + g(Cx, \gamma(x - v))$$

with input v, is input-to-state stable.

14.50 Verify that the output feedback controller of Section 14.5.3 recovers the performance of the state feedback controller for sufficiently small ε. In particular, show that the closed-loop system under output feedback has an exponentially stable equilibrium point at $(z, e_0, e, \hat{e}) = (0, \bar{e}_0, 0, 0)$.

In the next seven exercises, we present a few case studies.

14.51 A current-controlled induction motor can be modeled by[33]

$$\begin{aligned} J\dot{\omega} &= k_t(\lambda_a i_b - \lambda_b i_a) - T_L \\ \dot{\lambda}_a &= -\frac{R_r}{L_r}\lambda_a - p\omega\lambda_b + \frac{R_r M}{L_r} i_a \\ \dot{\lambda}_b &= -\frac{R_r}{L_r}\lambda_b + p\omega\lambda_a + \frac{R_r M}{L_r} i_b \end{aligned}$$

where ω is the rotor speed, T_L is the load torque, λ_a and λ_b are the components of the rotor flux vector, i_a and i_b are the components of the stator current vector, L_r and M are the rotor and mutual inductances, R_r is the rotor resistance, J is the moment of inertia, p is the number of pole pairs, and k_t is a positive constant. The load torque T_L can be taken as $T_L = T_o + \phi(\omega)$, where $\phi \in [0, \infty]$ is a locally Lipschitz function that models the load due to friction, while T_o models the speed-independent load. The currents i_a and i_b are the control variables. It is desired to design feedback control so that the speed ω tracks a reference speed ω_r in the presence of unknown load torque. In this exercise, we will pursue the design by transforming the equations into field-oriented coordinates. This transformation decouples the control problem into two separate control problems, one for the speed and the other for the rotor flux. The transformation should be applied on-line and requires measurement of the rotor flux. We will start by assuming that the rotor flux is available. Then we will use an observer to estimate the rotor flux and analyze the performance of the controller when the rotor flux is replaced by its estimate. The analysis will take into consideration uncertainty in the rotor resistance R_r, which may change significantly with temperature during the operation of the motor.

(a) Let ρ be the angle of the rotor flux vector and λ_d be its magnitude; that is, $\rho = \tan^{-1}(\lambda_b/\lambda_a)$ and $\lambda_d = \sqrt{\lambda_a^2 + \lambda_b^2}$. Replacing λ_a and λ_b by λ_d and ρ as state variables and transforming i_a and i_b into i_d and i_q, defined by

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\rho & \sin\rho \\ -\sin\rho & \cos\rho \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} \tag{14.117}$$

show that the motor can be represented by the state model

$$J\dot{\omega} = k_t\,\lambda_d i_q - T_L, \qquad \dot{\lambda}_d = -\frac{R_r}{L_r}\lambda_d + \frac{R_r M}{L_r} i_d, \qquad \dot{\rho} = p\omega + \frac{R_r M i_q}{L_r \lambda_d}$$

provided $\lambda_d > 0$.

(b) The first two equations of the foregoing model are independent of ρ. Therefore, to design the feedback control laws for i_d and i_q, we can drop the state equation for ρ. However, we still need ρ to calculate i_a and i_b by using the inverse of the transformation given by (14.117). Show that for constant ω, i_q, and λ_d, ρ will be unbounded. Explain why an unbounded ρ does not cause a problem.

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[33] See, for example, [50, Appendix C] or [117].

(c) Note that the torque generated by the motor is proportional to $\lambda_d i_q$. Therefore, we can control the speed by first regulating λ_d to a desired constant flux λ_r using the control input i_d; then we can design i_q assuming $\lambda_d = \lambda_r$. This assumption will be justified if the dynamics of the flux loop are much faster than the dynamics of the speed loop.[34] In this part of the exercise, we design the flux control. Show that $i_d = (\lambda_r/M) - k(\lambda_d - \lambda_r)$, with $k \geq 0$, achieves the desired regulation.

(d) Since the currents i_a and i_b must be limited to certain maximum values, in our design, we would like to assume that i_d and i_q are limited to I_d and I_q, respectively. Show that if $I_d > \lambda_r/M$ and $0 < \lambda_d(0) < \lambda_r$, then under the saturated control

$$i_d = I_d \ \mathrm{sat}\left(\frac{\lambda_r/M - k(\lambda_d - \lambda_r)}{I_d}\right) \tag{14.118}$$

$\lambda_d(t)$ changes monotonically from $\lambda_d(0)$ to λ_r. Estimate the settling time.

(e) For speed control, let us assume that $\omega_r(t)$, $\dot{\omega}_r(t)$, and $T_o(t)$ are bounded. Design a sliding mode controller of the form $i_q = -I_q \ \mathrm{sat}(s/\varepsilon)$ with appropriately chosen s and ε. Give conditions under which the controller will work and estimate the ultimate bound on the tracking error.

(f) Suppose, in addition to the foregoing assumption, $\omega_r(t)$ satisfies $\lim_{t\to\infty} \omega_r(t) = \bar{\omega}_r$ and $\lim_{t\to\infty} \dot{\omega}_r(t) = 0$ and we want to achieve zero steady-state error when T_o is constant. Using integral control, design a sliding mode controller of the form $i_q = -I_q \ \mathrm{sat}(s/\varepsilon)$ with appropriately chosen s and ε. Give conditions under which the controller will work and verify that it achieves zero steady-state error.

(g) The assumption that the rotor flux can be measured is not a practical one. In applications, it is common to estimate the flux by using the observer

$$\dot{\hat{\lambda}}_a = -\frac{\hat{R}_r}{L_r}\,\hat{\lambda}_a - p\omega\hat{\lambda}_b + \frac{\hat{R}_r M}{L_r}\,i_a, \qquad \dot{\hat{\lambda}}_b = -\frac{\hat{R}_r}{L_r}\,\hat{\lambda}_b + p\omega\hat{\lambda}_a + \frac{\hat{R}_r M}{L_r}\,i_b$$

where $\hat{R}_r$ is a nominal value (or estimate) of R_r. The field orientation angle ρ and the flux magnitude λ_d are calculated by using $\rho = \tan^{-1}(\hat{\lambda}_b/\hat{\lambda}_a)$ and $\lambda_d = \sqrt{\hat{\lambda}_a^2 + \hat{\lambda}_b^2}$; i_d and i_q are defined by using (14.117) with the newly defined ρ. To write down a state model for the overall system, we define the flux estimation errors e_d and e_q by

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} \cos\rho & \sin\rho \\ -\sin\rho & \cos\rho \end{bmatrix} \begin{bmatrix} \hat{\lambda}_a - \lambda_a \\ \hat{\lambda}_b - \lambda_b \end{bmatrix} \tag{14.119}$$

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[34] This can be justified by the singular perturbation theory of Chapter 11.

Using ω, λ_d, ρ, e_d, and e_q as state variables, show that the overall system can be represented by the state model

$$\begin{aligned}
J\dot{\omega} &= k_t(\lambda_d i_q + e_q i_d - e_d i_q) - T_L \\
\dot{\lambda}_d &= -\frac{\hat{R}_r}{L_r}\lambda_d + \frac{\hat{R}_r M}{L_r} i_d \\
\dot{\rho} &= p\omega + \frac{\hat{R}_r M i_q}{L_r \lambda_d} \\
\dot{e}_d &= -\frac{R_r}{L_r} e_d + \frac{\hat{R}_r M i_q}{L_r \lambda_d} e_q + \left(\frac{\hat{R}_r - R_r}{L_r}\right)(M i_d - \lambda_d) \\
\dot{e}_q &= -\frac{\hat{R}_r M i_q}{L_r \lambda_d} e_d - \frac{R_r}{L_r} e_q + \left(\frac{\hat{R}_r - R_r}{L_r}\right) M i_q
\end{aligned}$$

(h) Verify that the flux control (14.118) still regulates λ_d to λ_r, provided $I_d > \lambda_r/M$ and $0 < \lambda_d(0) < \lambda_r$. Furthermore, show that

$$|M i_d(t) - \lambda_d(t)| \le (1 + kM)[M I_d - \lambda_d(0)] \exp[-(\hat{R}_r/\hat{L}_r)t]$$

(i) Using $V = (1/2)(e_d^2 + e_q^2)$ and the comparison lemma, show that $\|e\| = \sqrt{e_d^2 + e_q^2}$ satisfies the bound

$$\|e(t)\| \le k_1 e^{-\gamma t} + k_2 \left|\frac{\hat{R}_r - R_r}{R_r}\right| M I_q$$

for some positive constants γ, k_1, and k_2.

(j) Find conditions under which the sliding mode controller with integral action, designed in part (f), will still achieve zero steady-state error.

14.52 The nonlinear dynamic equations of an m-link robot are given by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u \tag{14.120}$$

where all variables are defined in Exercise 1.4. We assume that M, C, and g are continuous functions of their arguments and

$$0 < \lambda_m y^T y \le y^T M(q) y \le \lambda_M y^T y, \quad \forall\ q, y \in R^m,\ y \neq 0$$

for some positive constants λ_m and λ_M. We want to design a state feedback control law such that $q(t)$ asymptotically tracks a reference trajectory $q_r(t)$, where $q_r(t)$, $\dot{q}_r(t)$, and $\ddot{q}_r(t)$ are continuous and bounded. In this exercise, we design a sliding mode controller. Take the sliding surface as $s = \Lambda e + \dot{e} = 0$, where Λ is a positive diagonal matrix, and let

$$u = \hat{M}(q)v + L[\hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) + \hat{M}(q)\ddot{q}_r - \hat{M}(q)\Lambda\dot{e}]$$

where $\hat{M}$, $\hat{C}$, and $\hat{g}$ are nominal models of M, C, and g, respectively, and L is either zero or identity, leading to two different control laws.

(a) Show that s satisfies an equation of the form

$$\dot{s} = v + \Delta(q, \dot{q}, q_r, \dot{q}_r, \ddot{q}_r, v)$$

and give expressions for Δ when $L = 0$ and $L = I$.

(b) Assuming that

$$\|M^{-1}(q)\hat{M}(q) - I\|_\infty \le \kappa_0 < 1, \quad \forall\ q \in R^m \tag{14.121}$$

show that Δ_i satisfies the inequality

$$|\Delta_i| \le \rho(\cdot) + \kappa_0\|v\|_\infty, \quad \text{for } 1 \le i \le m$$

where ρ may depend on $(q, \dot{q}, q_r, \dot{q}_r, \ddot{q}_r)$.

(c) Let

$$v_i = -\beta(\cdot)\ \mathrm{sat}(s_i/\varepsilon), \quad \varepsilon > 0, \text{ for } 1 \le i \le m$$

where β may depend on $(q, \dot{q}, q_r, \dot{q}_r, \ddot{q}_r)$. Show how to choose β to ensure that the error e is globally uniformly ultimately bounded and give an estimate of the ultimate bound in terms of ε.

(d) What properties can you prove for the sliding mode controller when β is taken as a constant?

14.53 Consider the m-link robot of the previous exercise. In the current exercise, we derive a different sliding mode controller [180] that uses the skew symmetric property of $(\dot{M} - 2C)$ and avoids the condition (14.121).

(a) Taking s as in the previous exercise and $W = (1/2)s^TM(q)s$ as a Lyapunov function candidate for the $\dot{s}$-equation, show that

$$\dot{W} = s^T[M\Lambda\dot{e} + C(\Lambda e - \dot{q}_r) - D\dot{q} - g - M\ddot{q}_r + u]$$

(b) Let

$$u = v + L[-\hat{M}(q)\Lambda\dot{e} - \hat{C}(q,\dot{q})(\Lambda e - \dot{q}_r) + \hat{g}(q) + \hat{M}(q)\ddot{q}_r]$$

where $L = 0$ or $L = I$, leading to two different control laws. Show that

$$\dot{W} = s^T[v + \Delta(q, \dot{q}, q_r, \dot{q}_r, \ddot{q}_r)]$$

and give expressions for Δ when $L = 0$ and $L = I$.

(c) Let

$$v = -\beta(\cdot)\varphi(s/\varepsilon), \quad \varepsilon > 0, \quad \varphi(y) = \begin{cases} y/\|y\|_2, & \text{for } \|y\|_2 \geq 1 \\ y, & \text{for } \|y\|_2 < 1 \end{cases}$$

where β may depend on $(q, \dot{q}, q_r, \dot{q}_r, \ddot{q}_r)$. Show how to choose β to ensure that the error e is globally uniformly ultimately bounded and give an estimate of the ultimate bound in terms of ε.

(d) What properties can you prove for the sliding mode controller when β is taken as a constant?

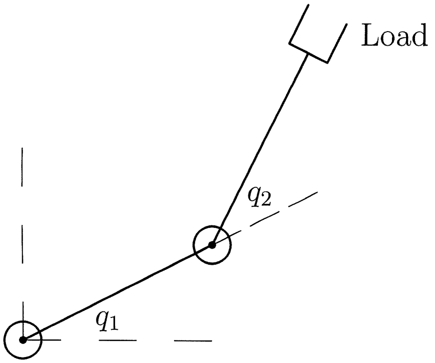

Figure 14.23: Two-link robot.

14.54 The two-link robot, shown in Figure 14.23, can be modeled [171] by equation (14.120) with

$$M = \begin{bmatrix} a_1 + 2a_4 \cos q_2 & a_2 + a_4 \cos q_2 \\ a_2 + a_4 \cos q_2 & a_3 \end{bmatrix}, \quad C = a_4 \sin q_2 \begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} b_1 \cos q_1 + b_2 \cos(q_1 + q_2) \\ b_2 \cos(q_1 + q_2) \end{bmatrix}$$

where a_1 through a_4, b_1, and b_2 are positive constants which depend on masses, moments of inertial, and lengths of the two links, as well as the acceleration due to gravity. We neglect damping and take $D = 0$. Let the nominal values of the parameters be

$$a_1 = 200.01, \quad a_2 = 23.5, \quad a_3 = 122.5, \quad a_4 = 25, \quad b_1 = 784.8, \quad b2 = 245.25$$

To move the arm from the initial position ($q_1 = 0, q_2 = 0$) to the terminal position ($q_1 = \pi/2, q_2 = \pi/2$), we take the reference trajectory as

$$q_{r1}(t) = q_{r2}(t) = (\pi/2)[1 - \exp(-5t)(1 + 5t)]$$

We assume that the control inputs are constrained to $|u_1| \leq 6000$ Nm and $|u_2| \leq 5000$ Nm. In the previous two exercises, we designed four different sliding mode control laws, given by

$$\begin{aligned} u &= -\hat{M}\beta \,\text{sat}(s/\varepsilon) \\ u &= -\hat{M}\beta \,\text{sat}(s/\varepsilon) + \hat{C}\dot{q} + \hat{g} + \hat{M}\ddot{q}_r - \hat{M}\Lambda\dot{e} \\ u &= -\beta \,\varphi(s/\varepsilon) \\ u &= -\beta \,\varphi(s/\varepsilon) + \hat{C}(\dot{q}_r - \Lambda e) + \hat{g} + \hat{M}\ddot{q}_r - \hat{M}\Lambda\dot{e} \end{aligned}$$

For convenience, we take β to be constant.

(a) Using simulation, choose the design parameters Λ, β, and ε for each of the four controllers. Include limiters in your simulation to impose the control constraints.

(b) Compare the performance of the four controllers when, due to an unknown load, the actual system parameters are perturbed to

$$a_1 = 259.7,\ a_2 = 58.19,\ a_3 = 157.19,\ a_4 = 56.25,\ b_1 = 1030.1,\ b2 = 551.8125$$

(c) Suppose we only measure the angles q_1 and q_2. Design a high-gain observer to implement the state feedback controllers. Using simulation, compare the performance of the output and state feedback controllers, for any one of the four control laws.

14.55 Reconsider the two-link robot of the previous exercise.

(a) Following Example 14.16, design a passivity-based controller to regulate the angles (q_1, q_2) from $(0, 0)$ to $(\pi/2, \pi/2)$. Use simulation to choose the design parameters K_p and K_d and compare with the sliding mode controllers of the previous exercise.

(b) Suppose we only measure the angles q_1 and q_2. Design a high-gain observer to implement the state feedback controller. Using simulation, compare the performance of the output and state feedback controllers.

14.56 Consider the TORA system of Exercise 1.16. In this exercise, we design a passivity-based control law [146] to globally stabilize the origin.

(a) Using the sum of the potential energy $(1/2)kx_c^2$ and kinetic energy $(1/2)v^T D(\theta)v$, where $v = [\dot{\theta},\ \dot{x}_c]^T$, as the storage function, show that the system with input u and output $\dot{\theta}$ is passive. Is it zero-state observable?

(b) Let $u = -\phi_1(\theta) + w$ where ϕ_1 is locally Lipschitz, $\phi_1(0) = 0$, $y\phi_1(y) > 0$ for all $y \neq 0$, and $\lim_{|y|\to\infty} \int_0^y \phi_1(\lambda)\, d\lambda = \infty$. Using

$$V = \frac{1}{2}v^T D(\theta)v + \frac{1}{2}kx_c^2 + \int_0^\theta \phi_1(\lambda)\, d\lambda, \qquad v = \begin{bmatrix} \dot{\theta} \\ \dot{x}_c \end{bmatrix}$$

as the storage function, show that the system with input w and output $\dot{\theta}$ is passive. Show also that it is zero-state observable.

(c) Let ϕ_2 be any locally Lipschitz function such that $\phi_2(0) = 0$ and $y\phi_2(y) > 0$ for all $y \neq 0$. Show that $u = -\phi_1(\theta) - \phi_2(\dot{\theta})$ globally stabilizes the origin.

(d) Let $M = 1.3608$ Kg, $m = 0.096$ Kg, $L = 0.0592$ m, $I = 0.0002175$ Kg/m^2, and $k = 186.3$ N/m. Verify that

$$u = -U_p \text{ sat}(K_p\theta) - U_v \text{ sat}(K_v\dot{\theta})$$

will be globally stabilizing for any positive constants U_p, U_v, K_p, and K_v. Choose U_p and U_v such that $U_p + U_v \leq 0.1$ to guarantee that u satisfies the constraint $|u| \leq 0.1$. We want to design these four positive constants to reduce the settling time. We will use simulation and local analysis to choose the constants. By linearizing the closed-loop system about the origin, show that the closed-loop characteristic equation is given by

$$1 + \beta_0 \frac{(s^2 + \beta_2)(s + \beta_1)}{s^2(s^2 + \beta_3)} = 0$$

where

$$\beta_0 = \frac{U_v K_v (m + M)}{\Delta_0}, \quad \beta_1 = \frac{U_p K_p}{U_v K_v}, \quad \beta_2 = \frac{k}{m + M}, \quad \beta_3 = \frac{k(I + mL^2)}{\Delta_0}$$

and $\Delta_0 = \Delta(0)$. Verify that the characteristic polynomial is Hurwitz and construct the root locus as β_0 varies from zero to infinity.

(e) By simulating the closed-loop system with the initial states $\theta(0) = \pi$, $\dot{\theta}(0) = 0$, $x_c(0) = 0.025$, and $\dot{x}_c(0) = 0$, and by using the root locus analysis of part (d), choose the constants U_p, U_v, K_p, and K_v to make the settling time as small as you can. You should be able to achieve a settling time of about 30 sec.

(f) Suppose now that we can only measure θ. Using Exercise 14.48, show that the origin is globally stabilized by the output feedback controller

$$u = -U_p \text{ sat}(K_p\theta) - U_v \text{ sat}(K_v z)$$

where z is the output of the transfer function $s/(\varepsilon s+1)$ driven by θ and ε is any positive constant. By viewing this transfer function as the transfer function of a reduced-order high-gain observer, use the analysis of Section 14.5 to show that the output feedback controller recovers the performance of the state feedback controller as ε tends to zero. Simulate the closed-loop system for different values of ε and compare with the performance under state feedback.

14.57 Consider the TORA system of Exercise 1.16. In this exercise, we design a sliding mode controller[35] to stabilize the origin.

(a) Show that the change of variables

$$\eta_1 = x_c + \frac{mL\sin\theta}{m+M}, \quad \eta_2 = \dot{x}_c + \frac{mL\dot{\theta}\cos\theta}{m+M}, \quad \eta_3 = \theta, \quad \xi = \dot{\theta}$$

transforms the system into the regular form (14.4)–(14.5).

(b) Using

$$V_0(\eta) = \frac{(m+M)k_1}{2mL}\left(\eta_1 - \frac{mL}{m+M}\sin\eta_3\right)^2 + \frac{(m+M)^2 k_1}{2kmL}\eta_2^2 + \frac{k_2}{2}\eta_3^2$$

with positive constants k_1 and k_2, show that

$$\xi = \phi(\eta) \overset{\text{def}}{=} k_1\left(\eta_1 - \frac{mL}{m+M}\sin\eta_3\right)\cos\eta_3 - k_2\eta_3$$

globally stabilizes the origin of

$$\dot{\eta} = \eta_2, \quad \dot{\eta}_2 = -\frac{k}{m+M}\left(\eta_1 - \frac{mL}{m+M}\sin\eta_3\right), \quad \dot{\eta}_3 = \xi$$

Verify that the sliding surface can be taken as

$$s = \dot{\theta} + k_2\theta - k_1 x_c\cos\theta = 0$$

and note that s is independent of the system parameters.

(c) Choose $\beta(x)$ such that $u = -\beta(x)\ \text{sat}(s/\mu)$ globally stabilizes the origin for sufficiently small μ.

(d) The expression of $\beta(x)$ in the previous part could be complicated. To simplify the control law, we take β as a positive constant and write the control law as

$$u = -\beta\ \text{sat}\left(\frac{\dot{\theta} + k_2\theta - k_1 x_c\cos\theta}{\mu}\right)$$

where the positive constants k_1, k_2, β and μ are the design parameters. Show that, for sufficiently small μ, this control law stabilizes the origin and guarantees that the region of attraction includes a compact set around the origin.

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[35] The design uses ideas from the passivity-based design of [172].

(e) Let the system parameters be as in part (d) of exercise 14.56. We will use simulation and local analysis to choose the design parameters to reduce the settling time. By linearizing the closed-loop system about the origin, show that the closed-loop characteristic equation is given by

$$1 + \gamma_0 \frac{s^3 + \gamma_1 s^2 + \gamma_2 s + \gamma_3}{s^2(s^2 + \gamma_4)} = 0$$

where

$$\gamma_0 = \frac{\beta(m+M)}{\mu\Delta_0}, \quad \gamma_1 = k_2 + \frac{mLk_1}{m+M}, \quad \gamma_2 = \frac{k}{m+M}$$

$$\gamma_3 = \frac{kk_2}{m+M}, \quad \gamma_4 = \frac{k(I+mL^2)}{\Delta_0}, \quad \Delta_0 = \Delta(0)$$

Verify that the characteristic polynomial is Hurwitz and construct the root locus as γ_0 varies from zero to infinity. Compare the root locus with the one obtained in exercise 14.56 and comment on the role played by the term $-k_1 x_c \cos\theta$ in the choice of s.

(f) By simulating the closed-loop system with the initial states $\theta(0) = \pi$, $\dot{\theta}(0) = 0$, $x_c(0) = 0.025$, and $\dot{x}_c(0) = 0$, and by using the root locus analysis of part (e), choose the constants k_1, k_2, β, and μ to make the settling time as small as you can. You should be able to achieve a settling time of about 4 sec. Compare with exercise 14.56.

(g) Suppose now that we can only measure θ and x_c. Using the analysis of Section 14.5, show that, for sufficiently small μ and ε, the origin is stabilized by the output feedback controller

$$u = -\beta \,\mathrm{sat}\left(\frac{z + k_2\theta - k_1 x_c \cos\theta}{\mu}\right)$$

where z is the output of the transfer function $s/(\varepsilon s + 1)$ driven by θ, which corresponds to a reduced-order high-gain observer. Verify that the output feedback controller recovers the performance of the state feedback controller as ε tends to zero. Simulate the closed-loop system for different values of ε and compare with the performance under state feedback.

Appendix A

Mathematical Review

Euclidean Space

The set of all n-dimensional vectors $x = [x_1, \ldots, x_n]^T$, where $x_1, \ldots, x_n$ are real numbers, defines the n-dimensional Euclidean space denoted by R^n. The one-dimensional Euclidean space consists of all real numbers and is denoted by R. Vectors in R^n can be added by adding their corresponding components. They can be multiplied by a scalar by multiplying each component by the scalar. The inner product of two vectors x and y is $x^T y = \sum_{i=1}^{n} x_i y_i$.

Vector and Matrix Norms

The norm $\|x\|$ of a vector x is a real-valued function with the properties

- $\|x\| \geq 0$ for all $x \in R^n$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x,\ y \in R^n$.
- $\|\alpha x\| = |\alpha|\ \|x\|$, for all $\alpha \in R$ and $x \in R^n$.

The second property is the triangle inequality. We consider the class of p-norms, defined by

$$\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_i |x_i|$$

The three most commonly used norms are $\|x\|_1$, $\|x\|_\infty$, and the Euclidean norm

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} = \left(x^T x\right)^{1/2}$$

All p-norms are equivalent in the sense that if $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two different p-norms, then there exist positive constants c_1 and c_2 such that

$$c_1\|x\|_\alpha \le \|x\|_\beta \le c_2\|x\|_\alpha$$

for all $x \in R^n$. For the 1-, 2-, and ∞-norms, these inequalities take the form

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\;\|x\|_2, \quad \|x\|_\infty \le \|x\|_2 \le \sqrt{n}\;\|x\|_\infty, \quad \|x\|_\infty \le \|x\|_1 \le n\;\|x\|_\infty$$

An important result concerning p-norms is the *Hölder inequality*

$$|x^T y| \le \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1$$

for all $x \in R^n$, $y \in R^n$. Quite often when we use norms, we only use properties deduced from the three basic properties satisfied by any norm. In those cases, the subscript p is dropped, indicating that the norm can be any p-norm.

An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m. The induced p-norm of A is defined by[1]

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$$

which for $p = 1$, 2, and ∞ is given by

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_2 = \left[\lambda_{\max}(A^T A)\right]^{1/2}, \quad \text{and} \quad \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$. Some useful properties of induced matrix norms for real matrices A and B of dimensions $m \times n$ and $n \times \ell$, respectively, are as follows:

$$\frac{1}{\sqrt{n}}\;\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\;\|A\|_\infty, \quad \frac{1}{\sqrt{m}}\;\|A\|_1 \le \|A\|_2 \le \sqrt{n}\;\|A\|_1$$

$$\|A\|_2 \le \sqrt{\|A\|_1\;\|A\|_\infty}, \quad \|AB\|_p \le \|A\|_p\;\|B\|_p$$

Topological Concepts in R^n

Convergence of Sequences: A sequence of vectors x_0, x_1, …, x_k, … in R^n, denoted by $\{x_k\}$, is said to converge to a limit vector x if

$$\|x_k - x\| \to 0 \text{ as } k \to \infty$$

which is equivalent to saying that, given any $\varepsilon > 0$, there is an integer N such that

$$\|x_k - x\| < \varepsilon, \quad \forall\; k \ge N$$

---

[1] sup denotes supremum, the least upper bound; inf denotes infimum, the greatest lower bound.

The symbol "$\forall$" reads "for all." A vector x is an accumulation point of a sequence $\{x_k\}$ if there is a subsequence of $\{x_k\}$ that converges to x; that is, if there is an infinite subset K of the nonnegative integers such that $\{x_k\}_{k \in K}$ converges to x. A bounded sequence $\{x_k\}$ in R^n has at least one accumulation point in R^n. A sequence of real numbers $\{r_k\}$ is said to be increasing (monotonically increasing or nondecreasing) if $r_k \leq r_{k+1}$ $\forall$ k. If $r_k < r_{k+1}$, it is said to be strictly increasing. Decreasing (monotonically decreasing or nonincreasing) and strictly decreasing sequences are defined similarly with $r_k \geq r_{k+1}$. An increasing sequence of real numbers that is bounded from above converges to a real number. Similarly, a decreasing sequence of real numbers that is bounded from below converges to a real number.

Sets: A subset $S \subset R^n$ is said to be *open* if, for every vector $x \in S$, one can find an ε-neighborhood of x

$$N(x, \varepsilon) = \{z \in R^n \mid \|z - x\| < \varepsilon\}$$

such that $N(x, \varepsilon) \subset S$. A set S is *closed* if and only if its complement in R^n is open. Equivalently, S is closed if and only if every convergent sequence $\{x_k\}$ with elements in S converges to a point in S. A set S is *bounded* if there is $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. A set S is *compact* if it is closed and bounded. A point p is a *boundary point* of a set S if every neighborhood of p contains at least one point of S and one point not belonging to S. The set of all boundary points of S, denoted by ∂S, is called the boundary of S. A closed set contains all its boundary points. An open set contains none of its boundary points. The *interior* of a set S is $S - \partial S$. An open set is equal to its interior. The *closure* of a set S, denoted by $\bar{S}$, is the union of S and its boundary. A closed set is equal to its closure. An open set S is connected if every pair of points in S can be joined by an arc lying in S. A set S is called a *region* if it is the union of an open connected set with some, none, or all of its boundary points. If none of the boundary points are included, the region is called an open region or *domain*. A set S is *convex* if, for every $x, y \in S$ and every real number θ, $0 < \theta < 1$, the point $\theta x + (1 - \theta)y \in S$. If $x \in X \subset R^n$ and $y \in Y \subset R^m$, we say that (x, y) belongs to the product set $X \times Y \subset R^n \times R^m$.

Continuous Functions: A function f mapping a set S_1 into a set S_2 is denoted by $f : S_1 \rightarrow S_2$. A function $f : R^n \rightarrow R^m$ is said to be *continuous* at a point x if $f(x_k) \rightarrow f(x)$ whenever $x_k \rightarrow x$. Equivalently, f is continuous at x if, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

The symbol "$\Rightarrow$" reads "implies." A function f is continuous on a set S if it is continuous at every point of S, and it is *uniformly continuous* on S if, given $\varepsilon > 0$ there is $\delta > 0$ (dependent only on ε) such that the inequality holds for all x, $y \in S$. Note that uniform continuity is defined on a set, while continuity is defined at a point. For uniform continuity, the same constant δ works for all points in the set.

Clearly, if f is uniformly continuous on a set S, then it is continuous on S. The opposite statement is not true in general. However, if S is a compact set, then continuity and uniform continuity on S are equivalent. The function

$$(a_1 f_1 + a_2 f_2)(\cdot) = a_1 f_1(\cdot) + a_2 f_2(\cdot)$$

is continuous for any two scalars a_1 and a_2 and any two continuous functions f_1 and f_2. If S_1, S_2, and S_3 are any sets and $f_1 : S_1 \rightarrow S_2$ and $f_2 : S_2 \rightarrow S_3$ are functions, then the function $f_2 \circ f_1 : S_1 \rightarrow S_3$, defined by

$$(f_2 \circ f_1)(\cdot) = f_2(f_1(\cdot))$$

is called the *composition* of f_1 and f_2. The composition of two continuous functions is continuous. If $S \subset R^n$ and $f : S \rightarrow R^m$, then the set of $f(x)$ such that $x \in S$ is called the image of S under f and is denoted by $f(S)$. If f is a continuous function defined on a compact set S, then $f(S)$ is compact; hence, continuous functions on compact sets are bounded. Moreover, if f is real valued, that is, $f : S \rightarrow R$, then there are points p and q in the compact set S such that $f(x) \leq f(p)$ and $f(x) \geq f(q)$ for all $x \in S$. If f is a continuous function defined on a connected set S, then $f(S)$ is connected. A function f defined on a set S is said to be *one to one* on S if whenever x, $y \in S$, and $x \neq y$, then $f(x) \neq f(y)$. If $f : S \rightarrow R^m$ is a continuous, one-to-one function on a compact set $S \subset R^n$, then f has a continuous inverse f^{-1} on $f(S)$. The composition of f and f^{-1} is identity; that is, $f^{-1}(f(x)) = x$. A function $f : R \rightarrow R^n$ is said to be *piecewise continuous* on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous for all $x \in J_0$, except, possibly, at a finite number of points where f may have discontinuities. Moreover, at each point of discontinuity x_0, the right-side limit $\lim_{h \rightarrow 0} f(x_0 + h)$ and the left-side limit $\lim_{h \rightarrow 0} f(x_0 - h)$ exist; that is, the function has a finite jump at x_0.

Differentiable functions: A function $f : R \rightarrow R$ is said to be *differentiable* at x if the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists. The limit $f'(x)$ is called the derivative of f at x. A function $f : R^n \rightarrow R^m$ is said to be *continuously differentiable* at a point x_0 if the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous at x_0 for $1 \leq i \leq m$, $1 \leq j \leq n$. A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S. For a continuously differentiable function $f : R^n \rightarrow R$, the row vector $\partial f / \partial x$ is defined by

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right]$$

The *gradient vector*, denoted by $\nabla f(x)$, is

$$\nabla f(x) = \left[\frac{\partial f}{\partial x} \right]^T$$

For a continuously differentiable function $f : R^n \to R^m$, the *Jacobian matrix* $[\partial f/\partial x]$ is an $m \times n$ matrix whose element in the ith row and jth column is $\partial f_i/\partial x_j$. Suppose $S \subset R^n$ is open, f maps S into R^m, f is continuously differentiable at $x_0 \in S$, g maps an open set containing $f(S)$ into R^k, and g is continuously differentiable at $f(x_0)$. Then the mapping h of S into R^k, defined by $h(x) = g(f(x))$, is continuously differentiable at x_0 and its Jacobian matrix is given by the *chain rule*

$$\left.\frac{\partial h}{\partial x}\right|_{x=x_0} = \left.\frac{\partial g}{\partial f}\right|_{f=f(x_0)} \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$

Mean Value and Implicit Function Theorems

If x and y are two distinct points in R^n, then the *line segment* $L(x,y)$ joining x and y is

$$L(x,y) = \{z \mid z = \theta x + (1-\theta)y,\ 0 < \theta < 1\}$$

Mean Value Theorem

Assume that $f : R^n \to R$ is continuously differentiable at each point x of an open set $S \subset R^n$. Let x and y be two points of S such that the line segment $L(x,y) \subset S$. Then there exists a point z of $L(x,y)$ such that

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x)$$

Implicit Function Theorem

Assume that $f : R^n \times R^m \to R^n$ is continuously differentiable at each point (x,y) of an open set $S \subset R^n \times R^m$. Let (x_0, y_0) be a point in S for which $f(x_0, y_0) = 0$ and for which the Jacobian matrix $[\partial f/\partial x](x_0, y_0)$ is nonsingular. Then there exist neighborhoods $U \subset R^n$ of x_0 and $V \subset R^m$ of y_0 such that for each $y \in V$ the equation $f(x,y) = 0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.

The proof of these two theorems, as well as the other facts stated earlier in this appendix, can be found in any textbook on advanced calculus or mathematical analysis.[2]

Gronwall–Bellman Inequality

Lemma A.1 *Let $\lambda : [a,b] \to R$ be continuous and $\mu : [a,b] \to R$ be continuous and nonnegative. If a continuous function $y : [a,b] \to R$ satisfies*

$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s)\, ds$$

---

[2] See, for example, [10].

for $a \le t \le b$, then on the same interval

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau)\, d\tau\right] \, ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \le \lambda \exp\left[\int_a^t \mu(\tau)\, d\tau\right]$$

If, in addition, $\mu(t) \equiv \mu \ge 0$ is a constant, then

$$y(t) \le \lambda \exp[\mu(t-a)]$$

◇

Proof: Let $z(t) = \int_a^t \mu(s)y(s)\, ds$ and $v(t) = z(t) + \lambda(t) - y(t) \ge 0$. Then, z is differentiable and

$$\dot{z} = \mu(t)y(t) = \mu(t)z(t) + \mu(t)\lambda(t) - \mu(t)v(t)$$

This is a scalar linear state equation with the state transition function

$$\phi(t,s) = \exp\left[\int_s^t \mu(\tau)\, d\tau\right]$$

Since $z(a) = 0$, we have

$$z(t) = \int_a^t \phi(t,s)[\mu(s)\lambda(s) - \mu(s)v(s)]\, ds$$

The term

$$\int_a^t \phi(t,s)\mu(s)v(s)\, ds$$

is nonnegative. Therefore,

$$z(t) \le \int_a^t \exp\left[\int_s^t \mu(\tau)\, d\tau\right] \mu(s)\lambda(s)\, ds$$

Since $y(t) \le \lambda(t) + z(t)$, this completes the proof in the general case. In the special case when $\lambda(t) \equiv \lambda$, we have

$$\begin{aligned} \int_a^t \mu(s) \exp\left[\int_s^t \mu(\tau)\, d\tau\right] \, ds &= -\int_a^t \frac{d}{ds}\left\{\exp\left[\int_s^t \mu(\tau)\, d\tau\right]\right\} \, ds \\ &= -\left\{\exp\left[\int_s^t \mu(\tau)\, d\tau\right]\right\}\Bigg|_{s=a}^{s=t} \\ &= -1 + \exp\left[\int_a^t \mu(\tau)\, d\tau\right] \end{aligned}$$

which proves the lemma when λ is a constant. The proof when both λ and μ are constants follows by integration. □

Appendix B

Contraction Mapping

Consider an equation of the form $x = T(x)$. A solution x^* to this equation is said to be a *fixed point* of the mapping T, since T leaves x^* invariant. A classical idea for finding a fixed point is the successive approximation method. We begin with an initial trial vector x_1 and compute $x_2 = T(x_1)$. Continuing in this manner iteratively, we compute successive vectors $x_{k+1} = T(x_k)$. The contraction mapping theorem gives sufficient conditions under which there is a fixed point x^* of $x = T(x)$ and the sequence $\{x_k\}$ converges to x^*. It is a powerful analysis tool for proving the existence of a solution of an equation of the form $x = T(x)$. The theorem is valid, not only when T is a mapping from one Euclidean space into another, but also when T is a mapping between *Banach spaces*. We will use the contraction mapping theorem in this general setting. We start by introducing Banach spaces.[1]

Vector Spaces: A linear vector space $\mathcal{X}$ over the field R is a set of elements $x,\ y,\ z,\ \ldots$ called vectors such that for any two vectors $x,\ y \in \mathcal{X}$, the sum $x+y$ is defined, $x+y \in \mathcal{X}$, $x+y = y+x$, $(x+y)+z = x+(y+z)$, and there is a zero vector $0 \in \mathcal{X}$ such that $x+0 = x$ for all $x \in \mathcal{X}$. Also for any numbers $\alpha,\ \beta \in R$, the scalar multiplication αx is defined, $\alpha x \in \mathcal{X}$, $1 \cdot x = x$, $0 \cdot x = 0$, $(\alpha\beta)x = \alpha(\beta x)$, $\alpha(x+y) = \alpha x + \alpha y$, and $(\alpha+\beta)x = \alpha x + \beta x$, for all $x,\ y \in \mathcal{X}$.

Normed Linear Space: A linear space $\mathcal{X}$ is a normed linear space if, to each vector $x \in \mathcal{X}$, there is a real-valued norm $\|x\|$ that satisfies

- $\|x\| \geq 0$ for all $x \in \mathcal{X}$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x+y\| \leq \|x\| + \|y\|$ for all $x,\ y \in \mathcal{X}$.
- $\|\alpha x\| = |\alpha|\ \|x\|$ for all $\alpha \in R$ and $x \in \mathcal{X}$.

---

[1] For a complete treatment of Banach spaces, consult any textbook on functional analysis. A lucid treatment can be found in [121, Chapter 2].

If it is not clear from the context whether $\|\cdot\|$ is a norm on $\mathcal{X}$ or a norm on R^n, we will write $\|\cdot\|_{\mathcal{X}}$ for the norm on $\mathcal{X}$.

Convergence: A sequence $\{x_k\} \in \mathcal{X}$, a normed linear space, converges to $x \in \mathcal{X}$ if

$$\|x_k - x\| \to 0 \ \text{ as } \ k \to \infty$$

Closed Set: A set $S \subset \mathcal{X}$ is closed if and only if every convergent sequence with elements in S has its limit in S.

Cauchy Sequence: A sequence $\{x_k\} \in \mathcal{X}$ is said to be a *Cauchy sequence* if

$$\|x_k - x_m\| \to 0 \ \text{ as } \ k,\ m \to \infty$$

Every convergent sequence is Cauchy, but not vice versa.

Banach Space: A normed linear space $\mathcal{X}$ is complete if every Cauchy sequence in $\mathcal{X}$ converges to a vector in $\mathcal{X}$. A complete normed linear space is a Banach space.

Example B.1 Consider the set of all continuous functions $f : [a, b] \to R^n$, denoted by $C[a, b]$. This set forms a vector space on R. The sum $x + y$ is defined by $(x + y)(t) = x(t) + y(t)$. The scalar multiplication is defined by $(\alpha x)(t) = \alpha x(t)$. The zero vector is the function that is identically zero on $[a, b]$. We define a norm by

$$\|x\|_C = \max_{t \in [a,b]} \|x(t)\|$$

where the right-hand side norm is any p-norm on R^n. Clearly $\|x\|_C \geq 0$ and is zero only for the zero vector. The triangle inequality follows from

$$\max \|x(t) + y(t)\| \leq \max[\|x(t)\| + \|y(t)\|] \leq \max \|x(t)\| + \max \|y(t)\|$$

Also,

$$\max \|\alpha x(t)\| = \max |\alpha| \ \|x(t)\| = |\alpha| \max \|x(t)\|$$

where all maxima are taken over $[a, b]$. Hence, $C[a, b]$, together with the norm $\|\cdot\|_C$, is a normed linear space. It is also a Banach space. To prove this claim, we need to show that every Cauchy sequence in $C[a, b]$ converges to a vector in $C[a, b]$. Suppose that $\{x_k\}$ is a Cauchy sequence in $C[a, b]$. For each fixed $t \in [a, b]$,

$$\|x_k(t) - x_m(t)\| \leq \|x_k - x_m\|_C \to 0 \ \text{ as } \ k,\ m \to \infty$$

So $\{x_k(t)\}$ is a Cauchy sequence in R^n. But R^n with any p-norm is complete because convergence implies componentwise convergence and R is complete. Therefore, there is a real vector $x(t)$ to which the sequence converges: $x_k(t) \to x(t)$. This proves pointwise convergence. We prove next that the convergence is uniform in $t \in [a, b]$. Given $\varepsilon > 0$, choose N such that $\|x_k - x_m\|_C < \varepsilon/2$ for $k,\ m > N$. Then for $k > N$

$$\begin{aligned} \|x_k(t) - x(t)\| &\leq \|x_k(t) - x_m(t)\| + \|x_m(t) - x(t)\| \\ &\leq \|x_k - x_m\|_C + \|x_m(t) - x(t)\| \end{aligned}$$

By choosing m sufficiently large (which may depend on t), each term on the right-hand side can be made smaller than $\varepsilon/2$; so $\|x_k(t) - x(t)\| < \varepsilon$ for $k > N$. Hence, $\{x_k\}$ converges to x, uniformly in $t \in [a, b]$. To complete the proof, we need to show that $x(t)$ is continuous and $\{x_k\}$ converges to x in the norm of $C[a, b]$. To prove continuity, consider

$$\begin{aligned} \|x(t+\delta) - x(t)\| \;\; &\leq \;\; \|x(t+\delta) - x_k(t+\delta)\| \\ &\quad + \|x_k(t+\delta) - x_k(t)\| + \|x_k(t) - x(t)\| \end{aligned}$$

Since $\{x_k\}$ converges uniformly to x, given any $\varepsilon > 0$, we can choose k large enough to make both the first and third terms on the right-hand side less than $\varepsilon/3$. Because $x_k(t)$ is continuous, we can choose δ small enough to make the second term less than $\varepsilon/3$. Therefore, $x(t)$ is continuous. The convergence of x_k to x in $\|\cdot\|_C$ is a direct consequence of the uniform convergence. △

Theorem B.1 (Contraction Mapping) *Let S be a closed subset of a Banach space $\mathcal{X}$ and let T be a mapping that maps S into S. Suppose that*

$$\|T(x) - T(y)\| \leq \rho\|x - y\|, \quad \forall\; x,\; y \in S, \quad 0 \leq \rho < 1$$

then

- *there exists a unique vector $x^* \in S$ satisfying $x^* = T(x^*)$.*
- *x^* can be obtained by the method of successive approximation, starting from any arbitrary initial vector in S.* ◇

Proof: Select an arbitrary $x_1 \in S$ and define the sequence $\{x_k\}$ by the formula $x_{k+1} = T(x_k)$. Since T maps S into S, $x_k \in S$ for all $k \geq 1$. The first step of the proof is to show that $\{x_k\}$ is Cauchy. We have

$$\begin{aligned} \|x_{k+1} - x_k\| \;\; &= \;\; \|T(x_k) - T(x_{k-1})\| \\ &\leq \;\; \rho\|x_k - x_{k-1}\| \;\leq\; \rho^2\|x_{k-1} - x_{k-2}\| \;\leq \cdots \leq\; \rho^{k-1}\|x_2 - x_1\| \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{k+r} - x_k\| \;\; &\leq \;\; \|x_{k+r} - x_{k+r-1}\| + \|x_{k+r-1} - x_{k+r-2}\| + \cdots + \|x_{k+1} - x_k\| \\ &\leq \;\; \left[\rho^{k+r-2} + \rho^{k+r-3} + \cdots + \rho^{k-1}\right] \|x_2 - x_1\| \\ &\leq \;\; \rho^{k-1}\sum_{i=0}^{\infty}\rho^i \;\|x_2 - x_1\| \;=\; \frac{\rho^{k-1}}{1-\rho}\|x_2 - x_1\| \end{aligned}$$

The right-hand side tends to zero as $k \to \infty$. Thus, the sequence is Cauchy. Because $\mathcal{X}$ is a Banach space,

$$x_k \to x^* \in \mathcal{X} \;\text{ as }\; k \to \infty$$

Moreover, since S is closed, $x^* \in S$. We now show that $x^* = T(x^*)$. For any $x_k = T(x_{k-1})$, we have

$$\|x^* - T(x^*)\| \leq \|x^* - x_k\| + \|x_k - T(x^*)\| \leq \|x^* - x_k\| + \rho\|x_{k-1} - x^*\|$$

By choosing k large enough, the right-hand side of the inequality can be made arbitrarily small. Thus, $\|x^* - T(x^*)\| = 0$; that is, $x^* = T(x^*)$. It remains to show that x^* is the unique fixed point of T in S. Suppose that x^* and y^* are fixed points. Then,

$$\|x^* - y^*\| = \|T(x^*) - T(y^*)\| \leq \rho\|x^* - y^*\|$$

Since $\rho < 1$, we have $x^* = y^*$. □

Appendix C

Proofs

C.1 Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1: We start by noting that if $x(t)$ is a solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{C.1}$$

then, by integration, we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{C.2}$$

Conversely, if $x(t)$ satisfies (C.2), then $x(t)$ satisfies (C.1). Thus, the study of existence and uniqueness of the solution of the differential equation (C.1) is equivalent to the study of existence and uniqueness of the solution of the integral equation (C.2). We proceed with (C.2). Viewing the right-hand side of (C.2) as a mapping of the continuous function $x : [t_0, t_1] \rightarrow R^n$, and denoting it by $(Px)(t)$, we can rewrite (C.2) as

$$x(t) = (Px)(t) \tag{C.3}$$

Note that $(Px)(t)$ is continuous in t. A solution of (C.3) is a fixed point of the mapping P that maps x into Px. Existence of a fixed point of (C.3) can be established by using the contraction mapping theorem. This requires defining a Banach space $\mathcal{X}$ and a closed set $S \subset \mathcal{X}$ such that P maps S into S and is a contraction over S. Let

$$\mathcal{X} = C[t_0, t_0 + \delta], \quad \text{with norm } \|x\|_C = \max_{t \in [t_0, t_0+\delta]} \|x(t)\|$$

and

$$S = \{x \in \mathcal{X} \mid \|x - x_0\|_C \le r\}$$

where r is the radius of the ball B and δ is a positive constant to be chosen. We will restrict the choice of δ to satisfy $\delta \le t_1 - t_0$ so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$. Notice that $\|x(t)\|$ denotes a norm on R^n, while $\|x\|_C$ denotes a norm on $\mathcal{X}$. Also, B is a ball in R^n, while S is a ball in $\mathcal{X}$. By definition, P maps $\mathcal{X}$ into $\mathcal{X}$. To show that it maps S into S, write

$$(Px)(t) - x_0 = \int_{t_0}^{t} f(s, x(s))\, ds = \int_{t_0}^{t} [f(s, x(s)) - f(s, x_0) + f(s, x_0)]\, ds$$

By piecewise continuity of f, we know that $f(t, x_0)$ is bounded on $[t_0, t_1]$. Let

$$h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\|$$

Using the Lipschitz condition (3.2) and the fact that for each $x \in S$,

$$\|x(t) - x_0\| \leq r, \qquad \forall\, t \in [t_0, t_0 + \delta]$$

we obtain

$$\begin{aligned} \|(Px)(t) - x_0\| &\leq \int_{t_0}^{t} [\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|]\, ds \\ &\leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h]\, ds \;\leq\; \int_{t_0}^{t} (Lr + h)\, ds \\ &= (t - t_0)(Lr + h) \;\leq\; \delta(Lr + h) \end{aligned}$$

and

$$\|Px - x_0\|_C = \max_{t \in [t_0, t_0 + \delta]} \|(Px)(t) - x_0\| \leq \delta(Lr + h)$$

Hence, choosing $\delta \leq r/(Lr + h)$ ensures that P maps S into S. To show that P is a contraction mapping over S, let x and $y \in S$ and consider

$$\begin{aligned} \|(Px)(t) - (Py)(t)\| &= \left\| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))]\, ds \right\| \\ &\leq \int_{t_0}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \\ &\leq \int_{t_0}^{t} L\|x(s) - y(s)\|\, ds \;\leq\; \int_{t_0}^{t} ds\; L\|x - y\|_C \end{aligned}$$

Therefore,

$$\|Px - Py\|_C \leq L\delta\|x - y\|_C \leq \rho\|x - y\|_C \quad \text{for} \;\; \delta \leq \frac{\rho}{L}$$

Thus, choosing $\rho < 1$ and $\delta \leq \rho/L$ ensures that P is a contraction mapping over S. By the contraction mapping theorem, we can conclude that if δ is chosen to satisfy

$$\delta \leq \min\left\{ t_1 - t_0, \frac{r}{Lr + h}, \frac{\rho}{L} \right\} \quad \text{for} \quad \rho < 1 \tag{C.4}$$

then (C.2) will have a unique solution in S. This is not the end of the proof, though, since we are interested in establishing uniqueness of the solution among all continuous functions $x(t)$, that is, uniqueness in $\mathcal{X}$. It turns out that any solution of (C.2) in $\mathcal{X}$ will lie in S. To see this, note that since $x(t_0) = x_0$ is inside the ball B, any continuous solution $x(t)$ must lie inside B for some interval of time. Suppose that $x(t)$ leaves the ball B and let $t_0 + \mu$ be the first time $x(t)$ intersects the boundary of B. Then,

$$\|x(t_0 + \mu) - x_0\| = r$$

On the other hand, for all $t \leq t_0 + \mu$,

$$\begin{aligned} \|x(t) - x_0\| &\leq \int_{t_0}^{t} [\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|] \, ds \\ &\leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h] \, ds \leq \int_{t_0}^{t} (Lr + h) \, ds \end{aligned}$$

Therefore,

$$r = \|x(t_0 + \mu) - x_0\| \leq (Lr + h)\mu \quad \Rightarrow \quad \mu \geq \frac{r}{Lr + h} \geq \delta$$

Hence, the solution $x(t)$ cannot leave the set B within the time interval $[t_0, t_0 + \delta]$, which implies that any solution in $\mathcal{X}$ lies in S. Consequently, uniqueness of the solution in S implies uniqueness in $\mathcal{X}$.

Proof of Theorem 3.2: The key point of the proof is to show that the constant δ of Theorem 3.1 can be made independent of the initial state x_0. From (C.4), we see that the dependence of δ on the initial state comes through the constant h in the term $r/(Lr + h)$. Since in the current case the Lipschitz condition holds globally, we can choose r arbitrarily large. Therefore, for any finite h, we can choose r large enough so that $r/(Lr + h) > \rho/L$. This reduces (C.4) to the requirement

$$\delta \leq \min\left\{t_1 - t_0, \frac{\rho}{L}\right\} \quad \text{for } \rho < 1$$

If $t_1 - t_0 \leq \rho/L$, we could choose $\delta = t_1 - t_0$ and be done. Otherwise, we choose δ to satisfy $\delta \leq \rho/L$. Now, divide $[t_0, t_1]$ into a finite number of subintervals of length $\delta \leq \rho/L$, and apply Theorem 3.1 repeatedly.[1] □

C.2 Proof of Lemma 3.4

The upper right-hand derivative $D^+v(t)$ is defined by

$$D^+v(t) = \lim \sup_{h \to 0^+} \frac{v(t + h) - v(t)}{h}$$

---

[1] Note that the initial state of each subinterval x_1, say, will satisfy $\|f(t, x_1)\| \leq h_1$ for some finite h_1.

where $\limsup_{n\to\infty}$ (the limit superior) of a sequence of real numbers $\{x_n\}$ is a real number y satisfying the following two conditions:

- for every $\varepsilon > 0$, there exists an integer N such that $n > N$ implies $x_n < y + \varepsilon$;
- given $\varepsilon > 0$ and $m > 0$, there exists an integer $n > m$ such that $x_n > y - \varepsilon$.

The first statement means that ultimately all terms of the sequence are less than $y + \varepsilon$, and the second one means that infinitely many terms are greater than $y - \varepsilon$. One of the properties of lim sup[2] is that if $z_n \le x_n$ for each $n = 1, 2 \ldots$, then $\limsup_{n\to\infty} z_n \le \limsup_{n\to\infty} x_n$. From this property, we see that if $|v(t+h) - v(t)|/h \le g(t,h)$, $\forall\ h \in (0,b]$ and $\lim_{h\to 0^+} g(t,h) = g_0(t)$, then $D^+v(t) \le g_0(t)$.

To prove lemma 3.4, consider the differential equation

$$\dot{z} = f(t,z) + \lambda, \qquad z(t_0) = u_0 \tag{C.5}$$

where λ is a positive constant. On any compact interval $[t_0, t_1]$, we conclude from Theorem 3.5 that for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\lambda < \delta$, then (C.5) has a unique solution $z(t,\lambda)$ defined on $[t_0, t_1]$ and

$$|z(t,\lambda) - u(t)| < \varepsilon, \quad \forall\ t \in [t_0, t_1] \tag{C.6}$$

Claim 1: $v(t) \le z(t,\lambda)$ for all $t \in [t_0, t_1]$.
This claim can be shown by contradiction, for if it were not true, there would be times $a, b \in (t_0, t_1]$ such that $v(a) = z(a,\lambda)$ and $v(t) > z(t,\lambda)$ for $a < t \le b$. Consequently,

$$v(t) - v(a) > z(t,\lambda) - z(a,\lambda), \qquad \forall\ t \in (a,b]$$

which implies

$$D^+v(a) \ge \dot{z}(a,\lambda) = f(a, z(a,\lambda)) + \lambda > f(a, v(a))$$

which contradicts the inequality $D^+v(t) \le f(t, v(t))$.

Claim 2: $v(t) \le u(t)$ for all $t \in [t_0, t_1]$.
Again, this claim can be shown by contradiction, for if it were not true, there would exist $a \in (t_0, t_1]$ such that $v(a) > u(a)$. Taking $\varepsilon = [v(a) - u(a)]/2$ and using (C.6), we obtain

$$v(a) - z(a,\lambda) = v(a) - u(a) + u(a) - z(a,\lambda) \ge \varepsilon$$

which contradicts the statement of Claim 1.
Thus, we have shown that $v(t) \le u(t)$ for all $t \in [t_0, t_1]$. Since this is true on every compact interval, we conclude that it holds for all $t \ge t_0$. If it was not the case, let $T < \infty$ be the first time the inequality is violated. We have $v(t) \le u(t)$ for all $t \in [t_0, T)$ and, by continuity, $v(T) = u(T)$. Consequently, we can extend the inequality to the interval $[T, T + \Delta]$ for some $\Delta > 0$, which contradicts the claim that T is the first time the inequality is violated.

---

[2] See [10, Theorem 12-4].

C.3 Proof of Lemma 4.1

Since $x(t)$ is bounded, by the Bolzano–Weierstrass theorem,[3] it has an accumulation point as $t \to \infty$; hence the positive limit set L^+ is nonempty. For each $y \in L^+$, there is a sequence t_i with $t_i \to \infty$ as $i \to \infty$ such that $x(t_i) \to y$ as $i \to \infty$. Because $x(t_i)$ is bounded uniformly in i, the limit y is bounded; that is, L^+ is bounded. To show that L^+ is closed, let $\{y_i\} \in L^+$ be some sequence such that $y_i \to y$ as $i \to \infty$ and prove that $y \in L^+$. For every i, there is a sequence $\{t_{ij}\}$ with $t_{ij} \to \infty$ as $j \to \infty$ such that $x(t_{ij}) \to y_i$ as $j \to \infty$. We will construct a particular sequence $\{\tau_i\}$. Given the sequences t_{ij}, choose $\tau_2 > t_{12}$ such that $\|x(\tau_2) - y_2\| < 1/2$; choose $\tau_3 > t_{13}$ such that $\|x(\tau_3) - y_3\| < 1/3$; and so on for $i = 4, 5, \ldots$. Of course, $\tau_i \to \infty$ as $i \to \infty$ and $\|x(\tau_i) - y_i\| < 1/i$ for every i. Now, given $\varepsilon > 0$, there are positive integers N_1 and N_2 such that

$$\|x(\tau_i) - y_i\| < \frac{\varepsilon}{2}, \quad \forall\ i > N_1 \quad \text{and} \quad \|y_i - y\| < \frac{\varepsilon}{2}, \quad \forall\ i > N_2$$

The first inequality follows from $\|x(\tau_i) - y_i\| < 1/i$ and the second one from the limit $y_i \to y$. Thus,

$$\|x(\tau_i) - y\| < \varepsilon, \quad \forall\ i > N = \max\{N_1, N_2\}$$

which shows that $x(\tau_i) \to y$ as $i \to \infty$. Hence, L^+ is closed. This proves that the set L^+ is compact, because it is closed and bounded.

To show that L^+ is an invariant set, let $y \in L^+$ and $\phi(t; y)$ be the solution of (4.1) that passes through y at $t = 0$; that is, $\phi(0; y) = y$, and show that $\phi(t; y) \in L^+$, $\forall\ t \in R$. There is a sequence $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ such that $x(t_i) \to y$ as $i \to \infty$. Write $x(t_i) = \phi(t_i; x_0)$, where x_0 is the initial state of $x(t)$ at $t = 0$. By uniqueness of the solution,

$$\phi(t + t_i; x_0) = \phi(t; \phi(t_i; x_0)) = \phi(t; x(t_i))$$

where, for sufficiently large i, $t + t_i \ge 0$. By continuity,

$$\lim_{i\to\infty} \phi(t + t_i; x_0) = \lim_{i\to\infty} \phi(t; x(t_i)) = \phi(t; y)$$

which shows that $\phi(t; y) \in L^+$.

Finally, to show that $x(t) \to L^+$ as $t \to \infty$, use a contradiction argument. Suppose this is not the case; then, there is an $\varepsilon > 0$ and a sequence $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ such that $\text{dist}(x(t_i), L^+) > \varepsilon$. Since the sequence $x(t_i)$ is bounded, it contains a convergent subsequence $x(t_i') \to x^*$ as $i \to \infty$. The point x^* must belong to L^+ and at the same time be at a distance ε from L^+, which is a contradiction.

---

[3] See [10].

C.4 Proof of Lemma 4.3

Define $\psi(s)$ by

$$\psi(s) = \inf_{s \le \|x\| \le r} V(x) \quad \text{for } 0 \le s \le r$$

The function $\psi(\cdot)$ is continuous, positive definite, and increasing. Moreover, $V(x) \ge \psi(\|x\|)$ for $0 \le \|x\| \le r$. But $\psi(\cdot)$ is not necessarily strictly increasing. Let $\alpha_1(s)$ be a class $\mathcal{K}$ function such that $\alpha_1(s) \le k\psi(s)$ with $0 < k < 1$. Then,

$$V(x) \ge \psi(\|x\|) \ge \alpha_1(\|x\|) \quad \text{for } \|x\| \le r$$

On the other hand, define $\phi(s)$ by

$$\phi(s) = \sup_{\|x\| \le s} V(x) \quad \text{for } 0 \le s \le r$$

The function $\phi(\cdot)$ is continuous, positive definite, and increasing (not necessarily strictly increasing). Moreover, $V(x) \le \phi(\|x\|)$ for $\|x\| \le r$. Let $\alpha_2(s)$ be a class $\mathcal{K}$ function such that $\alpha_2(s) \ge k\phi(s)$ with $k > 1$. Then

$$V(x) \le \phi(\|x\|) \le \alpha_2(\|x\|) \quad \text{for } \|x\| \le r$$

If $D = R^n$, the definitions of $\psi(s)$ and $\phi(s)$ are changed to

$$\psi(s) = \inf_{\|x\| \ge s} V(x), \quad \phi(s) = \sup_{\|x\| \le s} V(x), \quad \text{for } s \ge 0$$

The functions ψ and ϕ are continuous, positive definite, increasing, and

$$\psi(\|x\|) \le V(x) \le \phi(\|x\|), \quad \forall\, x \in R^n$$

The functions α_1 and α_2 can be chosen as before. If $V(x)$ is radially unbounded, $\psi(s)$ and $\phi(s)$ tend to infinity as $s \to \infty$. Hence, we can choose α_1 and α_2 to belong to class $\mathcal{K}_\infty$.

C.5 Proof of Lemma 4.4

Since $\alpha(\cdot)$ is locally Lipschitz, the equation has a unique solution for every initial state $y_0 \ge 0$. Because $\dot{y}(t) < 0$ whenever $y(t) > 0$, the solution has the property that $y(t) \le y_0$ for all $t \ge t_0$. Therefore, the solution is bounded and can be extended for all $t \ge t_0$. By integration, we have

$$-\int_{y_0}^{y} \frac{dx}{\alpha(x)} = \int_{t_0}^{t} d\tau$$

Let b be any positive number less than a, and define

$$\eta(y) = -\int_{b}^{y} \frac{dx}{\alpha(x)}$$

The function $\eta(y)$ is a strictly decreasing differentiable function on $(0, a)$. Moreover, $\lim_{y \to 0} \eta(y) = \infty$. This limit follows from two facts. First, the solution of the differential equation $y(t) \to 0$ as $t \to \infty$, since $\dot{y}(t) < 0$ whenever $y(t) > 0$. Second, the limit $y(t) \to 0$ can happen only asymptotically as $t \to \infty$; it cannot happen in finite time due to the uniqueness of the solution. Let $c = -\lim_{y \to a} \eta(y)$ (c may be ∞). The range of the function η is $(-c, \infty)$. Since η is strictly decreasing, its inverse η^{-1} is defined on $(-c, \infty)$. For any $y_0 > 0$, the solution $y(t)$ satisfies

$$\eta(y(t)) - \eta(y_0) = t - t_0$$

Hence,

$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0)$$

On the other hand, if $y_0 = 0$, then $y(t) \equiv 0$, because $y = 0$ is an equilibrium point. Define a function $\sigma(r, s)$ by

$$\sigma(r, s) = \begin{cases} \eta^{-1}(\eta(r) + s), & r > 0 \\ 0, & r = 0 \end{cases}$$

Then, $y(t) = \sigma(y_0, t - t_0)$ for all $t \geq t_0$ and $y_0 \geq 0$. The function σ is continuous, since both η and η^{-1} are continuous in their domains and $\lim_{x \to \infty} \eta^{-1}(x) = 0$. It is strictly increasing in r for each fixed s because

$$\frac{\partial}{\partial r}\sigma(r, s) = \frac{\alpha(\sigma(r, s))}{\alpha(r)} > 0$$

and strictly decreasing in s for each fixed r because

$$\frac{\partial}{\partial s}\sigma(r, s) = -\alpha(\sigma(r, s)) < 0$$

Furthermore, $\sigma(r, s) \to 0$ as $s \to \infty$. Thus, σ is a class $\mathcal{KL}$ function.

C.6 Proof of Lemma 4.5

Uniform Stability: Suppose there is a class $\mathcal{K}$ function α such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall\, t \geq t_0 \geq 0, \ \forall\, \|x(t_0)\| < c$$

Given $\varepsilon > 0$, let $\delta = \min\{c, \alpha^{-1}(\varepsilon)\}$. Then, for $\|x(t_0)\| < \delta$, we have

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) < \alpha(\delta) \leq \alpha(\alpha^{-1}(\varepsilon)) = \varepsilon$$

Now assume that given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0$$

For a fixed ε, let $\bar{\delta}(\varepsilon)$ be the supremum of all applicable $\delta(\varepsilon)$. The function $\bar{\delta}(\varepsilon)$ is positive and nondecreasing, but not necessarily continuous. We choose a class $\mathcal{K}$ function $\zeta(r)$ such that $\zeta(r) \leq k\bar{\delta}(r)$, with $0 < k < 1$. Let $\alpha(r) = \zeta^{-1}(r)$. Then, $\alpha(r)$ is class $\mathcal{K}$. Let $c = \lim_{r\to\infty} \zeta(r)$. Given $x(t_0)$, with $\|x(t_0)\| < c$, let $\varepsilon = \alpha(\|x(t_0)\|)$. Then, $\|x(t_0)\| < \bar{\delta}(\varepsilon)$ and

$$\|x(t)\| < \varepsilon = \alpha(\|x(t_0)\|) \tag{C.7}$$

Uniform Asymptotic Stability: Suppose there is a class $\mathcal{KL}$ function $\beta(r,s)$ such that (4.20) is satisfied. Then,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, 0)$$

which implies that $x = 0$ is uniformly stable. Moreover, for $\|x(t_0)\| < c$, the solution satisfies

$$\|x(t)\| \leq \beta(c, t - t_0)$$

which shows that $x(t) \to 0$ as $t \to \infty$, uniformly in t_0. Suppose now that $x = 0$ is uniformly stable and $x(t) \to 0$ as $t \to \infty$, uniformly in t_0, and show that there is a class $\mathcal{KL}$ function $\beta(r,s)$ for which (4.20) is satisfied. Due to uniform stability, there is a constant $c > 0$ and a class $\mathcal{K}$ function α such that for any $r \in (0, c]$, the solution $x(t)$ satisfies

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) < \alpha(r), \quad \forall\, t \geq t_0, \ \forall\, \|x(t_0)\| < r \tag{C.8}$$

Moreover, given $\eta > 0$, there exists $T = T(\eta, r) \geq 0$ (dependent on η and r, but independent of t_0) such that

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta, r)$$

Let $\bar{T}(\eta, r)$ be the infimum of all applicable $T(\eta, r)$. The function $\bar{T}(\eta, r)$ is nonnegative and nonincreasing in η, nondecreasing in r, and $\bar{T}(\eta, r) = 0$ for all $\eta \geq \alpha(r)$. Let

$$W_r(\eta) = \frac{2}{\eta}\int_{\eta/2}^{\eta} \bar{T}(s, r)\; ds + \frac{r}{\eta} \geq \bar{T}(\eta, r) + \frac{r}{\eta}$$

The function $W_r(\eta)$ is positive and has the following properties:

- for each fixed r, $W_r(\eta)$ is continuous, strictly decreasing, and $W_r(\eta) \to 0$ as $\eta \to \infty$, and
- for each fixed η, $W_r(\eta)$ is strictly increasing in r.

Take $U_r = W_r^{-1}$. Then, U_r inherits the foregoing two properties of W_r and $\bar{T}(U_r(s), r) < W_r(U_r(s)) = s$. Therefore,

$$\|x(t)\| \leq U_r(t - t_0), \quad \forall\, t \geq t_0, \ \forall\, \|x(t_0\| < r \tag{C.9}$$

From (C.8) and (C.9), it is clear that

$$\|x(t)\| \leq \sqrt{\alpha(\|x(t_0)\|)U_c(t-t_0)}, \quad \forall\ t \geq t_0,\ \forall\ \|x(t_0)\| < c$$

Thus, inequality (4.20) is satisfied with $\beta(r,s) = \sqrt{\alpha(r)U_c(s)}$.

Global Uniform Asymptotic Stability: If (4.20) holds for all $x(t_0) \in R^n$, then it can be easily seen, as in the previous case, that the origin is globally uniformly asymptotically stable. To prove the opposite direction, notice that in the current case the function $\bar{\delta}(\varepsilon)$ has the additional property $\bar{\delta}(\varepsilon) \to \infty$ as $\varepsilon \to \infty$. Consequently, the class $\mathcal{K}$ function α can be chosen to belong to class $\mathcal{K}_\infty$, and inequality (C.7) holds for all $x(t_0) \in R^n$. Moreover, inequality (C.9) holds for any $r > 0$. Let

$$\psi(r,s) = \min\left\{\alpha(r),\ \inf_{\rho \in (r,\infty)} U_\rho(s)\right\}$$

Then,

$$\|x(t)\| \leq \psi(\|x(t_0)\|, t-t_0), \quad \forall\ t \geq t_0,\ \forall\ x(t_0) \in R^n$$

If ψ would be class $\mathcal{KL}$, we would be done. This may not be the case, so we define the function

$$\phi(r,s) = \int_r^{r+1} \psi(\lambda,s)\ d\lambda + \frac{r}{(r+1)(s+1)}$$

which is positive and has the following properties:

- for each fixed $s \geq 0$, $\phi(r,s)$ is continuous and strictly increasing in r,
- for each fixed $r \geq 0$, $\phi(r,s)$ is strictly decreasing in s and tends to zero as $s \to \infty$, and
- $\phi(r,s) \geq \psi(r,s)$.

Thus,

$$\|x(t)\| \leq \phi(\|x(t_0)\|, t-t_0), \quad \forall\ t \geq t_0,\ \forall\ x(t_0) \in R^n \tag{C.10}$$

From (C.10) and the global version of (C.7), we see that

$$\|x(t)\| \leq \sqrt{\alpha(\|x(t_0)\|)\phi(\|x(t_0)\|, t-t_0)}, \quad \forall\ t \geq t_0,\ \forall\ x(t_0) \in R^n$$

Thus, the inequality (4.20) is satisfied globally with $\beta(r,s) = \sqrt{\alpha(r)\phi(r,s)}$.

C.7 Proof of Theorem 4.16

The construction of a Lyapunov function is accomplished by using a lemma, known as Massera's lemma. We start by stating and proving Massera's lemma.

Lemma C.1 *Let $g : [0, \infty) \to R$ be a positive, continuous, strictly decreasing function with $g(t) \to 0$ as $t \to \infty$. Let $h : [0, \infty) \to R$ be a positive, continuous, nondecreasing function. Then, there exists a function $G(t)$ such that*

- *$G(t)$ and its derivative $G'(t)$ are class $\mathcal{K}$ functions defined for all $t \geq 0$.*
- *For any continuous function $u(t)$ that satisfies $0 \leq u(t) \leq g(t)$ for all $t \geq 0$, there exist positive constants k_1 and k_2, independent of u, such that*

$$\int_0^\infty G(u(t))\ dt \leq k_1; \quad \int_0^\infty G'(u(t))h(t)\ dt \leq k_2$$

◇

Proof of Lemma C.1: Since $g(t)$ is strictly decreasing, we can choose a sequence t_n such that

$$g(t_n) \leq \frac{1}{n+1}, \quad n = 1, 2, \ldots$$

We use this sequence to define a function $\eta(t)$ as follows:

(a) $\eta(t_n) = 1/n$.

(b) Between t_n and t_{n+1}, $\eta(t)$ is linear.

(c) In the interval $0 < t \leq t_1$, $\eta(t) = (t_1/t)^p$, where p is a positive integer chosen so large that the derivative $\eta'(t)$ has a positive jump at t_1, $\eta'(t_1^-) < \eta'(t_1^+)$.

The function $\eta(t)$ is strictly decreasing, and for $t \geq t_1$, we have $g(t) < \eta(t)$. As $t \to 0^+$, $\eta(t)$ grows unbounded. The inverse of $\eta(t)$, denoted by $\eta^{-1}(s)$, is a strictly decreasing function that grows unbounded as $s \to 0^+$. Obviously,

$$\eta^{-1}(u(t)) \geq \eta^{-1}(g(t)) > \eta^{-1}(\eta(t)) = t, \quad \forall\ t \geq t_1$$

for any nonnegative function $u(t) \leq g(t)$. Define

$$H(s) = \frac{\exp[-\eta^{-1}(s)]}{h(\eta^{-1}(s))}, \quad s \geq 0$$

Since η^{-1} is continuous and h is positive, $H(s)$ is continuous on $0 < s < \infty$, while $\eta^{-1}(s) \to \infty$ as $s \to 0^+$. Hence, $H(s)$ defines a class $\mathcal{K}$ function on $[0, \infty)$. It follows that the integral

$$G(r) = \int_0^r H(s)\ ds$$

exists and both $G(r)$ and $G'(r) = H(r)$ are class $\mathcal{K}$ functions on $[0, \infty)$. Now, let $u(t)$ be a continuous nonnegative function such that $u(t) \leq g(t)$. We have

$$G'(u(t)) = \frac{\exp[-\eta^{-1}(u(t))]}{h(\eta^{-1}(u(t)))} \leq \frac{e^{-t}}{h(t)}, \quad \forall\ t \geq t_1$$

Thus,

$$\int_{t_1}^{\infty} G'(u(t))h(t)\ dt \le \int_{t_1}^{\infty} e^{-t} \le 1$$

and

$$\int_{0}^{\infty} G'(u(t))h(t)\ dt \le \int_{0}^{t_1} G'(g(t))h(t)\ dt + 1 \le k_2$$

which shows that the second integral in the lemma is bounded. For the first integral, we have

$$\int_{t_1}^{\infty} G(u(t))\ dt = \int_{t_1}^{\infty} \int_{0}^{u(t)} \frac{\exp[-\eta^{-1}(s)]}{h(\eta^{-1}(s))}\ ds\ dt \le \int_{t_1}^{\infty} \int_{0}^{\eta(t)} \frac{\exp[-\eta^{-1}(s)]}{h(0)}\ ds\ dt$$

For $0 \le s \le \eta(t)$, we have

$$-\eta^{-1}(s) \le -t$$

Hence,

$$\int_{0}^{\eta(t)} \frac{\exp[-\eta^{-1}(s)]}{h(0)}\ ds \le \int_{0}^{\eta(t)} \frac{e^{-t}}{h(0)}\ ds = \frac{e^{-t}}{h(0)}\eta(t) \le \frac{e^{-t}}{h(0)}$$

for $t \ge t_1$. Consequently,

$$\int_{0}^{\infty} G(u(t))\ dt \le \int_{0}^{t_1} G(g(t))\ dt + \int_{t_1}^{\infty} \frac{e^{-t}}{h(0)}\ dt \le k_1$$

Therefore, the first integral of the lemma is bounded. The proof of the lemma is complete. □

To prove the theorem, let

$$V(t,x) = \int_{t}^{\infty} G(\|\phi(\tau;t,x)\|_2)\ d\tau$$

where $\phi(\tau;t,x)$ is the solution that starts at (t,x) and G is a class $\mathcal{K}$ function to be selected using Lemma C.1. To see how to choose G, let us start by checking the upper bound on

$$\frac{\partial V}{\partial x} = \int_{t}^{\infty} G'(\|\phi\|_2)\frac{\phi^T}{\|\phi\|_2}\phi_x\ d\tau$$

We saw in the proof of Theorem 4.14 that the assumption $\|\partial f/\partial x\|_2 \le L$, uniformly in t, implies that $\|\phi_x(\tau;t,x)\|_2 \le \exp[L(\tau - t)]$. Therefore,

$$\begin{aligned}
\left\|\frac{\partial V}{\partial x}\right\|_2 &\le \int_{t}^{\infty} G'(\|\phi(\tau;t,x)\|_2)\exp[L(\tau-t)]\ d\tau \\
&\le \int_{t}^{\infty} G'(\beta(\|x\|_2, \tau-t))\exp[L(\tau-t)]\ d\tau \\
&\le \int_{0}^{\infty} G'(\beta(\|x\|_2, s))\exp(Ls)\ ds
\end{aligned}$$

With $\beta(r_0, s)$ and $\exp(Ls)$ as the functions g and h of Lemma C.1, we take G as the class $\mathcal{K}$ function provided by the lemma. Hence, the integral

$$\int_0^\infty G'(\beta(\|x\|_2, s)) \exp(Ls)\; ds \overset{\text{def}}{=} \alpha_4(\|x\|_2)$$

is bounded for all $\|x\|_2 \le r_0$, uniformly in x. Moreover, it is a continuous and strictly increasing function of $\|x\|_2$, because $\beta(\|x\|_2, s)$ is a class $\mathcal{K}$ function in $\|x\|_2$ for every fixed s. Thus, α_4 is a class $\mathcal{K}$ function, which proves the last inequality in the theorem statement. Consider now

$$\begin{aligned} V(t,x) &= \int_t^\infty G(\|\phi(\tau; t, x)\|_2)\; d\tau \\ &\le \int_t^\infty G(\beta(\|x\|_2, \tau - t))\; d\tau \;=\; \int_0^\infty G(\beta(\|x\|_2, s))\; ds \;\overset{\text{def}}{=}\; \alpha_2(\|x\|_2) \end{aligned}$$

By Lemma C.1, the last integral is bounded for all $\|x\|_2 \le r_0$. The function α_2 is a class $\mathcal{K}$ function. Recall from the proof of Theorem 4.14 that the assumption $\|\partial f/\partial x\|_2 \le L$, uniformly in t, implies that $\|\phi(\tau; t, x)\|_2 \ge \|x\|_2 \exp[-L(\tau - t)]$. Therefore,

$$\begin{aligned} V(t,x) &\ge \int_t^\infty G(\|x\|_2 e^{-L(\tau - t)})\; d\tau \;=\; \int_0^\infty G(\|x\|_2 e^{-Ls})\; ds \\ &\ge \int_0^{(\ln 2)/L} G(\tfrac{1}{2}\|x\|_2)\; ds \;=\; \frac{\ln 2}{L} G(\tfrac{1}{2}\|x\|_2) \;\overset{\text{def}}{=}\; \alpha_1(\|x\|_2) \end{aligned}$$

Clearly, $\alpha_1(\|x\|_2)$ is a class $\mathcal{K}$ function. Hence, V satisfies the inequality

$$\alpha_1(\|x\|_2) \le V(t,x) \le \alpha_2(\|x\|_2)$$

for all $\|x\|_2 \le r_0$. Finally, the derivative of V along the trajectories of the system is given by

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) = \\ &\qquad -G(\|x\|_2) + \int_t^\infty G'(\|\phi\|_2) \frac{\phi^T}{\|\phi\|_2} [\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x)]\; d\tau \end{aligned}$$

Since

$$\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x) \equiv 0, \quad \forall\; \tau \ge t$$

we have

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) = -G(\|x\|_2)$$

Consequently, the three inequalities of the theorem statement are satisfied for all $\|x\|_2 \le r_0$. Notice that, due to the equivalence of norms, we can state the inequalities in any p-norm. If the system is autonomous, the solution depends only on $\tau - t$;

that is, $\phi(\tau; t, x) = \psi(\tau - t; x)$. Therefore,

$$V = \int_t^\infty G(\|\psi(\tau - t; x)\|_2)\ d\tau = \int_0^\infty G(\|\psi(s; x)\|_2)\ ds$$

which is independent of t.

C.8 Proof of Theorem 4.17

For any given solution $x(t)$ of

$$\dot{x} = f(x), \qquad x(0) = x_0 \in R_A \tag{C.11}$$

the change of time variable from t to $\tau = \int_0^t (1 + \|f(x(s))\|)\ ds$ results in the system

$$\frac{d\bar{x}}{d\tau} = \frac{1}{1 + \|f(\bar{x})\|} f(\bar{x}) \overset{\text{def}}{=} \bar{f}(\bar{x}), \quad \bar{x}(0) = x_0 \tag{C.12}$$

where $\bar{x}(\tau) = x(t)$ with t substituted in terms of τ. The origin is an asymptotically stable equilibrium point of (C.12), and R_A is its region of attraction. If $V(x)$ is a Lyapunov function for (C.12) that satisfies

$$\frac{\partial V}{\partial x} \bar{f}(x) \leq -W(x)$$

for some positive definite function $W(x)$, then

$$\frac{\partial V}{\partial x} f(x) = (1 + \|f(x)\|) \frac{\partial V}{\partial x} \bar{f}(x) \leq -(1 + \|f(x)\|) W(x) \leq -W(x)$$

because $1 + \|f(x)\| \geq 1$. Therefore, it is sufficient to construct a Lyapunov function for (C.12). It is easier to work with (C.12) because the property $\|\bar{f}\| \leq 1$ implies that there is no finite escape time for $t < 0$. For the rest of the proof, we work with (C.12), which we rewrite as

$$\dot{x} = \bar{f}(x) \tag{C.13}$$

by dropping the bar from $\bar{x}$ and using $\dot{x}$ to denote the derivative with respect to τ.

We know from Lemma 8.1 that R_A is an open set. When $R_A \neq R^n$, let F be the complement of R_A in R^n. For every $x \in R_A$ define

$$\omega(x) = \max\left\{ \|x\|,\ \frac{1}{\text{dist}(x, F)} - \frac{2}{\text{dist}(0, F)} \right\} \tag{C.14}$$

if $R_A \neq R^n$ and $\omega(x) = \|x\|$ if $R_A = R^n$. It is easy to verify that $\omega(x)$ is positive definite and locally Lipschitz. Since $\text{dist}(x, F) \to 0$ as x approaches ∂R_A, $\omega(x) \to \infty$ as x approaches ∂R_A. Moreover, for $r_0 = (1/2)\text{dist}(0, F)$, we have

$$\inf_{y \in F} \{\|x - y\|\} \geq \inf_{y \in F} \{\|y\| - \|x\|\} \geq \inf_{y \in F} \{\|y\| - r_0\}, \quad \forall\ \|x\| \leq r_0$$

Hence,

$$\text{dist}(x, F) \geq \text{dist}(0, F) - \tfrac{1}{2}\text{dist}(0, F) = \tfrac{1}{2}\text{dist}(0, F), \quad \forall\ \|x\| \leq r_0$$

Therefore, $\omega(x) = \|x\|$ for all $\|x\| \leq r_0$.

Lemma C.2 *The solution of* (C.13) *satisfies*

$$\omega(x(t)) \leq \beta(\omega(x(0)), t), \quad \forall\ t \geq 0,\ \forall\ x(0) \in R_A \tag{C.15}$$

where $\beta(r, s)$ is a class $\mathcal{KL}$ function defined for all $r \geq 0$ and $s \geq 0$, and $\beta(r, 0)$ is a class $\mathcal{K}_\infty$ function. ◇

Proof of Lemma C.2: We start by showing that for any constant $r > 0$, there is a constant $b = b(r) > 0$ such that the solution of (C.13) with $\omega(x(0)) \leq r$ satisfies $\omega(x(t)) \leq b$ for all $t \geq 0$. Suppose to the contrary that this is not the case, then there is a sequence of solutions $x^{(i)}(t)$ of (C.13) and constants t_i such that $\omega(x^{(i)}(0)) \leq r$ for $i = 1, 2, 3, \cdots$ and $\omega(x^{(i)}(t_i) > i$ for $i = 1, 2, 3, \cdots$. Let T^* be the supremum of all $T \geq 0$ such that $x^{(i)}(t)$, for $i = 1, 2, 3, \cdots$, are finite for all $t \in [0, T]$; that is,

$$T^* = \sup\{T \geq 0 \mid \limsup_{i\to\infty}\{\max_{0\leq t\leq T}\{\omega(x^{(i)}(t))\} < \infty\} \tag{C.16}$$

We consider the cases $T^* < \infty$ and $T^* = \infty$ separately. When $T^* < \infty$, let τ_i be a sequence of positive constants such that for every $i \geq 1$ the solution of (C.13) satisfies

$$\omega(x(0)) \leq i \ \Rightarrow\ \omega(x(t)) \leq i + 1,\ \forall\ 0 \leq t \leq 2\tau_i \tag{C.17}$$

This sequence is not empty because of continuity of $x(t)$. We can always choose τ_i such that $T^* > \tau_1 > \tau_2 > \cdots$ and $\lim_{i\to\infty} \tau_i = 0$. Let $x^{(1,i)}(t)$ be the sequence of all functions $x^{(i)}(t)$ such that $\omega(x^{(i)}(T^* - \tau_1)) > 1$. The functions $x^{(1,i)}(t)$ form an infinite sequence. To see this fact, let I_1 be the set of indices i such that $x^{(i)}(t)$ does not belong to the sequence $x^{(1,i)}(t)$. The choice of τ_1 implies that

$$\omega(x^{(i)}(t)) \leq 2 \ \text{ for } T^* - \tau_1 \leq t \leq T^* + \tau_1,\ \forall\ i \in I \tag{C.18}$$

If there was only a finite number of functions $x^{(i)}(t)$ in the sequence $x^{(1,i)}(t)$, (C.18) would imply that

$$\limsup_{i\to\infty}\{\max_{0\leq t\leq T^*+\tau_1}\{\omega(x^{(i)}(t))\}\} < \infty \tag{C.19}$$

which contradicts the definition (C.16) of T^*. Thus, the sequence $x^{(1,i)}(t)$ is infinite. Let $x^{(2,i)}(t)$ be the sequence of all functions $x^{(1,i)}(t)$ such that $\omega(x^{(1,i)}(T^* - \tau_2)) > 2$. We can repeat the foregoing argument to show that the sequence $x^{(2,i)}(t)$ is infinite. Proceeding in the same manner, we can construct a family of subsequences until we end up with a sequence $\tilde{x}^{(i)}(t)$ such that

$$\omega(\tilde{x}^{(i)}(T^* - \tau_j)) \geq j, \quad \forall\ j = 1, 2, 3, \cdots, \quad \forall\ i = 1, 2, 3, \cdots \tag{C.20}$$

Since $\|\tilde{x}^{(i)}(0)\| \leq \omega(\tilde{x}^{(i)}(0)) \leq r$ and $\|\bar{f}(x)\| \leq 1$, the solutions $\tilde{x}^{(i)}(t)$ belong to the compact set $\{\|x\| \leq r + T\}$ for any $0 < T < T^*$. Thus, the sequence $\omega(\tilde{x}^{(i)}(t))$ is bounded on the interval $t \in [0, T]$, uniformly in i. We can select from the sequence $\tilde{x}^{(i)}(t)$ a subsequence $\bar{x}^{(i)}(t)$ that converges uniformly on the interval $[0, T]$, $0 < T < T^*$, to a solution $x(t)$, defined for $t \in [0, T^*)$. From (C.20) and the fact that $\lim_{i\to\infty} \tau_i = 0$, we conclude that $\lim_{t\to T^*} \omega(x(t)) = \infty$. Similarly, when $T^* = \infty$, the solutions $x^{(i)}(t)$ belong to the compact set $\{\|x\| \leq r + T\}$ for any $T > 0$. Hence, we can select a subsequence $\bar{x}^{(i)}(t)$ that converges uniformly on the interval $[0, T]$ to a solution $x(t)$, defined for $t \in [0, \infty)$, with $\lim_{t\to\infty} \omega(\bar{x}(t)) = \infty$. Thus, we have shown that there is a constant T^*, $0 < T^* \leq \infty$, and a solution $x(t)$ such that $\omega(x(0)) \leq r$ and $\lim_{t\to T^*} \omega(x(t)) = \infty$. However, this is impossible, because $x(0) \in R_A$. Therefore, we conclude that for any $r > 0$, there is $b = b(r) > 0$ such that the solution of (C.13) with $\omega(x(0)) \leq r$ satisfies $\omega(x(t)) \leq b$ for all $t \geq 0$. The constant $b(r)$ can be chosen as an increasing function of r and, in view of the fact that the origin is stable, $b(r) \to 0$ as $r \to 0$. Moreover, $b(r) \to \infty$ as $r \to \infty$, since $b(r) \geq r$. We can find a class $\mathcal{K}_\infty$ function $\alpha(r)$ such that $b(r) \leq \alpha(r)$ for all $r \geq 0$. Thus, the solution of (C.13) satisfies

$$\omega(x(t)) \leq \alpha(\omega(x(0))), \quad \forall\, t \geq 0, \ \forall\, x(0) \in R_A \tag{C.21}$$

On the other hand, given any positive constants r and η, we can show that there is $T = T(\eta, r) > 0$ such that

$$\omega(x(0)) < r \ \Rightarrow \ \omega(x(t)) < \eta, \ \forall\, t \geq T \tag{C.22}$$

If this were not the case, there would exist a sequence of solutions $x^{(i)}(t)$ of (C.13) and constants τ_i such that

$$\lim_{i\to\infty} \tau_i = \infty, \quad \omega(x^{(i)}(0)) \leq r, \quad \text{and} \quad \omega(x^{(i)}(\tau_i)) \geq \eta$$

However, from (C.21), we see that for any positive constant $\delta < \alpha^{-1}(\eta)$, every solution of (C.13) with $\omega(x(\tau)) \leq \delta$, satisfies $\omega(x(t)) < \eta$ for all $t \geq \tau$. Hence, $\omega(x^{(i)}(t)) \geq \delta$ for $0 \leq t \leq \tau_i$. Because $\omega(x^{(i)}(t)) \leq \alpha(r)$ for all $t \geq 0$, we select from the sequence $x^{(i)}(t)$ a subsequence $\tilde{x}^{(i)}(t)$ that converges uniformly in every interval $[0, T]$ with $0 < T < \infty$. The function $x(t) = \lim_{i\to\infty} \tilde{x}^{(i)}(t)$ is a solution of (C.13) for which $\omega(x(t)) \geq \delta$ for all $t \geq 0$. This, however, is impossible since $x(0) \in R_A$. Thus, $T(\eta, r)$ exists. By repeating the steps of the proof of Lemma 4.5 (the global uniform asymptotic stability case), we can use (C.21) and (C.22) to show that there is a class $\mathcal{KL}$ function $\beta(r, s)$ with $\beta(r, 0) \geq \alpha(r)$ such that (C.15) is satisfied. □

Let $\phi(t; x)$ denote the solution of (C.13) that starts from x at $t = 0$. Because $\|\bar{f}(x)\|$ is bounded, $\phi(t; x)$ is defined for all $t \leq 0$. Moreover, since R_A is an invariant set (Lemma 8.1), $\phi(t; x) \in R_A$ for all $t \leq 0$. Define $g : R_A \to R$ by

$$g(x) = \inf_{t\leq 0} \{\omega(\phi(t; x))\} \tag{C.23}$$

It is clear from the definition that

$$g(\phi(t;x)) \le g(x), \quad \forall\, t \ge 0,\ \forall\, x \in R_A \tag{C.24}$$

$$\alpha^{-1}(\omega(x)) \le g(x) \le \omega(x), \quad \forall\, x \in R_A \tag{C.25}$$

The first inequality of (C.25) holds because, from (C.21), $\omega(x) \le \alpha(\omega(\phi(t;x)))$, $\forall\, t \le 0$. Let us show that $g(x)$ is locally Lipschitz for $x \in R_A$, $x \ne 0$. Equivalently, we show that $g(x)$ is Lipschitz on the compact set $H = \{x \in R_A \mid c_1 \le \omega(x) \le c_2\}$, with $c_2 > c_1 > 0$. Inequality (C.15) implies that

$$c_1 \le \omega(x) \le \beta(\omega(\phi(t;x)), -t), \quad \forall\, t < 0,\ \forall\, x \in H$$

Let T_1 satisfy $\beta(2c_2, T_1) = c_1$. Then, for all $t \le -T_1$,

$$\beta(2c_2, T_1) = c_1 \le \beta(\omega(\phi(t;x)), -t) \le \beta(\omega(\phi(t;x)), T_1)$$

Hence,

$$\omega(\phi(t;x)) \ge 2c_2 \ge 2\omega(x) \ge 2g(x), \quad \forall\, t \le -T_1,\ \forall\, x \in H$$

This shows that, for all $x \in H$, the infimum defining $g(x)$ is reached within the interval $[-T_1, 0]$. Using the facts that $\phi(t;x)$ is Lipschitz in x on H for any compact time interval (Theorem 3.4) and ω is locally Lipschitz on R_A, we conclude that $g(x)$ is Lipschitz on H. Notice that this argument does not hold for $c_1 = 0$ and that is why we do not show that $g(x)$ is locally Lipschitz at $x = 0$. However, $g(x)$ is continuous for all $x \in R_A$. This is so because $g(0) = 0$, $g(x) \le \omega(x)$ (from (C.25)), and $\omega(x)$ is continuous.

Define the function $\tilde{V} : R_A \to R$ by

$$\tilde{V}(x) = \sup_{t \ge 0} \left\{ g(\phi(t;x)) \frac{1+2t}{1+t} \right\} \tag{C.26}$$

Using (C.24) and (C.25), it is easy to verify that

$$\alpha^{-1}(\omega(x)) \le g(x) \le \tilde{V}(x) \le 2g(x) \le 2\omega(x) \tag{C.27}$$

Let us show that $\tilde{V}(x)$ is locally Lipschitz for $x \in R_A$, $x \ne 0$, by showing that $\tilde{V}(x)$ is Lipschitz on $H = \{x \in R_A \mid c_1 \le \omega(x) \le c_2\}$ with $0 < c_1 < c_2$. Using (C.15) and (C.25), we have

$$g(\phi(t;x)) \frac{1+2t}{1+t} \le 2\omega(\phi(t;x)) \le 2\beta(\omega(x), t) \le 2\beta(c_2, t)$$

for all $x \in H$. Let $T_2 > 0$ satisfy $4\beta(c_2, T_2) = \alpha^{-1}(c_1)$. Then, for all $t \ge T_2$,

$$g(\phi(t;x)) \frac{1+2t}{1+t} \le \tfrac{1}{2}\alpha^{-1}(c_1) \le \tfrac{1}{2}\alpha^{-1}(\omega(x)) \le \tfrac{1}{2}\tilde{V}(x)$$

Thus, the supremum defining $\tilde{V}(x)$ is reached within the interval $[0, T_2]$. By repeating the argument used with $g(x)$, we can show that $\tilde{V}(x)$ is Lipschitz on H. Since $\tilde{V}(0) = 0$, it follows from (C.27) that $\tilde{V}(x)$ is continuous for all $x \in R_A$.

Next, we show that $\tilde{V}(x(t))$ is decreasing along the solution of (C.13). Because $\tilde{V}(x)$ is only locally Lipschitz, its derivative along the solution of (C.13) is calculated by using

$$\dot{\tilde{V}}(x) = \lim\sup_{h\to 0^+} \frac{1}{h}[\tilde{V}(\phi(h;x)) - \tilde{V}(x)] \tag{C.28}$$

For $x \neq 0$, take $r > \omega(x)$. From the properties of class $\mathcal{KL}$ functions, we can find a function $\gamma_r(\rho)$, defined for $0 < \rho < \infty$ and $0 < r < \infty$, such that for each fixed r, $\gamma_r(\rho)$ is continuous and decreasing in ρ, for each fixed ρ, $\gamma_r(\rho)$ is increasing in r, and $4\beta(r, \gamma_r(\rho)) \le \alpha^{-1}(\rho/2)$ for all $0 < \rho < \infty$. Let h_0 be such that $\omega(\phi(t;x)) \ge (1/2)\omega(x)$ for all $t \in [0, h_0]$ and pick $h \in [0, h_0]$. Then,

$$\tilde{V}(\phi(h;x)) = \sup_{t\ge 0}\left\{g(\phi(t;\phi(h;x)))\frac{1+2t}{1+t}\right\} = \sup_{t\ge 0}\left\{g(\phi(t+h;x))\frac{1+2t}{1+t}\right\}$$

Using

$$g(\phi(t+h;x))\frac{1+2t}{1+t} \le 2\omega(\phi(t+h;x)) \ \le\ 2\beta(\omega(x), t+h) < 2\beta(r, t+h)$$

we see that for all $t + h \ge \gamma_r(\omega(x))$,

$$g(\phi(t+h;x))\frac{1+2t}{1+t} \le \tfrac{1}{2}\alpha^{-1}\left(\tfrac{1}{2}\omega(x)\right) \le \tfrac{1}{2}\alpha^{-1}(\omega(\phi(h;x))) \le \tfrac{1}{2}\tilde{V}(\phi(h;x))$$

Hence, the supremum defining $\tilde{V}(\phi(h;x))$ is reached at some time t' such that $t' + h \le \gamma_r(\omega(x))$. Therefore,

$$\begin{aligned}
\tilde{V}(\phi(h;x)) &= g(\phi(t'+h;x))\frac{1+2t'}{1+t'} \\
&= g(\phi(t'+h;x))\frac{1+2t'+2h}{1+t'+h}\left[1 - \frac{h}{(1+2t'+2h)(1+t')}\right] \\
&\le \tilde{V}(x)\left[1 - \frac{h}{2[1+\gamma_r(\omega(x)]^2}\right]
\end{aligned}$$

Setting

$$\eta_r(s) = \frac{\alpha^{-1}(s)}{2[1+\gamma_r(s)]^2}$$

when $s > 0$ and $\eta_r(0) = 0$, it can be easily verified that $\eta_r(s)$ is a class $\mathcal{K}_\infty$ function and

$$\dot{\tilde{V}}(x) \le -\eta_r(\omega(x))$$

Since the preceding inequality holds for all $r > \omega(x)$, it will hold with $\bar{\eta}(s) = \sup_{r>s} \eta_r(s)$; that is, $\dot{\tilde{V}}(x) \le -\bar{\eta}(\omega(x))$ for all $x \ne 0$. Define

$$\eta(s) = \int_{2s}^{2s+1} \eta_r(s)\ dr$$

when $s > 0$ and $\eta(0) = 0$. Then $\eta(s)$ is a continuous, positive definite function on $[0, \infty)$ and $\eta(s) \le \bar{\eta}(s)$. It follows that

$$\dot{\tilde{V}}(x) \le -\eta(\omega(x)), \quad \forall\ x \in R_A,\ x \ne 0 \tag{C.29}$$

The function $\tilde{V}(x)$ satisfies all the conditions stated in Theorem 4.17, except that it is not smooth. To finish the proof, we smooth out $\tilde{V}(x)$ by using the next two lemmas, which we quote without proof.

Lemma C.3 *Let D be an open subset of R^n, and suppose there are locally Lipschitz functions $\Phi : D \to R$ and $g : D \to R^n$ and a continuous function $\psi : D \to R$ such that the derivative of $\Phi(x)$ along the trajectories of $\dot{x} = g(x)$ satisfies $\dot{\Phi}(x) \le \psi(x)$ for all $x \in D$. Then, given any continuous functions $\mu : D \to (0, \infty)$ and $\nu : D \to (0, \infty)$, there is a smooth function $\Psi : D \to R$ such that $|\Phi(x) - \Psi(x)| \le \mu(x)$ and $\dot{\Psi}(x) \le \psi(x) + \nu(x)$ for all $x \in D$.* ◇

Proof of Lemma C.3: See [118, Theorem B.1].

Lemma C.4 *Let $D \subset R^n$ be a domain that contains the origin and $\Phi : D \to [0, \infty)$ be a locally Lipschitz, positive definite function such that $\Phi(x)$ is smooth for $x \ne 0$. Then, there exists a class $\mathcal{K}_\infty$ function σ, smooth on $(0, \infty)$, such that $\sigma^{(i)}(r) \to 0$ as $r \to 0^+$ for each $i = 0, 1, \cdots$, $\sigma'(r) > 0$ for all $r > 0$, and $\Psi(x) = \sigma(\Phi(x))$ is smooth on D.* ◇

Proof of Lemma C.4: See [118, Lemma 4.3] (with the domain restricted to D instead of R^n).

Apply Lemma C.3 with $D = R_A - \{0\}$, $\Phi(x) = \tilde{V}(x)$, $g(x) = \tilde{f}(x)$, $\psi(x) = -\eta(\omega(x))$, $\mu(x) = (1/2)\alpha^{-1}(\omega(x))$, and $\nu(x) = (1/2)\eta(\omega(x))$ to find a function $\hat{V}(x)$, smooth on $R_A - \{0\}$, such that

$$\hat{\alpha}_1(\omega(x)) \le \hat{V}(x) \le \hat{\alpha}_2(\omega(x)) \quad \text{and} \quad \dot{\hat{V}}(x) \le -\hat{\alpha}_3(\omega(x))$$

where $\hat{\alpha}_1(r) = (1/2)\alpha^{-1}(r)$ and $\hat{\alpha}_2(r) = 2r + (1/2)\alpha^{-1}(r)$ are class $\mathcal{K}_\infty$ functions and $\hat{\alpha}_3(r) = (1/2)\eta(r)$ is continuous and positive definite on $[0, \infty)$. Now apply Lemma C.4 with $D = R_A$ and $\Phi = \hat{V}$ to find a class $\mathcal{K}_\infty$ function σ such that $V(x) = \sigma(\hat{V}(x))$ is smooth on R_A. It is easy to verify that $\alpha_i(r) = \sigma(\hat{\alpha}_i(r))$, $i = 1, 2$, are class $\mathcal{K}_\infty$ functions,

$$\alpha_3(r) = \hat{\alpha}_3(r) \min_{t \in [\hat{\alpha}_1(r), \hat{\alpha}_2(r)]} \sigma'(t)$$

is continuous and positive definite on $[0, \infty)$,

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x))$$

and

$$\dot{V}(x) = \sigma'(\hat{V}(x))\dot{\hat{V}}(x) \le -\sigma'(\hat{V}(x))\hat{\alpha}_3(\omega(x)) \le -\alpha_3(\omega(x))$$

The function V satisfies all the conditions stated in Theorem 4.17. The fact that for any $c > 0$ the set $\{V(x) \le c\}$ is a compact subset of R_A follows from $\{V(x) \le c\} \subset \{\omega(x) \le \alpha_1^{-1}(c)\}$.

C.9 Proof of Theorem 4.18

The statement of this theorem reduces to that of Theorem 4.9 when $\mu = 0$. Therefore, the current proof shares some ideas and terminology with the proof of Theorem 4.9. Let $\rho = \alpha_1(r)$. Then $\alpha_2(\mu) < \rho$ and $\alpha_2(\|x(t_0)\|) \le \rho$. Let $\eta = \alpha_2(\mu)$ and define $\Omega_{t,\eta} = \{x \in B_r \mid V(t,x) \le \eta\}$ and $\Omega_{t,\rho} = \{x \in B_r \mid V(t,x) \le \rho\}$. Then

$$B_\mu \subset \Omega_{t,\eta} \subset \{\alpha_1(\|x\|) \le \eta\} \subset \{\alpha_1(\|x\|) \le \rho\} = B_r \subset D$$

and

$$\Omega_{t,\eta} \subset \Omega_{t,\rho} \subset B_r \subset D$$

The sets $\Omega_{t,\rho}$ and $\Omega_{t,\eta}$ have the property that a solution starting in either set cannot leave it because $\dot{V}(t,x)$ is negative on the boundary. Since

$$\alpha_2(\|x(t_0)\|) \le \rho \;\Rightarrow x(t_0) \in \Omega_{t_0,\rho}$$

we conclude that $x(t) \in \Omega_{t,\rho}$ for all $t \ge t_0$. A solution starting in $\Omega_{t,\rho}$ must enter $\Omega_{t,\eta}$ in finite time because in the set $\{\Omega_{t,\rho} - \Omega_{t,\eta}\}$, $\dot{V}$ satisfies

$$\dot{V}(t,x) \le -k < 0$$

where $k = \min\{W_3(x)\}$ over the set $\{\mu \le \|x\| \le r\}$, which contains $\{\Omega_{t,\rho} - \Omega_{t,\eta}\}$. The foregoing inequality implies that

$$V(t,x(t)) \le V(t_0,x(t_0)) - k(t-t_0) \le \rho - k(t-t_0)$$

which shows that $V(t,x(t))$ reduces to η within the time interval $[t_0, t_0 + (\rho - \eta)/k]$. For a solution starting inside $\Omega_{t,\eta}$, inequality (4.43) is satisfied for all $t \ge t_0$, because $\Omega_{t,\eta} \subset \{\alpha_1(\|x\|) \le \alpha_2(\mu)\}$. For a solution starting inside $\Omega_{t,\rho}$, but outside $\Omega_{t,\eta}$, let $t_0 + T$ be the first time it enters $\Omega_{t,\eta}$. For all $t \in [t_0, t_0 + T]$

$$\dot{V} \le \; -W_3(x) \le -\alpha_3(\|x\|) \le -\alpha_3\left(\alpha_2^{-1}(V)\right) \stackrel{\text{def}}{=} -\alpha(V)$$

where α_3 and α are class $\mathcal{K}$ functions. The existence of α_3 follows from Lemma 4.3. Similar to the proof of Theorem 4.9, we can show that there is a class $\mathcal{KL}$ function σ such that

$$V(t,x(t)) \le \sigma(V(t_0,x(t_0)),t-t_0), \quad \forall\ t \in [t_0,t_0+T]$$

Defining $\beta(r,s) = \alpha_1^{-1}(\sigma(\alpha_2(r),s))$, we obtain

$$\|x(t)\| \le \beta(\|x(t_0)\|,t-t_0), \quad \forall\ t \in [t_0,t_0+T]$$

If $D = R^n$, α_3, and consequently β, can be chosen independent of ρ. If α_1 belongs to class $\mathcal{K}_\infty$, so does α_2 and the bound $\alpha_2^{-1}(\rho)$ can be made arbitrarily large by choosing ρ large enough. Hence, any initial state $x(t_0)$ can be included in the set $\{\|x\| \le \alpha_2^{-1}(\rho)\}$.

C.10 Proof of Theorem 5.4

We complete the proof of Theorem 5.4 by showing that the $\mathcal{L}_2$ gain is equal to $\sup_{\omega\in R}\|G(j\omega)\|_2$. Let c_1 be the $\mathcal{L}_2$ gain and $c_2 = \sup_{\omega\in R}\|G(j\omega)\|_2$. We know that $c_1 \le c_2$. Suppose $c_1 < c_2$ and set $\varepsilon = (c_2-c_1)/3$. Then, for any $u \in \mathcal{L}_2$ with $\|u\|_{\mathcal{L}_2} \le 1$, we have $\|y\|_{\mathcal{L}_2} \le c_2 - 3\varepsilon$. We will establish a contradiction by finding a signal u with $\|u\|_{\mathcal{L}_2} \le 1$ such that $\|y\|_{\mathcal{L}_2} \ge c_2 - 2\varepsilon$. It is easier to construct such signal if we define signals on the whole real line R. There is no loss of generality in doing so due to the fact (shown in Exercise 5.19) that the $\mathcal{L}_2$ gain is the same whether signals are defined on $[0,\infty)$ or on R. Now, select $\omega_0 \in R$ such that $\|G(j\omega_0)\|_2 \ge c_2 - \varepsilon$. Let $v \in C^m$ be the normalized eigenvector ($v^*v = 1$) corresponding to the maximum eigenvalue of the Hermitian matrix $G^T(-j\omega_0)G(j\omega_0)$. Hence, $v^*G^T(-j\omega_0)G(j\omega_0)v = \|G(j\omega_0)\|_2^2$. Write v as

$$v = \left[\alpha_1 e^{j\theta_1},\ \alpha_2 e^{j\theta_2},\ \ldots\ ,\ \alpha_m e^{j\theta_m}\right]^T$$

where $\alpha_i \in R$ is such that $\theta_i \in (-\pi,0]$. Take $0 \le \beta_i \le \infty$ such that $\theta_i = -2\tan^{-1}(\omega_0/\beta_i)$, with $\beta_i = \infty$ if $\theta_i = 0$. Define the $m \times 1$ transfer function $H(s)$ by

$$H(s) = \left[\alpha_1\frac{\beta_1 - s}{\beta_1 + s},\ \alpha_2\frac{\beta_2 - s}{\beta_2 + s},\ \ldots\ ,\ \alpha_m\frac{\beta_m - s}{\beta_m + s}\right]^T$$

with 1 replacing $(\beta_i - s)/(\beta_i + s)$ if $\theta_i = 0$. It can be easily seen that $H(j\omega_0) = v$ and $H^T(-j\omega)H(j\omega) = \sum_{i=1}^m \alpha_i^2 = v^*v = 1$ for all $\omega \in R$. Take $u_\sigma(t)$ as the output of $H(s)$ driven by the scalar function

$$z_\sigma(t) = \left(\frac{1}{1+e^{-\omega_0^2\sigma/2}}\right)^{1/2}\left(\frac{8}{\pi\sigma}\right)^{1/4} e^{-t^2/\sigma}\cos(\omega_0 t), \quad \sigma > 0, \quad t \in R$$

It can be verified that $z_\sigma \in \mathcal{L}_2$ and $\|z_\sigma\|_{\mathcal{L}_2} = 1$.[4] Consequently, $u_\sigma \in \mathcal{L}_2$ and $\|u_\sigma\|_{\mathcal{L}_2} \leq 1$. The Fourier transform of $z_\sigma(t)$ is given by

$$Z_\sigma(j\omega) = \left(\frac{1}{1+e^{-\omega_0^2\sigma/2}}\right)^{1/2} \left(\frac{\pi\sigma}{2}\right)^{1/4} \left[e^{-(\omega-\omega_0)^2\sigma/4} + e^{-(\omega+\omega_0)^2\sigma/4}\right]$$

Let $y_\sigma(t)$ be the output of $G(s)$ when the input is $u_\sigma(t)$. The Fourier transform of $y_\sigma(t)$ is given by $Y_\sigma(j\omega) = G(j\omega)U_\sigma(j\omega) = G(j\omega)H(j\omega)Z_\sigma(j\omega)$. By Parseval's theorem,

$$\begin{aligned} \|y_\sigma\|_{\mathcal{L}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_\sigma^T(-j\omega)H^T(-j\omega)G^T(-j\omega)G(j\omega)H(j\omega)Z_\sigma(j\omega)\ d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H^T(-j\omega)G^T(-j\omega)G(j\omega)H(j\omega)\ |Z_\sigma(j\omega)|^2\ d\omega \end{aligned}$$

Using

$$|Z_\sigma(j\omega)|^2 \geq \frac{1}{1+e^{-\omega_0^2\sigma/2}}\left(\frac{\pi\sigma}{2}\right)^{1/2}\left[e^{-(\omega-\omega_0)^2\sigma/2} + e^{-(\omega-\omega_0)^2\sigma/2}\right] \stackrel{\text{def}}{=} \psi_\sigma(\omega)$$

we obtain

$$\|y_\sigma\|_{\mathcal{L}_2}^2 \geq \frac{1}{2\pi}\int_{-\infty}^{\infty} H^T(-j\omega)G^T(-j\omega)G(j\omega)H(j\omega)\psi_\sigma(\omega)\ d\omega$$

By letting $\sigma \to \infty$, one can concentrate the frequency spectrum $\psi_\sigma(\omega)$ around the the two frequencies $\omega = \pm\omega_0$.[5] Hence, the right-hand side of the last inequality approaches

$$H^T(-j\omega_0)G^T(-j\omega_0)G(j\omega_0)H(j\omega_0) = \|G(j\omega_0)\|_2^2 \geq (c_2 - \varepsilon)^2$$

as $\sigma \to \infty$. Therefore, we can choose a finite σ large enough such that $\|y_\sigma\|_{\mathcal{L}_2} \geq c_2 - 2\varepsilon$. This, however, contradicts the inequality $\|y_\sigma\|_{\mathcal{L}_2} \leq c_2 - 3\varepsilon$. The contradiction shows that $c_1 = c_2$.

C.11 Proof of Lemma 6.1

Sufficiency: Suppose the conditions of the lemma are satisfied. Since $G(s)$ is Hurwitz, there exist positive constants δ and μ^* such that poles of all elements of $G(s-\mu)$ have real parts less than $-\delta$, for all $\mu < \mu^*$. To show that $G(s)$ is strictly positive real, it is sufficient to show that $G(j\omega - \mu) + G^T(-j\omega - \mu)$ is positive semidefinite for all $\omega \in R$. Let $\{A, B, C, D\}$ be a minimal realization of $G(s)$. Then,

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[4] For the purpose of this discussion, the $\mathcal{L}_2$-norm is defined by $\|z\|_{\mathcal{L}_2}^2 = \int_{-\infty}^{\infty} z^T(t)z(t)\ dt$.

[5] $\psi_\sigma(\omega)$ approaches $\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$ as $\sigma \to \infty$, where $\delta(\cdot)$ is the impulse function.

$$\begin{aligned}
G(s-\mu) &= D + C(sI - \mu I - A)^{-1}B \\
&= D + C(sI - A)^{-1}(sI - A)(sI - \mu I - A)^{-1}B \\
&= D + C(sI - A)^{-1}(\mu I + sI - \mu I - A)(sI - \mu I - A)^{-1}B \\
&= G(s) + \mu N(s) \qquad \text{(C.30)}
\end{aligned}$$

where

$$N(s) = C(sI - A)^{-1}(sI - \mu I - A)^{-1}B$$

Since A and $(A + \mu I)$ are Hurwitz, uniformly in μ, there is $k_0 > 0$ such that

$$\sigma_{\max}\left[N(j\omega) + N^T(-j\omega)\right] \le k_0, \quad \forall\ \omega \in R \qquad \text{(C.31)}$$

Moreover, $\lim_{\omega\to\infty} \omega^2 N(j\omega)$ exists. Hence, there exist $k_1 > 0$ and $\omega_1 > 0$ such that

$$\omega^2 \sigma_{\max}\left[N(j\omega) + N^T(-j\omega)\right] \le k_1, \quad \forall\ |\omega| \ge \omega_1 \qquad \text{(C.32)}$$

If $G(\infty) + G^T(\infty)$ is positive definite, there is $\sigma_0 > 0$ such that

$$\sigma_{\min}\left[G(j\omega) + G^T(-j\omega)\right] \ge \sigma_0, \quad \forall\ \omega \in R \qquad \text{(C.33)}$$

From (C.30), (C.31), and (C.33),

$$\sigma_{\min}\left[G(j\omega - \mu) + G^T(-j\omega - \mu)\right] \ge \sigma_0 - \mu k_0, \quad \forall\ \omega \in R$$

Choosing $\mu < \sigma_0/k_0$ ensures that $G(j\omega - \mu) + G^T(-j\omega - \mu)$ is positive definite for all $\omega \in R$. If $G(\infty) + G^T(\infty)$ is singular, the third condition of the lemma ensures that $G(j\omega) + G^T(-j\omega)$ has q singular values with $\lim_{\omega\to\infty} \sigma_i(\omega) > 0$ and $(p - q)$ singular values with $\lim_{\omega\to\infty} \sigma_i(\omega) = 0$ and $\lim_{\omega\to\infty} \omega^2 \sigma_i(\omega) > 0$. Therefore, there exist $\sigma_1 > 0$ and $\omega_2 > 0$ such that

$$\omega^2 \sigma_{\min}\left[G(j\omega) + G^T(-j\omega)\right] \ge \sigma_1, \quad \forall\ |\omega| \ge \omega_2 \qquad \text{(C.34)}$$

From (C.30), (C.32), and (C.34),

$$\omega^2 \sigma_{\min}\left[G(j\omega - \mu) + G^T(-j\omega - \mu)\right] \ge \sigma_1 - \mu k_1, \quad \forall\ |\omega| \ge \omega_3 \qquad \text{(C.35)}$$

where $\omega_3 = \max\{\omega_1, \omega_2\}$. On the compact frequency interval $[-\omega_3, \omega_3]$, we have

$$\sigma_{\min}\left[G(j\omega) + G^T(-j\omega)\right] \ge \sigma_2 > 0 \qquad \text{(C.36)}$$

Hence, from (C.30), (C.31), and (C.36),

$$\sigma_{\min}\left[G(j\omega - \mu) + G^T(-j\omega - \mu)\right] \ge \sigma_2 - \mu k_0, \quad \forall\ |\omega| \le \omega_3 \qquad \text{(C.37)}$$

Choosing $\mu < \min\{\sigma_1/k_1, \sigma_2/k_0\}$ ensures that $G(j\omega - \mu) + G^T(-j\omega - \mu)$ is positive definite for all $\omega \in R$.

Necessity: Suppose $G(s)$ is strictly positive real. There exists $\mu > 0$ such that $G(s-\mu)$ is positive real. It follows that $G(s)$ is Hurwitz and positive real. Consequently,

$$G(j\omega) + G^T(-j\omega) \geq 0, \quad \forall \ \omega \in R$$

Therefore,

$$G(\infty) + G^T(\infty) \geq 0$$

Let $\{A, B, C, D\}$ be a minimal realization of $G(s)$. By Lemma 6.3, there exist P, L, W, and ε that satisfy (6.14) through (6.16). Let $\Phi(s) = (sI - A)^{-1}$. We have

$$G(s) + G^T(-s) = D + D^T + C\Phi(s)B + B^T\Phi^T(-s)C^T$$

Substitute for C by using (6.15) and for $D + D^T$ by using (6.16). Then,

$$\begin{aligned} G(s) + G^T(-s) &= W^TW + (B^TP + W^TL)\Phi(s)B + B^T\Phi^T(-s)(PB + L^TW) \\ &= W^TW + W^TL\Phi(s)B + B^T\Phi^T(-s)L^TW \\ &\quad + B^T\Phi^T(-s)[-A^TP - PA]\Phi(s)B \end{aligned}$$

Using (6.14) yields

$$G(s) + G^T(-s) = \left[W^T + B^T\Phi^T(-s)L^T\right] \ [W + L\Phi(s)B] + \varepsilon B^T\Phi^T(-s)P\Phi(s)B$$

From the last equation, it can be seen that $G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega \in R$, for if it were singular at some frequency ω, there would exist $x \in C^p$, $x \neq 0$, such that

$$(x^*)^T[G(j\omega) + G^T(-j\omega)]x = 0 \Rightarrow (x^*)^TB^T\Phi^T(-j\omega)P\Phi(j\omega)Bx = 0 \Rightarrow Bx = 0$$

Also,

$$(x^*)^T[G(j\omega) + G^T(-j\omega)]x = 0 \Rightarrow (x^*)^T \left[W + L\Phi(-j\omega)B\right]^T \left[W + L\Phi(j\omega)B\right] x = 0$$

Since $Bx = 0$, the preceding equation implies that $Wx = 0$. Hence,

$$(x^*)^T[G(s) + G^T(-s)]x \equiv 0, \quad \forall \ s$$

which contradicts the assumption that $\det[G(s) + G^T(-s)]$ is not identically zero. Now if $G(\infty) + G^T(\infty)$ is positive definite, we are done. Otherwise, let M be any $p \times (p-q)$ full rank matrix such that $M^T(D + D^T)M = M^TW^TWM = 0$. Then, $WM = 0$ and

$$M^T[G(j\omega) + G^T(-j\omega)]M = M^TB^T\Phi^T(-j\omega)(L^TL + \varepsilon P)\Phi(j\omega)BM$$

We note that BM must have full column rank; otherwise, there is $x \neq 0$ such that $BMx = 0$. Taking $y = Mx$ yields

$$y^T[G(j\omega) + G^T(-j\omega)]y = 0, \quad \forall\ \omega \in R$$

which contradicts the positive definiteness of $G(j\omega) + G^T(-j\omega)$. Now

$$\lim_{\omega \to \infty} \omega^2 M^T[G(j\omega) + G^T(-j\omega)]M = M^T B^T (L^T L + \varepsilon P) BM$$

The fact that BM has full column rank implies that $M^T B^T (L^T L + \varepsilon P) BM$ is positive definite.

C.12 Proof of Lemma 6.2

Sufficiency: Suppose there exist $P = P^T > 0$, L, and W that satisfy (6.11) through (6.13). Using $V(x) = x^T P x$ as a Lyapunov function for $\dot{x} = Ax$, (6.11) shows that the origin of $\dot{x} = Ax$ is stable. Hence, A has no eigenvalues in $\text{Re}[s] > 0$. Let $\Phi(s) = (sI - A)^{-1}$. We have

$$G(s) + G^T(s^*) = D + D^T + C\Phi(s)B + B^T\Phi^T(s^*)C^T$$

Substitute for C by using (6.12) and for $D + D^T$ by using (6.13). Then,

$$\begin{aligned} G(s) + G^T(s^*) &= W^T W + (B^T P + W^T L)\Phi(s)B + B^T\Phi^T(s^*)(PB + L^T W) \\ &= W^T W + W^T L\Phi(s)B + B^T\Phi^T(s^*)L^T W \\ &\quad + B^T\Phi^T(s^*)[(s + s^*)P - A^T P - PA]\Phi(s)B \end{aligned}$$

Using (6.11) yields

$$\begin{aligned} G(s) + G^T(s^*) &= \left[W^T + B^T\Phi^T(s^*)L^T\right]\ [W + L\Phi(s)B] \\ &\quad + (s + s^*)B^T\Phi^T(s^*)P\Phi(s)B \end{aligned} \tag{C.38}$$

which shows that for all s in $\text{Re}[s] \geq 0$, $G(s) + G^T(s^*) \geq 0$. It follows that, for all ω for which $j\omega$ is not a pole of any element of $G(s)$, $G(j\omega) + G^T(-j\omega)$ is positive semidefinite. It remains to show that $G(s)$ satisfies the third condition of Definition 6.4. Suppose $j\omega_0$ is a pole of order m of any element of $G(s)$. Then, for any p-dimensional complex vector x, the values taken by $(x^*)^T G(s)x$ on a semicircular arc of arbitrarily small radius ρ, centered at $j\omega_0$, are

$$(x^*)^T G(s)x \approx (x^*)^T K_0 x \rho^{-m} e^{-jm\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Therefore,

$$\rho^m \text{Re}[(x^*)^T G(s)x] \approx \text{Re}[(x^*)^T K_0 x] \cos m\theta + \text{Im}[(x^*)^T K_0 x] \sin m\theta$$

For $m > 1$, this expression could have either sign, while (C.38) implies that it is nonnegative. Hence, m must be limited to one. For $m = 1$, choosing θ near $-\pi/2$, 0, and $\pi/2$ shows that $\text{Im}[(x^*)^T K_0 x] = 0$ and $\text{Re}[(x^*)^T K_0 x] \geq 0$. Hence, K_0 is positive semidefinite Hermitian.

Necessity: We prove necessity first for the special case when A is Hurwitz. Then we extend the proof to the general case when A may have eigenvalues on the imaginary axis.

Special case: The proof uses a spectral factorization result, which we quote without proof.

Lemma C.5 *Let the $p \times p$ proper rational transfer function matrix $U(s)$ be positive real and Hurwitz. Then, there exists an $r \times p$ proper rational Hurwitz transfer function matrix $V(s)$ such that*

$$U(s) + U^T(-s) = V^T(-s)V(s) \tag{C.39}$$

where r is the normal rank of $U(s) + U^T(-s)$, that is, the rank over the field of rational functions of s. Furthermore, rank $V(s) = r$ *for* $\text{Re}[s] > 0$. △

Proof: See [214, Theorem 2].

Suppose now that $G(s)$ is positive real and Hurwitz, and recall that $\{A, B, C, D\}$ is a minimal realization of $G(s)$. From Lemma C.5, there exists an $r \times p$ transfer function matrix $V(s)$ such that (C.39) is satisfied. Let $\{F, G, H, J\}$ be a minimal realization of $V(s)$. The matrix F is Hurwitz, since $V(s)$ is Hurwitz. It can be easily seen that $\{-F^T, H^T, -G^T, J^T\}$ is a minimal realization of $V^T(-s)$. Therefore,

$$\{\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1\} = \left\{ \begin{bmatrix} F & 0 \\ H^T H & -F^T \end{bmatrix}, \begin{bmatrix} G \\ H^T J \end{bmatrix}, \begin{bmatrix} J^T H & -G^T \end{bmatrix}, J^T J \right\}$$

is a realization of the cascade connection $V^T(-s)V(s)$. By checking controllability and observability and by using the property that rank $V(s) = r$ for $\text{Re}[s] > 0$, it can be seen that this realization is minimal. Let us show the controllability test;[6] the observability test is similar. By writing

$$\begin{bmatrix} I & 0 \\ H(sI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} sI - F & G \\ -H & J \end{bmatrix} = \begin{bmatrix} sI - F & G \\ 0 & H(sI - F)^{-1}G + J \end{bmatrix} = \begin{bmatrix} sI - F & G \\ 0 & V(s) \end{bmatrix}$$

---

[6] The controllability argument uses the fact that $V^T(-s)$ has no transmission zeros at the poles of $V(s)$.

it can be seen that

$$\operatorname{rank} V(s) = r, \ \forall \operatorname{Re}[s] > 0 \quad \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - F & G \\ \\ -H & J \end{bmatrix} = n_F + r, \ \forall \operatorname{Re}[s] > 0$$

where n_F is the dimension of F. Now, show controllability of $(\mathcal{A}_1, \mathcal{B}_1)$ by contradiction. Suppose $(\mathcal{A}_1, \mathcal{B}_1)$ is not controllable. Then, there are a complex number λ and a vector $w \in C^{n_F+r}$, partitioned into n_F and r subvectors, such that

$$(w_1^*)^T F + (w_2^*)^T H^T H = \lambda (w_1^*)^T \tag{C.40}$$

$$-(w_2^*)^T F^T = \lambda (w_2^*)^T \tag{C.41}$$

$$(w_1^*)^T G + (w_2^*)^T H^T J = 0 \tag{C.42}$$

Equation (C.41) shows that $\operatorname{Re}[\lambda] > 0$, since F is Hurwitz, and (C.40) and (C.42) show that

$$\begin{bmatrix} (w_1^*)^T & (w_2^*)^T H^T \end{bmatrix} \begin{bmatrix} \lambda I - F & G \\ \\ -H & J \end{bmatrix} = 0 \ \Rightarrow \ \operatorname{rank} V(\lambda) < r$$

which contradicts the fact that rank $V(s) = r$ for $\operatorname{Re}[s] > 0$. Thus, $(\mathcal{A}_1, \mathcal{B}_1)$ is controllable.

Consider the Lyapunov equation

$$KF + F^T K = -H^T H$$

Because the pair (F, H) is observable, there is a unique positive definite solution K. This fact is shown in Exercise 4.22. Using the similarity transformation

$$\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

we obtain the following alternative minimal realization of $V^T(-s)V(s)$:

$$\{\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2, \mathcal{D}_2\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix}, \begin{bmatrix} G \\ KG + H^T J \end{bmatrix}, \begin{bmatrix} J^T H + G^T K & -G^T \end{bmatrix}, J^T J \right\}$$

On the other hand, $\{-A^T, C^T, -B^T, D^T\}$ is a minimal realization of $U^T(-s)$. Therefore,

$$\{\mathcal{A}_3, \mathcal{B}_3, \mathcal{C}_3, \mathcal{D}_3\} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ C^T \end{bmatrix}, \begin{bmatrix} C & -B^T \end{bmatrix}, D + D^T \right\}$$

is a realization of the parallel connection $U(s) + U^T(-s)$. Since the eigenvalues of A are in the open left-half plane, while the eigenvalues of $-A^T$ are in the open right-half plane, it can be easily seen that this realization is minimal. Thus, due to (C.39), $\{\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2, \mathcal{D}_2\}$ and $\{\mathcal{A}_3, \mathcal{B}_3, \mathcal{C}_3, \mathcal{D}_3\}$ are equivalent minimal realizations of the same transfer function. Therefore, they have the same dimension and there is a nonsingular matrix T such that[7]

$$\mathcal{A}_2 = T\mathcal{A}_3T^{-1}, \ \mathcal{B}_2 = T\mathcal{B}_3, \ \mathcal{C}_2 = \mathcal{C}_3T^{-1}, \ J^TJ = \mathcal{D} + \mathcal{D}^T$$

The matrix T must be a block-diagonal matrix. To see this point, partition T compatibly as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

Then, the matrix T_{12} satisfies the equation

$$FT_{12} + T_{12}A^T = 0$$

Premultiplying by $\exp(Ft)$ and postmultiplying by $\exp(A^Tt)$, we obtain

$$0 = \exp(Ft)[FT_{12} + T_{12}A^T]\exp(A^Tt) = \frac{d}{dt}[\exp(Ft)T_{12}\exp(A^Tt)]$$

Hence, $\exp(Ft)T_{12}\exp(A^Tt)$ is constant for all $t \geq 0$. In particular, since $\exp(0) = I$,

$$T_{12} = \exp(Ft)T_{12}\exp(A^Tt) \rightarrow 0 \text{ as } t \rightarrow \infty$$

Therefore, $T_{12} = 0$. Similarly, we can show that $T_{21} = 0$. Consequently, the matrix T_{11} is nonsingular and

$$F = T_{11}AT_{11}^{-1}, \ G = T_{11}B, \ J^TH + G^TK = CT_{11}^{-1}$$

Define

$$P = T_{11}^TKT_{11}, \ L = HT_{11}, \ W = J$$

It can be easily verified that P, L, and W satisfy the equations

$$PA + A^TP = -L^TL, \ \ PB = C^T - L^TW, \ \ W^TW = D + D^T$$

which completes the proof for the special case.

General case: Suppose now that A has eigenvalues on the imaginary axis. There is a nonsingular matrix Q such that

$$QAQ^{-1} = \begin{bmatrix} A_0 & 0 \\ 0 & A_n \end{bmatrix}, \ QB = \begin{bmatrix} B_0 \\ B_n \end{bmatrix}, \ CQ^{-1} = \begin{bmatrix} C_0 & C_n \end{bmatrix}$$

where A_0 has eigenvalues on the imaginary axis and A_n has eigenvalues with negative real parts. The transfer function $G(s)$ can be written as $G(s) = G_0(s) + G_n(s)$, where $G_0(s) = C_0(sI - A_0)^{-1}B_0$ has all poles on the imaginary axis, and $G_n(s) =$

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[7] See [35, Theorem 5-20].

$C_n(sI - A_n)^{-1}b_n + D$ has all poles in the open left-half plane. It follows that the poles of $G_0(s)$ are simple and the corresponding residue matrices are positive semidefinite Hermitian. Due to this property, we can choose Q such that

$$A_0 + A_0^T = 0, \quad C_0 = B_0^T \tag{C.43}$$

To see this fact, notice that $G_0(s)$ can be written as

$$G_0(s) = \frac{1}{s}F_0 + \sum_{i=1}^{m} \frac{1}{s^2 + \omega_i^2}(F_i s + H_i) = \frac{1}{s}F_0 + \sum_{i=1}^{m} \left[\frac{1}{s - j\omega_i}R_i + \frac{1}{s + j\omega_i}R_i^*\right]$$

where F_0 is positive semidefinite symmetric and R_i is positive semidefinite Hermitian. If, for each term in this summation, we can find a minimal realization that has the property (C.43), the parallel connection will be a minimal realization of $G_0(s)$ that has the same property. It is sufficient to consider the terms $(1/s)F_0$ and $[1/(s^2 + \omega_i^2)](F_i s + H_i)$. If r_0 = rank F_0, $(1/s)F_0$ has a minimal realization of dimension r_0, given by $\{0, N_0, N_0^T\}$, where $F_0 = N_0^T N_0$. If r_i = rank R_i, $[1/(s^2 + \omega_i^2)](F_i s + H_i)$ has a minimal realization of dimension $2r_i$, given by

$$A_i = \begin{bmatrix} 0 & \omega_i I \\ -\omega_i I & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} M_{i1} \\ M_{i2} \end{bmatrix}, \quad C_i = \begin{bmatrix} M_{i1}^T & M_{i2}^T \end{bmatrix}$$

where

$$M_{i1} = \frac{1}{\sqrt{2}}(N_i + N_i^*), \quad M_{i2} = \frac{j}{\sqrt{2}}(N_i - N_i^*), \quad R_i = (N_i^*)^T N_i$$

It is clear that $\{A_i, B_i, C_i\}$ has the property (C.43).

Since $G_n(s)$ is positive real and Hurwitz, from the special case treated earlier, there exist matrices $P_n = P_n > 0$, L_n, and W such that

$$P_n A_n + A_n^T P_n = -L_n^T L_n, \quad P_n B_n = C_n^T - L_n^T W, \quad W^T W = D + D^T$$

It can be easily verified that

$$P = Q^T \begin{bmatrix} I & 0 \\ 0 & P_n \end{bmatrix} Q, \quad L = \begin{bmatrix} 0 & L_n \end{bmatrix} Q$$

and W satisfy equations (6.11) through (6.13).

C.13 Proof of Lemma 7.1

We start by writing a time-domain version of the infinite-dimensional equation (7.20). Consider the space $\mathcal{S}$ of all half-wave symmetric periodic signals of fundamental frequency ω, which have finite energy on any finite interval. A signal $y \in \mathcal{S}$ can be represented by its Fourier series

$$y(t) = \sum_{k \text{ odd}} a_k \exp(jk\omega t), \quad \sum_{k \text{ odd}} |a_k|^2 < \infty$$

Define a norm on $\mathcal{S}$ by

$$\|y\|^2 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} y^2(t)\ dt = 2\sum_{k\ \text{odd}} |a_k|^2$$

With this norm, $\mathcal{S}$ is a Banach space. Define $g_k(t-\tau)$ by

$$g_k(t-\tau) = \frac{\omega}{\pi}\{G(jk\omega)\exp[jk\omega(t-\tau)] + G(-jk\omega)\exp[-jk\omega(t-\tau)]\}$$

For odd integers m and $k > 0$, we have

$$\int_0^{\pi/\omega} g_k(t-\tau)\exp(jm\omega\tau)\ d\tau = \begin{cases} G(jk\omega)\exp(jk\omega t), & \text{if } m = k \\ G(-jk\omega)\exp(-jk\omega t), & \text{if } m = -k \\ 0, & \text{if } |m| \neq k \end{cases} \tag{C.44}$$

Define the linear mapping g and the nonlinear mapping $g\psi$ on $\mathcal{S}$ by

$$\begin{aligned} gy &= \int_0^{\pi/\omega} \sum_{k\ \text{odd};\ k>0} g_k(t-\tau)y(\tau)\ d\tau \\ g\psi y &= \int_0^{\pi/\omega} \sum_{k\ \text{odd};\ k>0} g_k(t-\tau)\psi(y(\tau))\ d\tau \end{aligned}$$

where

$$y(t) = \sum_{k\ \text{odd}} a_k\exp(jk\omega t) \quad \text{and} \quad \psi(y(t)) = \sum_{k\ \text{odd}} c_k\exp(jk\omega t)$$

Using (C.44), it can be seen that

$$\begin{aligned} gy &= \sum_{k\ \text{odd}} G(jk\omega)a_k\exp(jk\omega t) \\ g\psi y &= \sum_{k\ \text{odd}} G(jk\omega)c_k\exp(jk\omega t) \end{aligned}$$

With these definitions, the condition for existence of half-wave symmetric periodic oscillation can be written as

$$y = -g\psi y \tag{C.45}$$

Equation (C.45) is equivalent to (7.20). In order to separate the effect of the higher harmonics from that of the first harmonic, define a mapping P_1 by

$$P_1 y = y_1 = a_1\exp(j\omega t) + \bar{a}_1\exp(-j\omega t) = 2\text{Re}[a_1\exp(j\omega t)]$$

and a mapping P_h by

$$P_h y = y_h = y - y_1 = \sum_{k\ \text{odd};\ |k|\neq 1} a_k\exp(jk\omega t)$$

Without loss of generality, we take $a_1 = a/2j$ so that $y_1(t) = a\sin\omega t$. Solving (C.45) is equivalent to solving both (C.46) and (C.47):

$$\begin{aligned} y_h &= -P_h g\psi(y_1 + y_h) && \text{(C.46)} \\ y_1 &= -P_1 g\psi(y_1 + y_h) && \text{(C.47)} \end{aligned}$$

By evaluating the right-hand side of (C.47), it can be seen that this equation is equivalent to (7.35). The error term $\delta\Psi$, defined after (7.36), satisfies

$$P_1 g\psi y_1 - P_1 g\psi(y_1 + y_h) = 2\mathrm{Re}[G(j\omega)a_1\delta\Psi\exp(j\omega t)] \tag{C.48}$$

Thus, to obtain a bound on $\delta\Psi$, we need to find a bound on y_h, which we will find by using the contraction mapping theorem, without solving (C.46). By adding $[P_h g(\beta+\alpha)/2]y_h$ to both sides of (C.46), we can rewrite it as

$$\left(I + P_h g\frac{\beta+\alpha}{2}\right) y_h = -P_h g\left[\psi(y_1 + y_h) - \frac{\beta+\alpha}{2}y_h\right] \tag{C.49}$$

Let us consider the linear mapping $K = I + P_h g(\beta+\alpha)/2$, which appears on the left-hand side of (C.49). It maps $\mathcal{S}$ into $\mathcal{S}$. Given any $z \in \mathcal{S}$ defined by

$$z(t) = \sum_{k \text{ odd}} b_k \exp(jk\omega t)$$

we consider the linear equation $Kx = z$ and seek a solution x in $\mathcal{S}$. Representing x as

$$x(t) = \sum_{k \text{ odd}} d_k \exp(jk\omega t)$$

we have

$$\left(I + P_h g\frac{\beta+\alpha}{2}\right) x = x_1 + \sum_{k \text{ odd};\ |k|\neq 1} \left[1 + \frac{\beta+\alpha}{2}G(jk\omega)\right] d_k \exp(jk\omega t)$$

Hence, the linear equation $Kx = z$ has a unique solution if

$$\inf_{k \text{ odd};\ |k|\neq 1} \left|1 + \frac{\beta+\alpha}{2}G(jk\omega)\right| \neq 0 \tag{C.50}$$

In other words, condition (C.50) guarantees that the linear mapping K has an inverse. This condition is always satisfied if $\omega \in \Omega$, because the left-hand side of (C.50) can vanish only if $\rho(\omega) = 0$. Denote the inverse of K by K^{-1} and rewrite (C.49) as

$$y_h = -K^{-1}P_h g\left[\psi(y_1 + y_h) - \frac{\beta+\alpha}{2}(y_1 + y_h)\right] \overset{\text{def}}{=} Ty_h$$

where we have used the fact that $P_h g y_1 = 0$. We would like to apply the contraction mapping theorem to the equation $y_h = T y_h$. Clearly, T maps $\mathcal{S}$ into $\mathcal{S}$. We need to verify that T is a contraction on $\mathcal{S}$. To that end, consider

$$Ty^{(2)} - Ty^{(1)} = K^{-1} P_h g \left[\psi_T(y_1 + y^{(2)}) - \psi_T(y_1 + y^{(1)}) \right]$$

where

$$\psi_T(y) = \psi(y) - \frac{\beta + \alpha}{2} y$$

Let

$$\psi_T(y_1 + y^{(2)}) - \psi_T(y_1 + y^{(1)}) = \sum_{k \text{ odd; } |k| \neq 1} e_k \exp(jk\omega t)$$

Then,

$$\begin{aligned} \left\| Ty^{(2)} - Ty^{(1)} \right\|^2 &= 2 \sum_{k \text{ odd; } |k| \neq 1} \left| \frac{G(jk\omega)}{1 + [(\beta+\alpha)/2] G(jk\omega)} \right|^2 |e_k|^2 \\ &\leq \left\{ \sup_{k \text{ odd; } |k| \neq 1} \left| \frac{G(jk\omega)}{1 + [(\beta+\alpha)/2] G(jk\omega)} \right| \right\}^2 \\ &\quad \times \left\| \psi_T(y_1 + y^{(2)}) - \psi_T(y_1 + y^{(1)}) \right\|^2 \end{aligned}$$

Due to the slope restriction on ψ, we have

$$\left| \psi_T(y_1 + y^{(2)}) - \psi_T(y_1 + y^{(1)}) \right| \leq \left(\frac{\beta - \alpha}{2} \right) \left| y^{(2)} - y^{(1)} \right|$$

Moreover,

$$\sup_{k \text{ odd; } |k| \neq 1} \left| \frac{G(jk\omega)}{1 + [(\beta+\alpha)/2] G(jk\omega)} \right| \leq \frac{1}{\rho(\omega)}$$

where $\rho(\omega)$ is defined by (7.38). Hence,

$$\left\| Ty^{(2)} - Ty^{(1)} \right\| \leq \frac{1}{\rho(\omega)} \left(\frac{\beta - \alpha}{2} \right) \left\| y^{(2)} - y^{(1)} \right\|$$

Since

$$\frac{1}{\rho(\omega)} \left(\frac{\beta - \alpha}{2} \right) < 1, \quad \forall\, \omega \in \Omega$$

we conclude that as long as $\omega \in \Omega$, T is a contraction mapping. Thus, by the contraction mapping theorem, the equation $y_h = T y_h$ has a unique solution. Noting that $T(-y_1) = 0$, we rewrite the equation $y_h = T y_h$ as

$$y_h = T y_h - T(-y_1)$$

and conclude that

$$\|y_h\| \le \frac{1}{\rho(\omega)}\left(\frac{\beta-\alpha}{2}\right)\|y_h\| + \frac{1}{\rho(\omega)}\left(\frac{\beta-\alpha}{2}\right)a$$

Therefore,

$$\|y_h\| \le \frac{a[(\beta-\alpha)/2]\,/\rho(\omega)}{1-[(\beta-\alpha)/2]\,/\rho(\omega)} = \frac{a[(\beta-\alpha)/2]}{\rho(\omega)-[(\beta-\alpha)/2]} = \frac{2\sigma(\omega)a}{\beta-\alpha}$$

which proves (7.40). To prove (7.41), consider (C.47) rewritten as

$$y_1 + P_1 g\psi y_1 = P_1 g[\psi y_1 - \psi(y_1 + y_h)] \tag{C.51}$$

by adding $P_1 g\psi y_1$ to both sides. Taking norms on both sides of (C.51) gives

$$|1+G(j\omega)\Psi(a)|a \le |G(j\omega)|\left(\frac{\beta-\alpha}{2}\right)\|y_h\| \le |G(j\omega)|\sigma(\omega)a$$

from which, we can calculate the bound

$$|\delta\Psi| = \left|\frac{1}{G(j\omega)} + \Psi(a)\right| = \left|\frac{1+G(j\omega)\Psi(a)}{G(j\omega)}\right| \le \sigma(\omega)$$

which completes the proof of the lemma.

C.14 Proof of Theorem 7.4

If, in the proof of Lemma 7.1, we had defined $P_1 = 0$ and $P_h = I$, then the mapping T would still be a contraction mapping if $\omega \in \tilde{\Omega}$. Therefore, $y = y_h = 0$ would be the unique solution of $y_h = Ty_h$. This proves that there is no half-wave symmetric periodic solutions with fundamental frequency $\omega \in \tilde{\Omega}$. The necessity of the condition

$$\left|\frac{1}{G(j\omega)} + \Psi(a)\right| \le \sigma(\omega)$$

shows that there would be no half-wave symmetric periodic solutions with fundamental frequency $\omega \in \Omega'$ if the corresponding error circle does not intersect the $-\Psi(a)$ locus. Thus, we are left with the third part of the theorem where we would like to show that for each complete intersection defining Γ, there is a half-wave symmetric periodic solutions with $(\omega, a) \in \bar{\Gamma}$. The proof of this part uses a result from degree theory, so let us explain that result first.

Suppose we are given a continuously differentiable function $\phi : D \to R^n$, where $D \subset R^n$ is open and bounded. Let $p \in R^n$ be a point such that $\phi(x) = p$ for some x inside D, but $\phi(x) \ne p$ on the boundary ∂D of D. We are interested in showing that $\tilde{\phi}(x) = p$ has a solution in D, where $\tilde{\phi}(x)$ is a perturbation of $\phi(x)$. Degree theory achieves this by ensuring that no solution leaves D as ϕ is perturbed to $\tilde{\phi}$; that is why no solutions were allowed on the boundary ∂D. Assume that at every solution $x_i \in D$ of $\phi(x) = p$, the Jacobian matrix $[\partial\phi/\partial x]$ is nonsingular. Define the degree of ϕ at p relative to D by

$$d(\phi, D, p) = \sum_{x_i=\phi^{-1}(p)} \text{sgn}\left\{\det\left[\frac{\partial\phi}{\partial x}(x_i)\right]\right\}$$

Notice that if $\phi(x) \neq p \ \forall \ x \in D$, the degree is zero. The two basic properties of the degree are as follows:[8]

- If $d(\phi, D, p) \neq 0$, then $\phi(x) = p$ has at least one solution in D.
- If $\eta : \bar{D} \times [0,1] \to R^n$ is continuous and $\eta(x,\mu) \neq 0$ for all $x \in \partial D$ and all $\mu \in [0,1]$, then $d[\eta(\cdot,\mu), D, p]$ is the same for all $\mu \in [0,1]$.

The second property is known as the homotopic invariance of d.

Let us get back to our problem and define

$$\phi(\omega, a) = \Psi(a) + \frac{1}{G(j\omega)}$$

on Γ; ϕ is a complex variable. By taking the real and imaginary parts of ϕ as components of a second-order vector, we can view ϕ as a mapping from Γ into R^2. By assumption, the equation $\phi(w,a) = 0$ has a unique solution (ω_s, a_s) in Γ. The Jacobian of ϕ with respect to (ω, a) at (ω_s, a_s) is given by

$$\begin{bmatrix} \frac{d}{da}\Psi(a)\big|_{a=a_s} & -\Psi^2(a_s)\left\{\frac{d}{d\omega}\text{Re}[G(j\omega)]\right\}_{\omega=\omega_s} \\ 0 & -\Psi^2(a_s)\left\{\frac{d}{d\omega}\text{Im}[G(j\omega)]\right\}_{\omega=\omega_s} \end{bmatrix}$$

The assumptions

$$\left.\frac{d}{da}\Psi(a)\right|_{a=a_s} \neq 0; \quad \left.\frac{d}{d\omega}\text{Im}[G(j\omega)]\right|_{\omega=\omega_s} \neq 0$$

guarantee that the Jacobian is nonsingular. Thus,

$$d(\phi, \Gamma, 0) = \pm 1$$

We are interested in showing that

$$\tilde{\phi}(\omega, a) \overset{\text{def}}{=} \frac{1}{G(j\omega)} + \Psi(a) - \delta\Psi(\omega, a) = 0 \tag{C.52}$$

---

[8] See [26] for the proof of these properties.

has a solution in $\bar{\Gamma}$. This will be the case if we can show that

$$d(\tilde{\phi}, \Gamma, 0) \neq 0$$

To that end, define

$$\eta(\omega, a, \mu) = (1 - \mu)\phi(\omega, a) + \mu\tilde{\phi}(\omega, a) = \phi(\omega, a) - \mu\delta\Psi(\omega, a)$$

for $\mu \in [0, 1]$, so that $\eta = \phi$ at $\mu = 0$ and $\eta = \tilde{\phi}$ at $\mu = 1$. It can be verified that

$$\left|\Psi(a) + \frac{1}{G(j\omega)}\right| \geq \sigma(\omega), \quad \forall\ (\omega, a) \in \partial\Gamma \tag{C.53}$$

For example, if we take the boundary $a = a_1$, then, with reference to Figure 7.20, the left-hand side of (C.53) is the length of the line connecting the point on the locus of $-\Psi(a)$ corresponding to $a = a_1$ to the a point on the locus of $1/G(j\omega)$ with $\omega_1 \leq \omega \leq \omega_2$. By construction, the former point is outside (or on) the error circle centered at the latter one. Therefore, the length of the line connecting the two points must be greater than (or equal to) the radius of the error circle, that is, $\sigma(\omega)$. Using (C.53), we have

$$\begin{aligned} |\eta(\omega, a, \mu)| &\geq |\phi(\omega, a)| - \mu|\delta\Psi(\omega, a)| \\ &= \left|\Psi(a) + \frac{1}{G(j\omega)}\right| - \mu|\delta\Psi(\omega, a)| \ \geq \ \sigma(\omega) - \mu\sigma(\omega) \end{aligned}$$

where we have used the bound (7.41). Thus, for all $0 \leq \mu < 1$, the right-hand side of the last inequality is positive, which implies that $\eta(\omega, a, \mu) \neq 0$ on $\partial\Gamma$ when $\mu < 1$. There is no loss of generality in assuming that $\eta(\omega, a, 1) \neq 0$ on $\partial\Gamma$, because equality would imply we had found a solution as required. Thus, by the homotopic invariance property of d, we have

$$d(\tilde{\phi}, \Gamma, 0) = d(\phi, \Gamma, 0) \neq 0$$

Therefore, (C.52) has a solution in $\bar{\Gamma}$, which concludes the proof of the theorem.

C.15 Proof of Theorems 8.1 and 8.3

The main element in the proofs of these two theorems is a contraction mapping argument, which is used almost identically in the two proofs. To avoid repetition, we will state and prove a lemma that captures the needed contraction mapping argument and then use it to prove the two theorems. The statement of the lemma appears to be very similar to the statement of Theorem 8.1, but it has an additional claim that is needed in Theorem 8.3.

Lemma C.6 *Consider the system*

$$\dot{y} = Ay + f(y, z) \tag{C.54}$$
$$\dot{z} = Bz + g(y, z) \tag{C.55}$$

where $y \in R^k$, $z \in R^m$, *the eigenvalues of* A *have zero real parts, the eigenvalues of* B *have negative real parts, and* f *and* g *are twice continuously differentiable functions that vanish together with their first derivatives at the origin. Then, there exist* $\delta > 0$ *and a continuously differentiable function* $\eta(y)$, *defined for all* $\|y\| < \delta$, *such that* $z = \eta(y)$ *is a center manifold for* (C.54)–(C.55). *Moreover, if* $\|g(y, 0)\| \leq k\|y\|^p$ *for all* $\|y\| \leq r$, *where* $p > 1$ *and* $r > 0$, *then there is* $c > 0$ *such that* $\|\eta(y)\| \leq c\|y\|^p$. ◇

Proof: It is more convenient to show the existence of a center manifold when solutions in the manifold are defined for all $t \in R$. A center manifold for (C.54)–(C.55) may, in general, be only local; that is, a solution in the manifold may be defined only on an interval $[0, t_1) \subset R$. Therefore, the following idea is used in the proof: We consider a modified equation that is identical to (C.54)–(C.55) in the neighborhood of the origin, but that has some desired global properties which ensure that a solution in a center manifold will be defined for all t. We prove the existence of a center manifold for the modified equation. Since the two equations agree in the neighborhood of the origin, this proves the existence of a (local) center manifold for the original equation.

Let $\psi : R^k \to [0, 1]$ be a smooth (continuously differentiable infinitely many times) function[9] with $\psi(y) = 1$ when $\|y\| \leq 1$ and $\psi(y) = 0$ when $\|y\| \geq 2$. For $\varepsilon > 0$, define F and G by

$$F(y, z) = f\left(y\psi\left(\frac{y}{\varepsilon}\right), z\right); \quad G(y, z) = g\left(y\psi\left(\frac{y}{\varepsilon}\right), z\right)$$

The functions F and G are twice continuously differentiable, and they, together with their first partial derivatives, are globally bounded in y; that is, whenever $\|z\| \leq k_1$, the function is bounded for all $y \in R^k$. Consider the modified system

$$\dot{y} = Ay + F(y, z) \tag{C.56}$$
$$\dot{z} = Bz + G(y, z) \tag{C.57}$$

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[9] An example of such a function in the scalar ($k = 1$) case is $\psi(y) = 1$ for $|y| \leq 1$, $\psi(y) = 0$ for $|y| \geq 2$, and

$$\psi(y) = 1 - \frac{1}{b}\int_1^{|y|} \exp\left(\frac{-1}{x-1}\right) \exp\left(\frac{-1}{2-x}\right) dx, \quad \text{for } 1 < |y| < 2$$

where

$$b = \int_1^2 \exp\left(\frac{-1}{x-1}\right) \exp\left(\frac{-1}{2-x}\right) dx$$

We prove the existence of a center manifold for (C.56)–(C.57). Let X denote the set of all globally bounded, continuous functions $\eta : R^k \to R^m$. With $\sup_{y\in R^k} \|\eta(y)\|$ as a norm, X is a Banach space. For $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$, let $S \subset X$ be the set of all continuously differential functions $\eta : R^k \to R^m$ such that

$$\eta(0) = 0,\ \frac{\partial \eta}{\partial y}(0) = 0,\ \|\eta(y)\| \le c_1,\ \left\|\frac{\partial \eta}{\partial y}(y)\right\| \le c_2,\ \left\|\frac{\partial \eta}{\partial y}(y) - \frac{\partial \eta}{\partial y}(x)\right\| \le c_3\|y - x\|$$

for all $x,\ y \in R^k$. To show that S is closed, let $\eta_i(y)$ be a convergent sequence in S and show that $\eta(y) = \lim_{i\to\infty} \eta_i(y)$ belongs to S. The key step is to show that $\eta(y)$ is continuously differentiable. The rest can be shown by contradiction. Because continuous differentiability can be shown component wise, we do it for a scalar η. Let v be an arbitrary vector in R^k, with $\|v\| = 1$, and μ be a positive constant. By the mean value theorem,

$$\begin{aligned}\eta_i(y + \mu v) - \eta_i(y) &= \frac{\partial \eta_i}{\partial y}(y + \alpha_i \mu v)\mu v \\ \eta_j(y + \mu v) - \eta_j(y) &= \frac{\partial \eta_j}{\partial y}(y + \alpha_j \mu v)\mu v\end{aligned}$$

where $0 < \alpha_i < 1$ and $0 < \alpha_j < 1$. By adding and subtracting terms, we can write the following equation:

$$\begin{aligned}\left[\frac{\partial \eta_i}{\partial y}(y) - \frac{\partial \eta_j}{\partial y}(y)\right]\mu v &= \left[\frac{\partial \eta_i}{\partial y}(y) - \frac{\partial \eta_i}{\partial y}(y + \alpha_i \mu v)\right]\mu v \\ &\quad - \left[\frac{\partial \eta_j}{\partial y}(y) - \frac{\partial \eta_j}{\partial y}(y + \alpha_j \mu v)\right]\mu v \\ &\quad + [\eta_i(y + \mu v) - \eta_j(y + \mu v)] - [\eta_i(y) - \eta_j(y)]\end{aligned}$$

Given $\varepsilon > 0$, find integers i_0 and j_0 large enough that

$$\sup_{y\in R^k} \|\eta_i(y) - \eta_j(y)\| < \frac{\varepsilon^2}{16c_3}$$

for all $i > i_0$ and $j > j_0$, which is possible because $\{\eta_i\}$ is convergent. Using the preceding inequality and

$$\left\|\left[\frac{\partial \eta_\ell}{\partial y}(y) - \frac{\partial \eta_\ell}{\partial y}(y + \alpha_\ell \mu v)\right]\mu v\right\| \le c_3 \alpha_\ell \mu^2 \|v\|^2 < c_3 \mu^2 \quad \text{for } \ell = i \text{ or } j$$

it can be shown that

$$\left\|\left[\frac{\partial \eta_i}{\partial y}(y) - \frac{\partial \eta_j}{\partial y}(y)\right] v\right\| < 2c_3\mu + \frac{\varepsilon^2}{8c_3\mu}$$

Taking $\mu = \varepsilon/(4c_3)$ yields

$$\left\|\frac{\partial \eta_i}{\partial y}(y) - \frac{\partial \eta_j}{\partial y}(y)\right\| < \varepsilon$$

Hence, $\partial\eta_i/\partial y$ is a Cauchy sequence in the Banach space of globally bounded, continuous functions from R^k into R^k. Consequently, it converges to a continuous function $J(y)$. It follows that $\eta(y)$ is differentiable and $\partial\eta/\partial y = J(y)$.[10]

For a given $\eta \in S$, consider the system

$$\begin{aligned} \dot{y} &= Ay + F(y, \eta(y)) & \text{(C.58)} \\ \dot{z} &= Bz + G(y, \eta(y)) & \text{(C.59)} \end{aligned}$$

Due to the boundedness of $\eta(y)$ and $[\partial\eta/\partial y]$, the right-hand side of (C.58) is globally Lipschitz in y. Therefore, for every initial state $y_0 \in R^k$, (C.58) has a unique solution defined for all t. We denote this solution by $y(t) = \pi(t; y_0, \eta)$, where $\pi(0; y_0, \eta) = y_0$; the solution is parameterized in the fixed function η. Equation (C.59) is linear in z; therefore, its solution is given by

$$\begin{aligned} z(t) &= \exp[B(t-\tau)]z(\tau) \\ &\quad + \int_{\tau}^{t} \exp[B(t-\lambda)]G(\pi(\lambda-\tau; y(\tau), \eta), \eta(\pi(\lambda-\tau; y(\tau), \eta)))\; d\lambda \end{aligned}$$

Multiply through by $\exp[-B(t-\tau)]$, move the integral term to the other side, and change the integration variable from λ to $s = \lambda - \tau$ to obtain

$$\begin{aligned} z(\tau) &= \exp[-B(t-\tau)]z(t) \\ &\quad + \int_{t-\tau}^{0} \exp(-Bs)G(\pi(s; y(\tau), \eta), \eta(\pi(s; y(\tau), \eta)))\; ds \end{aligned}$$

This expression is valid for any $t \in R$. Taking the limit $t \to -\infty$ results in

$$z(\tau) = \int_{-\infty}^{0} \exp(-Bs)G(\pi(s; y(\tau), \eta), \eta(\pi(s; y(\tau), \eta)))\; ds \qquad \text{(C.60)}$$

Let us rewrite the right-hand side of (C.60) with $y(\tau)$ replaced by y and denote it by $(P\eta)(y)$.

$$(P\eta)(y) = \int_{-\infty}^{0} \exp(-Bs)G(\pi(s; y, \eta), \eta(\pi(s; y, \eta)))\; ds \qquad \text{(C.61)}$$

With this definition, (C.60) says that $z(\tau) = (P\eta)(y(\tau))$ for all τ. Hence, $z = (P\eta)(y)$ defines an invariant manifold for (C.58)–(C.59) parameterized in η.

Consider (C.56)–(C.57). If $\eta(y)$ is a fixed point of the mapping $(P\eta)(y)$, that is,

$$\eta(y) = (P\eta)(y)$$

then $z = \eta(y)$ is a center manifold for (C.56)–(C.57). This fact can be seen as follows: First, using the properties of $\eta \in S$ and the fact that $y = 0$ is an equilibrium point of (C.58), it can be seen from (C.61) that

$$(P\eta)(0) = 0; \quad \frac{\partial(P\eta)}{\partial y}(0) = 0$$

---

[10] See [111, Theorem 9.1].

Second, since $z = (P\eta)(y)$ is an invariant manifold for (C.58)–(C.59), $(P\eta)(y)$ satisfies the partial differential equation

$$\frac{\partial}{\partial y}(P\eta)(y)[Ay + F(y, (P\eta)(y))] = B(P\eta)(y) + G(y, (P\eta)(y))$$

If $\eta(y) = (P\eta)(y)$, then clearly $\eta(y)$ satisfies the same partial differential equation; hence, it is a center manifold for (C.56)–(C.57). It remains now to show that the mapping $(P\eta)$ has a fixed point, which will be done by an application of the contraction mapping theorem. We want to show that the mapping $P\eta$ maps S into itself and is a contraction mapping on S. From the definitions of F and G, there is a nonnegative continuous function $\rho(\varepsilon)$ with $\rho(0) = 0$ such that

$$\|F(y,z)\| \le \varepsilon\rho(\varepsilon); \quad \left\|\frac{\partial F}{\partial y}(y,z)\right\| \le \rho(\varepsilon); \quad \left\|\frac{\partial F}{\partial z}(y,z)\right\| \le \rho(\varepsilon) \tag{C.62}$$

$$\|G(y,z)\| \le \varepsilon\rho(\varepsilon); \quad \left\|\frac{\partial G}{\partial y}(y,z)\right\| \le \rho(\varepsilon); \quad \left\|\frac{\partial G}{\partial z}(y,z)\right\| \le \rho(\varepsilon) \tag{C.63}$$

for all $y \in R^k$ and all $z \in R^m$ with $\|z\| < \varepsilon$. Because the eigenvalues of B have negative real parts, there exist positive constants β and C such that for $s \le 0$,

$$\|\exp(-Bs)\| \le C\exp(\beta s) \tag{C.64}$$

Since the eigenvalues of A have zero real parts, for each $\alpha > 0$, there is a positive constant $M(\alpha)$ (which may tend to ∞ as $\alpha \to 0$) such that for $s \in R$,

$$\|\exp(As)\| \le M(\alpha)\exp(\alpha|s|) \tag{C.65}$$

To show that $P\eta$ maps S into itself, we need to show that there are positive constants c_1, c_2, and c_3 such that if $\eta(y)$ is continuously differentiable and satisfies

$$\|\eta(y)\| \le c_1; \quad \left\|\frac{\partial\eta}{\partial y}(y)\right\| \le c_2; \quad \left\|\frac{\partial\eta}{\partial y}(y) - \frac{\partial\eta}{\partial y}(x)\right\| \le c_3\|y - x\|$$

for all $x,\ y \in R^k$, then $(P\eta)(y)$ is continuously differentiable and satisfies the same inequalities. Continuous differentiability of $(P\eta)(y)$ is obvious from (C.61). To verify the inequalities, we need to use the estimates on F and G provided by (C.62) and (C.63); therefore, we choose c_1 to satisfy $0.5\varepsilon < c_1 < \varepsilon$. Using (C.64) and the estimates on G and η, we have from (C.61),

$$\|(P\eta)(y)\| \le \int_{-\infty}^{0} \|\exp(-Bs)\|\ \|G\|\ ds \le \int_{-\infty}^{0} C\exp(\beta s)\ \varepsilon\rho(\varepsilon)\ ds = \frac{C\varepsilon\rho(\varepsilon)}{\beta}$$

The upper bound on $(P\eta)(y)$ will be less than c_1 for sufficiently small ε. Let $\pi_y(t; y, \eta)$ denote the Jacobian of $\pi(t; y, \eta)$ with respect to y. It satisfies the variational equation

$$\dot{\pi}_y = \left[A + \left(\frac{\partial F}{\partial y}\right) + \left(\frac{\partial F}{\partial z}\right)\ \left(\frac{\partial\eta}{\partial y}\right)\right]\pi_y, \quad \pi_y(0; y, \eta) = I$$

where

$$\left(\frac{\partial F}{\partial y}\right) = \left(\frac{\partial F}{\partial y}\right)(\pi(t;y,\eta), \eta((\pi(t;y,\eta)))$$

with similar expressions for $\partial F/\partial z$ and $\partial \eta/\partial y$. Hence, for $t \leq 0$,

$$\pi_y(t;y,\eta) = \exp(At) - \int_t^0 \exp[A(t-s)]\left[\left(\frac{\partial F}{\partial y}\right) + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]\pi_y(s;y,\eta)\ ds$$

Using (C.65) and the estimates on F and η, we obtain

$$\|\pi_y(t;y,\eta)\| \leq M(\alpha)\exp(-\alpha t) + \int_t^0 M(\alpha)\exp[\alpha(s-t)](1+c_2)\rho(\varepsilon)\|\pi_y(s;y,\eta)\|\ ds$$

Multiply through by $\exp(\alpha t)$ and apply the Gronwall–Bellman inequality to show that[11]

$$\|\pi_y(t;y,\eta)\| \leq M(\alpha)\exp(-\gamma t)$$

where $\gamma = \alpha + M(\alpha)(1+c_2)\rho(\varepsilon)$. Using this bound, as well as (C.64) and the estimates on G and η, we proceed to calculate a bound on the Jacobian $[\partial(P\eta)(y)/\partial y]$. From (C.61),

$$\frac{\partial(P\eta)(y)}{\partial y} = \int_{-\infty}^0 \exp(-Bs)\left[\left(\frac{\partial G}{\partial y}\right) + \left(\frac{\partial G}{\partial z}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]\pi_y(s;y,\eta)\ ds$$

Thus,

$$\begin{aligned}\left\|\frac{\partial(P\eta)(y)}{\partial y}\right\| &\leq \int_{-\infty}^0 C\exp(\beta s)(1+c_2)\rho(\varepsilon)M(\alpha)\exp(-\gamma s)\ ds \\ &= \frac{C(1+c_2)\rho(\varepsilon)M(\alpha)}{\beta-\gamma}\end{aligned}$$

provided ε and α are small enough that $\beta > \gamma$. This bound on the Jacobian of $(P\eta)(y)$ will be less than c_2 for sufficiently small ε. To show that the Jacobian $[\partial(P\eta)(y)/\partial y]$ is Lipschitz with a Lipschitz constant c_3, we note that the Jacobian matrices $[\partial F/\partial y]$, $[\partial F/\partial z]$, $[\partial G/\partial y]$, and $[\partial G/\partial z]$ satisfy Lipschitz inequalities of the form

$$\left\|\frac{\partial F}{\partial y}(y,z) - \frac{\partial F}{\partial y}(x,w)\right\| \leq L[\|y-x\| + \|z-w\|]$$

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[11] Because t appears in the lower limit of the integral, you actually apply a version of the Gronwall–Bellman inequality that says: If

$$y(t) \leq \lambda(t) + \int_t^a \mu(s)y(s)\ ds$$

then

$$y(t) \leq \lambda(t) + \int_t^a \lambda(s)\mu(s)\exp\left[\int_t^s \mu(\tau)\ d\tau\right]\ ds$$

for all $x,\ y \in R^k$ and $z,\ w \in B_\varepsilon$. By working with $\varepsilon < \varepsilon^*$ for some $\varepsilon^* > 0$, we can choose the Lipschitz constant L independent of ε. Moreover, we can use the same L for all four Jacobian matrices. Using these inequalities and the Gronwall–Bellman inequality, we can repeat the foregoing derivations to show that

$$\|\pi_y(t; y, \eta) - \pi_y(t; x, \eta)\| \le L_1 \exp(-2\gamma t)\|y - x\|$$

for all $x,\ y \in R^k$ and $t \le 0$, where $L_1 = [(1 + c_2)^2 L + \rho(\varepsilon)c_3]M^3(\alpha)/\gamma$. The last inequality can be used to show that

$$\left\|\frac{\partial(P\eta)}{\partial y}(y) - \frac{\partial(P\eta)}{\partial y}(x)\right\| \le \frac{CL_1(2\gamma - \alpha)}{M(\beta - 2\gamma)}\|y - x\|$$

provided $\beta - 2\gamma > 0$. Choose α and ε small enough that $\beta - 2\gamma > \beta/2$. Then,

$$\left\|\frac{\partial(P\eta)}{\partial y}(y) - \frac{\partial(P\eta)}{\partial y}(x)\right\| \le [L_2 + \rho(\varepsilon)L_3 c_3]\|y - x\|$$

where L_2 and L_3 are independent of ε and c_3. Choosing $c_3 > L_2$, we can choose ε small enough that $L_2 + \rho(\varepsilon)L_3c_3 < c_3$. Thus, we have shown that, for sufficiently small c_1 and sufficiently large c_3, the mapping $P\eta$ maps S into itself. To show that it is a contraction on S, let $\eta_1(y)$ and $\eta_2(y)$ be two functions in S. Let $\pi_1(t)$ and $\pi_2(t)$ be the corresponding solutions of (C.58), which start at y; that is,

$$\pi_i(t) = \pi(t; y, \eta_i), \quad i = 1, 2$$

Using the estimates given by (C.62) and (C.63), it can be shown that

$$\|F(\pi_2, \eta_2(\pi_2)) - F(\pi_1, \eta_1(\pi_1))\| \le (1 + c_2)\rho(\varepsilon)\|\pi_2 - \pi_1\| + \rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\|$$

$$\|G(\pi_2, \eta_2(\pi_2)) - G(\pi_1, \eta_1(\pi_1))\| \le (1 + c_2)\rho(\varepsilon)\|\pi_2 - \pi_1\| + \rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\|$$

From (C.58), $\|\pi_2 - \pi_1\|$ satisfies

$$\begin{aligned}
\|\pi_2(t) - \pi_1(t)\| &\le \int_t^0 M(\alpha)\exp[\alpha(s - t)][\rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\| \\
&\quad + (1 + c_2)\rho(\varepsilon)\|\pi_2(s) - \pi_1(s)\|]\ ds \\
&\le \frac{1}{\alpha}M(\alpha)\rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\| \exp(-\alpha t) \\
&\quad + \int_t^0 (\gamma - \alpha)\exp[\alpha(s - t)]\|\pi_2(s) - \pi_1(s)\|\ ds
\end{aligned}$$

where we used $\gamma = \alpha + M(\alpha)(1 + c_2)\rho(\varepsilon)$ and the fact that $\pi_1(0) = \pi_2(0)$. Multiply through by $\exp(\alpha t)$ and apply the Gronwall–Bellman inequality to show that

$$\|\pi_2(t) - \pi_1(t)\| \le \frac{1}{\alpha}M(\alpha)\rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\| \exp(-\gamma t)$$

Using this inequality in

$$(P\eta_2)(y) - (P\eta_1)(y) = \int_{-\infty}^{0} \exp(-Bs)\,[G(\pi_2, \eta_2(\pi_2)) - G(\pi_1, \eta_1(\pi_1))]\ ds$$

we obtain

$$\begin{aligned} \|(P\eta_2)(y) - (P\eta_1)(y)\| &\leq \int_{-\infty}^{0} Ce^{\beta s}[(1+c_2)\rho(\varepsilon)\|\pi_2(s) - \pi_1(s)\| \\ &\quad + \rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\|\ ]\ ds \\ &\leq C\rho(\varepsilon) \sup_{y \in R^k} \|\eta_2 - \eta_1\| \left[\frac{1}{\beta}\right. \\ &\quad \left. + \int_{-\infty}^{0} e^{\beta s}(1+c_2)\frac{1}{\alpha}M(\alpha)\rho(\varepsilon)e^{-\gamma s}\ ds\right] \\ &\leq b \sup_{y \in R^k} \|\eta_2 - \eta_1\| \end{aligned}$$

where

$$b = C\rho(\varepsilon)\left[\frac{1}{\beta} + \frac{\gamma - \alpha}{\alpha(\beta - \gamma)}\right]$$

By choosing ε small enough, we can ensure that $b < 1$; hence, $P\eta$ is a contraction mapping on S. Thus, by the contraction mapping theorem, the mapping $P\eta$ has a fixed point in S.

Suppose now that $\|g(y,0)\| \leq k\|y\|^p$. The function $G(y,0)$ satisfies the same bound. Consider the closed subset

$$Y = \{\eta \in S \mid \|\eta(y)\| \leq c_4\|y\|^p\}$$

where c_4 is a positive constant to be chosen. To complete the proof of the lemma, we want to show that the fixed point of the mapping $P\eta$ is in Y, which will be the case if we can show that c_4 can be chosen such that $P\eta$ maps Y into itself. Using the estimate on G provided by (C.63), we have

$$\|G(y,\eta(y))\| \leq \|G(y,0)\| + \|G(y,\eta(y)) - G(y,0)\| \leq k\|y\|^p + \rho(\varepsilon)\|\eta(y)\|$$

Since, in the set Y, $\|\eta(y)\| \leq c_4\|y\|^p$,

$$\|G(y,\eta(y))\| \leq [k + c_4\rho(\varepsilon)]\|y\|^p$$

Using this estimate in (C.61) yields

$$\|(P\eta)(y)\| \leq \int_{-\infty}^{0} C\exp(\beta s)\ [k + c_4\rho(\varepsilon)]\|\pi(s; y, \eta)\|^p\ ds$$

Because $\pi(t;0,\eta) = 0$ and $\|\pi_y(t;y,\eta)\| \leq M(\alpha)\exp(-\gamma t)$, it can be shown, as in the proof of Lemma 3.1, that

$$\|\pi(t;y,\eta)\| \leq M(\alpha)\exp(-\gamma t)\|y\|$$

for $t \leq 0$. Thus,

$$\|(P\eta)(y)\| \leq \frac{C[k + c_4\rho(\varepsilon)]M^p(\alpha)}{\beta - p\gamma}\|y\|^p \overset{\text{def}}{=} c_5\|y\|^p$$

provided ε and α are small enough that $\beta - p\gamma > 0$. By choosing c_4 large enough and ε small enough, we have $c_5 < c_4$. Therefore, $(P\eta)$ maps Y into itself, which completes the proof of the lemma. □

Proof of Theorem 8.1

It follows from Lemma C.6 with $A = A_1$, $B = A_2$, $f = g_1$, and $g = g_2$. □

Proof of Theorem 8.3

Define $\mu(y) = h(y) - \phi(y)$. Using the fact that $\mathcal{N}(h(y)) = 0$ and $\mathcal{N}(\phi(y)) = O(\|y\|^p)$, where $\mathcal{N}(h(y))$ is defined by (8.11), we can show that $\mu(y)$ satisfies the partial differential equation

$$\frac{\partial \mu}{\partial y}(y)[A_1 + N(y,\mu(y))] - A_2\mu(y) - Q(y,\mu(y)) = 0 \tag{C.66}$$

where

$$N(y,z) = g_1(y,\phi(y)+z)$$

and

$$\begin{aligned} Q(y,z) &= g_2(y,\phi(y)+z) - g_2(y,\phi(y)) + \mathcal{N}(\phi(y)) \\ &\quad - \frac{\partial \phi}{\partial y}(y)[g_1(y,\phi(y)+z) - g_1(y,\phi(y))] \end{aligned}$$

A function $\mu(y)$ satisfying (C.66) is a center manifold for an equation of the form (C.54)–(C.55) with $A = A_1$, $B = A_2$, $f = N$, and $g = Q$. Furthermore, in this case,

$$Q(y,0) = \mathcal{N}(\phi(y)) = O(\|y\|^p)$$

Hence, by Lemma C.6, there exists a continuously differentiable function $\mu(y) = O(\|y\|^p)$ that satisfies (C.66). Therefore, $h(y) - \phi(y) = O(\|y\|^p)$. The reduced system is given by

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y,h(y)) \\ &= A_1 y + g_1(y,\phi(y)) + g_1(y,h(y)) - g_1(y,\phi(y)) \end{aligned}$$

Since g_1 is twice continuously differentiable and its first partial derivatives vanish at the origin, we have

$$\left\| \frac{\partial g_1}{\partial z}(y, z) \right\| \leq k_1 \|y\| + k_2 \|z\|$$

in the neighborhood of the origin. By the mean value theorem,

$$g_{1i}(y, h(y)) - g_{1i}(y, \phi(y)) = \frac{\partial g_{1i}}{\partial z}(y, \zeta(y))[h(y) - \phi(y)]$$

where

$$\|\zeta(y)\| \leq \|\mu(y)\| + \|\phi(y)\| \leq k_3 \|y\|^p \leq k_3 \|y\|$$

for $\|y\| < 1$. Therefore,

$$\|g_1(y, h(y)) - g_1(y, \phi(y))\| \leq k_4 \|y\| \; \|\mu(y)\| = O(\|y\|^{p+1})$$

which completes the proof of the theorem. □

C.16 Proof of Lemma 8.1

To show that R_A is invariant, we need to show that

$$x \in R_A \Rightarrow x(s) \stackrel{\text{def}}{=} \phi(s; x) \in R_A, \quad \forall\ s \in R$$

Since

$$\phi(t; \phi(s; x)) = \phi(t + s; x)$$

it is clear that $\lim_{t \to \infty} \phi(t; x(s)) = 0$ for all $s \in R$. Hence, R_A is invariant. To show that R_A is open, take any point $p \in R_A$ and show that every point in a neighborhood of p belongs to R_A. To that end, let $T > 0$ be large enough that $\|\phi(T; p)\| < a/2$, where a is chosen so small that the domain $\|x\| < a$ is contained in R_A. Consider the neighborhood $\|x - p\| < b$ of p. By continuous dependence of the solution on initial states, we can choose b small enough to ensure that for any point q in the neighborhood $\|x - p\| < b$, the solution at time T satisfies

$$\|\phi(T; p) - \phi(T; q)\| < \frac{a}{2}$$

Then,

$$\|\phi(T; q)\| \leq \|\phi(T; q) - \phi(T; p)\| + \|\phi(T; p)\| < a$$

This shows that the point $\phi(T; q)$ is inside R_A. Hence, the solution starting at q approaches the origin as $t \to \infty$. Thus, $q \in R_A$ and the set R_A is open. We leave it to the reader (Exercise 8.13) to show that R_A is connected. The statement about the boundary of R_A follows from the next lemma.

Lemma C.7 *The boundary of an open invariant set is an invariant set. Hence, it is formed of trajectories.* △

Proof: Let M be an open invariant set and x be a boundary point of M. There exists a sequence $x_n \in M$ that converges to x. Since M is an invariant set, the solution $\phi(t; x_n) \in M$ for all $t \in R$. The sequence $\phi(t; x_n)$ converges to $\phi(t; x)$ for all $t \in R$. Thus, $\phi(t; x)$ is an accumulation point of M for all t. On the other hand, $\phi(t; x) \notin M$, since x is a boundary point of M. Therefore, the solution $\phi(t; x)$ belongs to the boundary of M for all t. □

C.17 Proof of Theorem 11.1

We will work with the full problem given by (11.10) and (11.11) in the (x, y) variables. The error estimate for z will then follow from the change of variables (11.9). Let y belong to the domain D_y, where (11.9) maps $D_x \times D_y$ into D_z. When we analyze (11.12) with the slowly varying t and x, we want to use the uniform exponential stability property (11.15) of the boundary-layer model. Inequality (11.15) is valid only when $x \in D_x$, so to use it, we need to confirm that the slowly varying x will always be in D_x. We anticipate that this will be true because the solution $\bar{x}$ of the reduced problem (11.8) belongs to S, a compact subset of D_x, and we anticipate that the error $\|x(t, \varepsilon) - \bar{x}(t)\|$ will be $O(\varepsilon)$. Then, for sufficiently small ε, x will belong to D_x. However, the estimate $\|x(t, \varepsilon) - \bar{x}(t)\| = O(\varepsilon)$ has not been proven yet, so we cannot start by using it. We use a special technique to get around this difficulty.[12] If $D_x \neq R^n$, let E be the complement of D_x in R^n and define

$$k = \tfrac{1}{2} \inf \{\|x - y\| \mid x \in S,\ y \in E\} > 0$$

If $D_x = R^n$, take k to be any positive constant. The sets

$$S_1 = \{x \in R^n \mid \text{dist}(x, S) \leq k/2\} \quad \text{and} \quad S_2 = \{x \in R^n \mid \text{dist}(x, S) \leq k\}$$

are compact subsets of D_x and $S \subset S_1 \subset S_2$. Let $\psi : R^n \to [0, 1]$ be a smooth (continuously differentiable infinitely many times) function with $\psi(x) = 1$ when x belongs to S_1 and $\psi(x) = 0$ when x is outside S_2.[13] We define F and G by

$$\begin{aligned} F(t, x, y, \varepsilon) &= f(t, \varphi(x), y + h(t, \varphi(x)), \varepsilon) && \text{(C.67)} \\ G(t, x, y, \varepsilon) &= g(t, \varphi(x), y + h(t, \varphi(x)), \varepsilon) - \varepsilon \frac{\partial h}{\partial t}(t, \varphi(x)) \\ &\quad - \varepsilon \frac{\partial h}{\partial x}(t, \varphi(x)) f(t, \varphi(x), y + h(t, \varphi(x)), \varepsilon) && \text{(C.68)} \end{aligned}$$

where $\varphi(x) = (x - \xi_0)\psi(x) + \xi_0$. It can easily seen that, for all $x \in R^n$, $\varphi(x)$ is bounded and belongs to D_x, since D_x is convex. When $x \in S_1$, we have $\varphi(x) = x$; hence, the functions F and f are identical. The same is true for the functions G and $g - \varepsilon[(\partial h/\partial t) + (\partial h/\partial x) f]$. It can be verified that for all $(t, x, y, \varepsilon) \in [0, t_1] \times R^n \times \Omega_1 \times [0, \varepsilon_0]$, where Ω_1 is any compact subset of D_y, we have the following:

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[12] The same technique is used in the proof of the center manifold theorem. (See Appendix C.15.)

[13] The existence of ψ is shown in Lemma 6.2 of Chapter 23 of [111].

- F and G and their first partial derivatives with respect to ε are continuous and bounded.
- $F(t,x,y,0)$ has bounded first partial derivatives with respect to (x,y).
- $G(t,x,y,0)$ and $[\partial G(t,x,y,0)/\partial y]$ have bounded first partial derivatives with respect to (t,x,y).

Consider the modified singular perturbation problem

$$\dot{x} = F(t,x,y,\varepsilon), \quad x(t_0) = \xi(\varepsilon) \tag{C.69}$$

$$\varepsilon\dot{y} = G(t,x,y,\varepsilon), \quad y(t_0) = \eta(\varepsilon) - h(t_0,\xi(\varepsilon)) \tag{C.70}$$

The modified problem of (C.69) and (C.70) is identical to the original problem of (11.10) and (11.11) when $x \in S_1$. The set S_1 has been chosen in anticipation that the solution $x(t,\varepsilon)$ will be confined to S_1, which is based on the fact that $\bar{x}(t) \in S$. The boundary-layer model

$$\frac{dy}{d\tau} = G(t,x,y,0) \tag{C.71}$$

has an equilibrium point at $y = 0$. Since

$$G(t,x,y,0) = g(t,\varphi(x),y + h(t,\varphi(x)),0)$$

for any fixed $x \in R^n$, the boundary-layer model (C.71) can be represented as a boundary-layer model of the form (11.14) with $\varphi(x) \in D_x$ as the frozen parameter. Since inequality (11.15) holds uniformly in the frozen parameter, it is clear that the solutions of (C.71) satisfy the same inequality for all $x \in R^n$; that is,

$$\|y(\tau)\| \le k\|y(0)\|\exp(-\gamma\tau), \ \forall\ \|y(0)\| < \rho_0, \ \forall\ (t,x) \in [0,t_1] \times R^n, \ \forall\ \tau \ge 0 \tag{C.72}$$

The reduced problem for (C.69) and (C.70) is

$$\dot{x} = F(t,x,0,0), \quad x(t_0) = \xi_0 \tag{C.73}$$

This problem is identical to the reduced problem (11.8) whenever $x \in S_1$. Since (11.8) has a unique solution $\bar{x}(t)$ defined for all $t \in [t_0,t_1]$ and $\bar{x}(t) \in S$, it follows that $\bar{x}(t)$ is the unique solution of (C.73) for $t \in [t_0,t_1]$. We proceed to prove the theorem for the modified singular perturbation problem given by (C.69) and (C.70). Upon completion of this task, we will show that, for sufficiently small ε, the solution $x(t,\varepsilon)$ of (C.69)–(C.70) belongs to S_1. This will establish that the original and modified problems have the same solution and proves the theorem for the original problem given by (11.10) and (11.11).

Consider the boundary-layer model (C.71). Since $[\partial G/\partial y]$ has bounded first partial derivatives with respect to (t,x) and $G(t,x,0,0) = 0$ for all (t,x), the Jacobian matrices $[\partial G/\partial t]$ and $[\partial G/\partial x]$ satisfy

$$\left\|\frac{\partial G}{\partial t}\right\| \le L_1\|y\|; \quad \left\|\frac{\partial G}{\partial x}\right\| \le L_2\|y\|$$

Using these estimates and (C.72), we conclude from Lemma 9.8 that there is a Lyapunov function $V_1(t,x,y)$ which satisfies

$$c_1\|y\|^2 \le V_1(t,x,y) \le c_2\|y\|^2 \tag{C.74}$$

$$\frac{\partial V_1}{\partial y} G(t,x,y,0) \le -c_3\|y\|^2 \tag{C.75}$$

$$\left\| \frac{\partial V_1}{\partial y} \right\| \le c_4\|y\|; \quad \left\| \frac{\partial V_1}{\partial t} \right\| \le c_5\|y\|^2; \quad \left\| \frac{\partial V_1}{\partial x} \right\| \le c_6\|y\|^2 \tag{C.76}$$

for all $y \in \{\|y\| < \rho_0\}$ and all $(t,x) \in [0,t_1] \times R^n$. The derivative of V_1 along the trajectories of the full system (C.69)–(C.70) is given by

$$\begin{aligned} \dot{V}_1 &= \frac{1}{\varepsilon}\frac{\partial V_1}{\partial y} G(t,x,y,\varepsilon) + \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} F(t,x,y,\varepsilon) \\ &= \frac{1}{\varepsilon}\frac{\partial V_1}{\partial y} G(t,x,y,0) + \frac{1}{\varepsilon}\frac{\partial V_1}{\partial y}[G(t,x,y,\varepsilon) - G(t,x,y,0)] \\ &\quad + \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} F(t,x,y,\varepsilon) \end{aligned}$$

Using (C.75) and (C.76) and the estimates

$$\|F(t,x,y,\varepsilon)\| \le k_0; \quad \|G(t,x,y,\varepsilon) - G(t,x,y,0)\| \le \varepsilon L_3$$

we obtain

$$\begin{aligned} \dot{V}_1 &\le -\frac{c_3}{\varepsilon}\|y\|^2 + c_4 L_3\|y\| + c_5\|y\|^2 + c_6 k_0\|y\|^2 \\ &\le -\frac{c_3}{2\varepsilon}\|y\|^2 + c_4 L_3\|y\|, \quad \text{for } \varepsilon \le \frac{c_3}{2c_5 + 2c_6 k_0} \end{aligned}$$

Thus, if at some time $t^* \ge t_0$, $\|y(t^*,\varepsilon)\| < \rho_0\sqrt{c_1/c_2} \stackrel{\text{def}}{=} \mu$, the solution $y(t,\varepsilon)$ of the full problem will satisfy the exponentially decaying bound

$$\|y(t,\varepsilon)\| \le \mu\sqrt{c_2/c_1}\exp\left[\frac{-\alpha(t-t^*)}{\varepsilon}\right] + \varepsilon\delta, \quad \forall\ t \ge t^* \tag{C.77}$$

where $\alpha = c_3/4c_2$ and $\delta = 2c_2c_4L_3/c_1c_3$. On the other hand, $y(t_0,\varepsilon) = \eta(\varepsilon) - h(t_0,\xi(\varepsilon)) = \eta_0 - h(t_0,\xi_0) + O(\varepsilon)$ and $\eta_0 - h(t_0,\xi_0)$ belongs to Ω_y, a compact subset of the region of attraction of the boundary-layer model

$$\frac{dy}{d\tau} = G(t_0,\xi_0,y,0) = g(t_0,\xi_0,y + h(t_0,\xi_0),0) \tag{C.78}$$

We recall from (the converse Lyapunov) Theorem 4.17 that there is a Lyapunov function $V_0(y)$ such that

$$\frac{\partial V_0}{\partial y} g(t_0,\xi_0,y + h(t_0,\xi_0),0) \le -W_0(y)$$

over the region of attraction, where $W_0(y)$ is positive definite and $\{V_0(y) \le c\}$ is a compact subset of the region of attraction for any $c > 0$. Choose c_0 such that Ω_y is in the interior of $\{V_0(y) \le c_0\}$. The derivative of V_0 along the trajectories of the full system (C.69)–(C.70) is given by

$$\begin{aligned} \dot{V}_0 &= \frac{1}{\varepsilon}\frac{\partial V_0}{\partial y} G(t,x,y,\varepsilon) \\ &= \frac{1}{\varepsilon}\frac{\partial V_0}{\partial y} G(t_0,\xi_0,y,0) + \frac{1}{\varepsilon}\frac{\partial V_0}{\partial y}[G(t,x,y,\varepsilon) - G(t_0,\xi_0,y,0)] \end{aligned}$$

It can be verified that, for all $(t,y) \in [t_0, t_0 + \varepsilon T] \times \{V_0(y) \le c_0\}$,

$$\dot{V}_0 \le \frac{1}{\varepsilon}\left[-W_0(y) + a_0\varepsilon(1+T)\right]$$

for some $a_0 > 0$. Application of Theorem 4.18 shows that there is $\varepsilon_1^* > 0$ such that, for $0 < \varepsilon < \varepsilon_1^*$, $y(t,\varepsilon)$ satisfies the inequality

$$\|y(t,\varepsilon)\| \le \beta(\mu_2, (t-t_0)/\varepsilon) + \varrho(\varepsilon(1+T))$$

over the interval $[t_0, t_0 + \varepsilon T]$, where β is a class $\mathcal{KL}$ function, ϱ is a class $\mathcal{K}$ function, and μ_2 is a positive constant. Choose T large enough that $\beta(\mu_2, T) < \mu/2$; then choose $\varepsilon^* < \varepsilon_1^*$ small enough that $\varrho(\varepsilon^*(1+T)) < \mu/2$. It follows that, for $\varepsilon < \varepsilon^*$, $y(t,\varepsilon)$ satisfies

$$\|y(t,\varepsilon)\| < \mu_1 + \mu/2 \text{ for } t \in [t_0, t_0+\varepsilon T] \;\text{ and }\; \|y(t_0 + \varepsilon T, \varepsilon)\| < \mu \tag{C.79}$$

with $\mu_1 = \beta(\mu_2, 0)$. Then, (C.77) and (C.79) yield

$$\|y(t,\varepsilon)\| \le k_1 \exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right] + \varepsilon\delta, \;\; \forall\, t \ge t_0 \tag{C.80}$$

for some $k_1 > 0$.

Consider (C.69). By rewriting the right-hand side as

$$F(t,x,y,\varepsilon) = F(t,x,0,0) + [F(t,x,y,\varepsilon) - F(t,x,0,0)]$$

we view (C.69) as a perturbation of the reduced system (C.73). The bracketed perturbation term satisfies

$$\begin{aligned} \|F(t,x,y,\varepsilon) - F(t,x,0,0)\| &\le \|F(t,x,y,\varepsilon) - F(t,x,y,0)\| \\ &\quad + \|F(t,x,y,0) - F(t,x,0,0)\| \\ &\le L_4\varepsilon + L_5\|y\| \\ &\le \theta_1\varepsilon + \theta_2 \exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right] \end{aligned}$$

where $\theta_1 = L_4 + L_5\delta$ and $\theta_2 = L_5 k_1$. Define

$$u(t,\varepsilon) = x(t,\varepsilon) - \bar{x}(t)$$

Then,

$$\begin{aligned} u(t,\varepsilon) &= \xi(\varepsilon) - \xi(0) + \int_{t_0}^{t} [F(s,x(s,\varepsilon),y(s,\varepsilon),\varepsilon) - F(s,\bar{x}(s),0,0)]\, ds \\ &= \xi(\varepsilon) - \xi(0) + \int_{t_0}^{t} [F(s,x(s,\varepsilon),y(s,\varepsilon),\varepsilon) - F(s,x(s,\varepsilon),0,0)]\, ds \\ &\quad + \int_{t_0}^{t} [F(s,x(s,\varepsilon),0,0) - F(s,\bar{x}(s),0,0)]\, ds \end{aligned}$$

and

$$\begin{aligned} \|u(t,\varepsilon)\| &\le k_2\varepsilon + \int_{t_0}^{t} \left\{ \theta_1\varepsilon + \theta_2 \exp\left[\frac{-\alpha(s-t_0)}{\varepsilon}\right] \right\} ds + \int_{t_0}^{t} L_6 \|u(s,\varepsilon)\|\, ds \\ &\le k_2\varepsilon + \left[\theta_1\varepsilon(t_1 - t_0) + \frac{\theta_2\varepsilon}{\alpha}\right] + \int_{t_0}^{t} L_6\|u(s,\varepsilon)\|\, ds \end{aligned}$$

By the Gronwall–Bellman lemma, we arrive at the estimate

$$\|x(t,\varepsilon) - \bar{x}(t)\| \le \varepsilon k_3 [1 + t_1 - t_0] \exp[L_6(t_1 - t_0)] \tag{C.81}$$

which proves the error estimate for x. We can also conclude that, for sufficiently small ε, the solution $x(t,\varepsilon)$ is defined for all $t \in [t_0, t_1]$.

To prove the error estimate for y, consider (C.70), which, for convenience, is written in the τ time scale as

$$\frac{dy}{d\tau} = G(t_0 + \varepsilon\tau, x(t_0 + \varepsilon\tau, \varepsilon), y, \varepsilon)$$

Let $\hat{y}(\tau)$ denote the solution of the boundary-layer model

$$\frac{dy}{d\tau} = G(t_0, \xi_0, y, 0), \quad y(0) = \eta_0 - h(t_0, \xi_0)$$

and set

$$v(\tau,\varepsilon) = y(\tau,\varepsilon) - \hat{y}(\tau)$$

By differentiating both sides with respect to τ and substituting for the derivatives of y and $\hat{y}$, we obtain

$$\frac{dv}{d\tau} = G(t_0 + \varepsilon\tau, x(t_0 + \varepsilon\tau, \varepsilon), y(\tau,\varepsilon), \varepsilon) - G(t_0, \xi_0, \hat{y}(\tau), 0)$$

We add and subtract $G(t_0 + \varepsilon\tau, x(t_0 + \varepsilon\tau, \varepsilon), v, 0)$ to obtain

$$\frac{dv}{d\tau} = G(t, x, v, 0) + \Delta G \tag{C.82}$$

where $t = t_0 + \varepsilon\tau$, $x = x(t_0 + \varepsilon\tau, \varepsilon)$, $\Delta G = \Delta_1 + \Delta_2 + \Delta_3$, and

$$\begin{aligned} \Delta_1 &= G(t, x, y, 0) - G(t, x, \hat{y}, 0) - G(t, x, v, 0) \\ \Delta_2 &= G(t, x, y, \varepsilon) - G(t, x, y, 0) \\ \Delta_3 &= G(t, x, \hat{y}, 0) - G(t_0, \xi_0, \hat{y}, 0) \end{aligned}$$

It can be verified that

$$\begin{aligned} \|\Delta_1\| &\le k_4\|v\|^2 + k_5\|v\|\ \|\hat{y}\|, \qquad \|\Delta_2\| \le \varepsilon L_3 \\ \|\Delta_3\| &\le L_1|t - t_0|\ \|\hat{y}\| + L_2\|x - \xi_0\|\ \|\hat{y}\| \le (L_1\varepsilon\tau + L_2\varepsilon a + L_2\varepsilon\tau k_0)\|\hat{y}\| \end{aligned}$$

for some nonnegative constants k_4, k_5, and a. Repeating the derivation that led to (C.80), it can be shown that

$$\|\hat{y}(\tau)\| \le k_1 e^{-\alpha\tau}, \quad \forall\ \tau \ge 0 \tag{C.83}$$

Hence,

$$\begin{aligned} \|\Delta G\| &\le k_4\|v\|^2 + k_5 k_1\|v\|\ e^{-\alpha\tau} + \varepsilon L_3 + \varepsilon a_1 k_1(1 + \tau)e^{-\alpha\tau} \\ &\le k_4\|v\|^2 + k_5 k_1\|v\|e^{-\alpha\tau} + \varepsilon a_2 \end{aligned} \tag{C.84}$$

where $a_1 = \max\{L_2 a, L_1 + L_2 k_0\}$ and $a_2 = L_3 + a_1 k_1 \max\{1, 1/\alpha\}$. We have used the fact that $(1 + \tau)e^{-\alpha\tau} \le \max\{1, 1/\alpha\}$. Equation (C.82) can be viewed as a perturbation of

$$\frac{dv}{d\tau} = G(t, x, v, 0) \tag{C.85}$$

which, by Lemma 9.8, has a Lyapunov function $V_1(t, x, v)$ that satisfies (C.74) through (C.76). Calculating the derivative of V_1 along the trajectories of (C.82) and using the estimate (C.84), we obtain

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}F + \frac{1}{\varepsilon}\frac{\partial V_1}{\partial v}[G(t, x, v, 0) + \Delta G] \\ &\le c_5\|v\|^2 + c_6 k_0\|v\|^2 - \frac{c_3}{\varepsilon}\|v\|^2 + \frac{c_4}{\varepsilon}\|v\|\left(k_4\|v\|^2 + k_5 k_1\|v\|e^{-\alpha\tau} + \varepsilon a_2\right) \end{aligned}$$

For $\|v\| \le c_3/4c_4k_4$ and $0 < \varepsilon < c_3/4(c_5 + c_6 k_0)$, we have

$$\begin{aligned} \dot{V}_1 &\le -\frac{c_3}{2\varepsilon}\|v\|^2 + \frac{c_4 k_5 k_1}{\varepsilon}\|v\|^2 e^{-\alpha\tau} + c_4 a_2\|v\| \\ &\le -\frac{2}{\varepsilon}\left(k_a - k_b e^{-\alpha\tau}\right)V_1 + 2k_c\sqrt{V_1} \end{aligned}$$

where $k_a = c_3/4c_2$, $k_b = c_4 k_5 k_1/2c_1$, and $k_c = c_4 a_2/2\sqrt{c_1}$. Taking $W = \sqrt{V_1}$ yields

$$D^+W(\tau) \le -\left(k_a - k_b e^{-\alpha\tau}\right)W + \varepsilon k_c$$

where $D^+W(\tau)$ is the upper right-hand derivative of W with respect to τ. By the comparison principle (Lemma 3.4), we conclude that

$$W(\tau) \le \phi(\tau, 0)W(0) + \varepsilon \int_0^\tau \phi(\tau, \sigma)k_c \, d\sigma$$

where

$$\phi(\tau, \sigma) = \exp\left[-\int_\sigma^\tau \left(k_a - k_b e^{-\alpha\lambda}\right) \, d\lambda\right] \quad \text{and} \quad |\phi(\tau, \sigma)| \le k_g e^{-\alpha_g(\tau-\sigma)}$$

for some $k_g, \alpha_g > 0$. Using the fact that $v(0) = O(\varepsilon)$, we conclude that $v(\tau) = O(\varepsilon)$ for all $\tau \ge 0$. This completes the proof that the solution of (C.69)–(C.70) satisfies

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon), \qquad y(t, \varepsilon) - \hat{y}\left(\frac{t}{\varepsilon}\right) = O(\varepsilon)$$

$\forall\ t \in [t_0, t_1]$ for sufficiently small ε. Since $\bar{x}(t) \in S$, there is $\varepsilon_2^* > 0$ small enough such that $x(t, \varepsilon) \in S_1$ for all $t \in [t_0, t_1]$ and all $\varepsilon < \varepsilon_2^*$. Hence, $x(t, \varepsilon)$ and $y(t, \varepsilon)$ are the solutions of (11.10)–(11.11). From (11.9), we have

$$z(t, \varepsilon) - h(t, \bar{x}(t)) - \hat{y}\left(\frac{t}{\varepsilon}\right) = y(t, \varepsilon) - \hat{y}\left(\frac{t}{\varepsilon}\right) + h(t, x(t, \varepsilon)) - h(t, \bar{x}(t)) = O(\varepsilon)$$

where we have used the fact that h is Lipschitz in x. Finally, since $\hat{y}(\tau)$ satisfies (C.83) and

$$\exp\left[\frac{-\alpha(t - t_0)}{\varepsilon}\right] \le \varepsilon, \quad \forall\ \alpha(t - t_0) \ge \varepsilon \ln\left(\frac{1}{\varepsilon}\right)$$

the term $\hat{y}(t/\varepsilon)$ will be $O(\varepsilon)$ uniformly on $[t_b, t_1]$ if ε is small enough to satisfy

$$\varepsilon \ln\left(\frac{1}{\varepsilon}\right) \le \alpha(t_b - t_0)$$

The proof of Theorem 11.1 is now complete.

C.18 Proof of Theorem 11.2

The proof follows closely that of Theorem 11.1. We will only point out two main differences, one in showing that x belongs to D_x and the other in analyzing the error $x - \bar{x}$. The first point makes use of the Lyapunov function V of the reduced system, while the second one makes use of its stability properties.

Using the Lyapunov function V, we will argue that x belongs to the compact set $\{W_1(x) \le c\}$ for all $t \ge t_0$. Therefore, we do not need to truncate x by using the function $\psi(x)$, as we did in the proof of Theorem 11.1. The functions F and G are still defined by (C.67) and (C.68), but with $\varphi(x)$ replaced by x. They have the same properties as before for all $(t,x,y,\varepsilon) \in [0,\infty) \times \{W_1(x) \le c\} \times \Omega_1 \times [0,\varepsilon_0]$. Moreover, $[\partial F/\partial x](t,x,0,0)$ is Lipschitz in x, uniformly in t. For all $x \in \{W_1(x) \le c\}$, we can repeat the earlier derivation to show that $y(t,\varepsilon)$ satisfies (C.80). Hence,

$$\dot{V} \le -W_3(x) + k_6\varepsilon + k_7 \exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right]$$

Using the fact that $\xi_0 \in \{W_2(x) \le \rho c\}$, we can see that there is a time $T_1 > 0$, independent of ε, such that, for sufficiently small ε, $x(t,\varepsilon) \in \{W_2(x) \le c\}$ for all $t \in [t_0, t_0+T_1]$. For $t \ge t_0 + T_1$, the exponential term $\exp[-\alpha(t-t_0)/\varepsilon]$ is $O(\varepsilon)$. Thus,

$$\dot{V} \le -W_3(x) + k_8\varepsilon$$

Using this inequality, we can show that $\dot{V}$ is negative on the boundary $V(t,x) = c$. Therefore, $x(t,\varepsilon) \in \{W_1(x) \le c\}$ for all $t \ge t_0$.

To analyze the approximation error $u(t,\varepsilon) = x(t,\varepsilon) - \bar{x}(t)$, we view (C.69) as a perturbation of the reduced system (C.73). Instead of using the Gronwall–Bellman lemma to derive an estimate of u, we employ Lyapunov analysis that exploits the exponential stability of the origin of (C.73). The Lyapunov analysis is very similar to the boundary-layer analysis of Theorem 11.1's proof. Therefore, we will describe it only briefly. The error u satisfies the equation

$$\dot{u} = F(t,u,0,0) + \Delta F \tag{C.86}$$

where

$$\Delta F = [F(t,\bar{x}+u,0,0) - F(t,\bar{x},0,0) - F(t,u,0,0)] + [F(t,x,y,\varepsilon) - F(t,x,0,0)]$$

It can be verified that

$$\|\Delta F\| \le \tilde{k}_4\|u\|^2 + \tilde{k}_5\|u\|\ \|\bar{x}\| + \tilde{k}_6 \exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right] + \varepsilon\tilde{k}_7$$

The system (C.86) is viewed as a perturbation of

$$\dot{u} = F(t,u,0,0) \tag{C.87}$$

Because the origin of (C.87) is exponentially stable, we can obtain a Lyapunov function $\tilde{V}(t,u)$ for it by using Theorem 4.14. Using this function with (C.86), we obtain

$$\begin{aligned}\dot{\tilde{V}} &= \frac{\partial \tilde{V}}{\partial t} + \frac{\partial \tilde{V}}{\partial u}F(t,u,0,0) + \frac{\partial \tilde{V}}{\partial u}\Delta F \\ &\le -\tilde{c}_3\|u\|^2 + \tilde{c}_4\|u\|\left\{\tilde{k}_4\|u\|^2 + \tilde{k}_5\|u\|\ \|\bar{x}\| + \tilde{k}_6\exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right] + \varepsilon\tilde{k}_7\right\}\end{aligned}$$

For $\|u\| \le \tilde{c}_3/2\tilde{c}_4\tilde{k}_4$, we have

$$\dot{\tilde{V}} \le -2\left[\tilde{k}_a - \tilde{k}_b e^{-\tilde{\alpha}(t-t_0)}\right]\tilde{V} + 2\left\{\varepsilon\tilde{k}_c + \tilde{k}_d\exp\left[\frac{-\alpha(t-t_0)}{\varepsilon}\right]\right\}\sqrt{\tilde{V}}$$

for some $\tilde{k}_a$, $\tilde{\alpha} > 0$ and $\tilde{k}_b$, $\tilde{k}_c$, $\tilde{k}_d \geq 0$. Applying the comparison principle yields

$$\tilde{W}(t) \leq \tilde{\phi}(t, t_0)\tilde{W}(0) + \int_{t_0}^{t} \tilde{\phi}(t, s)\left\{\varepsilon\tilde{k}_c + \tilde{k}_d \exp\left[\frac{-\alpha(s - t_0)}{\varepsilon}\right]\right\} ds$$

where $\tilde{W} = \sqrt{\tilde{V}}$ and

$$|\tilde{\phi}(t, s)| \leq \tilde{k}_g e^{-\tilde{\sigma}(t - t_0)}, \quad \tilde{\sigma} > 0, \ \tilde{k}_g > 0$$

Since $u(t_0) = O(\varepsilon)$ and

$$\int_{t_0}^{t} \exp[-\tilde{\sigma}(t - s)] \ \exp\left[\frac{-\alpha(s - t_0)}{\varepsilon}\right] \ ds = O(\varepsilon)$$

we can show that $\tilde{W}(t) = O(\varepsilon)$ and, subsequently, $u(t, \varepsilon) = O(\varepsilon)$. The rest of the proof proceeds exactly as in Theorem 11.1. Notice that the boundary-layer analysis in that proof is valid for all $\tau \geq 0$.

C.19 Proof of Theorem 12.1

We analyze the closed-loop system (12.45) as a slowly varying system by using the results of Section 9.6. Since dependence on w does not play a role in the proof, we write $g(\mathcal{X}, \rho, w)$ as $g(\mathcal{X}, \rho)$. It can be verified that $g(\mathcal{X}, \rho)$ is continuously differentiable in a domain $D_{\mathcal{X}} \times D_\rho$ and $\mathcal{X}_{\text{ss}}(\alpha)$ and $A_{ms}(\alpha)$ are continuously differentiable in D_ρ. Because $A_{ms}(\alpha)$ is Hurwitz for all $\alpha \in D_\rho$, it is Hurwitz uniformly in α for all $\alpha \in S$ (a compact subset of D_ρ). Hence, A_{ms} satisfies all the assumptions of Lemma 9.9 for $\alpha \in S$. Let $P_{ms} = P_{ms}(\alpha)$ be the solution of the Lyapunov equation $P_{ms}A_{ms} + A_{ms}^T P_{ms} = -I$. We use $V(\mathcal{X}_\delta, \alpha) = \mathcal{X}_\delta^T P_{ms}\mathcal{X}_\delta$ as a Lyapunov function candidate for the frozen system $\dot{\mathcal{X}}_\delta = g(\mathcal{X}_\delta + \mathcal{X}_{\text{ss}}(\alpha), \alpha)$. Lemma 9.9 shows that $V(\mathcal{X}_\delta, \alpha)$ satisfies (9.41), (9.43), and (9.44). We only need to verify (9.42). The frozen system can be rewritten as

$$\dot{\mathcal{X}}_\delta = A_{ms}(\alpha)\mathcal{X}_\delta + \Delta g(\mathcal{X}_\delta, \alpha)$$

where

$$\|\Delta g(\mathcal{X}_\delta, \alpha)\|_2 = \|g(\mathcal{X}_\delta + \mathcal{X}_{\text{ss}}(\alpha), \alpha) - A_{ms}(\alpha)\mathcal{X}_\delta\|_2 \leq k_1\|\mathcal{X}_\delta\|_2^2$$

in some domain $\{\|\mathcal{X}_\delta\|_2 < r_1\}$. Thus, the derivative of V along the trajectories of $\dot{\mathcal{X}}_\delta = g(\mathcal{X}_\delta + \mathcal{X}_{\text{ss}}(\alpha), \alpha)$ satisfies

$$\dot{V} \leq -\|\mathcal{X}_\delta\|_2^2 + 2c_2k_1\|\mathcal{X}_\delta\|_2^3 \leq -\ \tfrac{1}{2}\|\mathcal{X}_\delta\|_2^2$$

for $\|\mathcal{X}_\delta\|_2 < 1/(4c_2k_1)$. Therefore, there exists $r > 0$ such that $V(\mathcal{X}_\delta, \alpha)$ satisfies (9.41) through (9.44) for all $(\mathcal{X}_\delta, \alpha) \in \{\|\mathcal{X}_\delta\|_2 < r\} \times S$. The conclusions of Theorem 12.1 follow from Theorem 9.3.

C.20 Proof of Theorem 12.2

To analyze the closed-loop system (12.58)–(12.59), we combine stability analysis of slowly varying systems from Section 9.6 with stability analysis of singularly perturbed systems from Section 11.5. Since dependence on w does not play a role in the proof, we write (12.58)–(12.59) as

$$\begin{aligned} \dot{\mathcal{X}} &= g(\mathcal{X}, \rho) + N(\rho)[\vartheta - \phi(\mathcal{X}, \rho)] \qquad &(\mathrm{C.88}) \\ \varepsilon\dot{\vartheta} &= -\vartheta + \phi(\mathcal{X}, \rho) \qquad &(\mathrm{C.89}) \end{aligned}$$

When $\varepsilon = 0$, we obtain the reduced system $\dot{\mathcal{X}} = g(\mathcal{X}, \rho)$, which was analyzed in the proof of Theorem 12.1 by using the quadratic Lyapunov function $\mathcal{X}_\delta^T P_{ms} \mathcal{X}_\delta$. It can be verified that $\phi(\mathcal{X}, \rho)$ is continuously differentiable in a domain $D_{\mathcal{X}} \times D_\rho$. The change of variables

$$\mathcal{Y} = \mathcal{X} - \mathcal{X}_{\mathrm{ss}}(\rho), \qquad \mathcal{Z} = \vartheta - \phi(\mathcal{X}, \rho)$$

transforms the system (C.88)–(C.89) into

$$\begin{aligned} \dot{\mathcal{Y}} &= g(\mathcal{Y} + \mathcal{X}_{\mathrm{ss}}(\rho), \rho) + N(\rho)\mathcal{Z} - \frac{\partial \mathcal{X}_{\mathrm{ss}}}{\partial \rho}\dot{\rho} \qquad &(\mathrm{C.90}) \\ \varepsilon\dot{\mathcal{Z}} &= -\mathcal{Z} - \varepsilon\frac{\partial \phi}{\partial \mathcal{X}}[g(\mathcal{Y} + \mathcal{X}_{\mathrm{ss}}(\rho), \rho) + N(\rho)\mathcal{Z}] - \varepsilon\frac{\partial \phi}{\partial \rho}\dot{\rho} \qquad &(\mathrm{C.91}) \end{aligned}$$

Using $\mathcal{V} = \mathcal{Y}^T P_{ms} \mathcal{Y} + (1/2)\mathcal{Z}^T\mathcal{Z}$ as a Lyapunov function for (C.90)–(C.91), we obtain

$$\begin{aligned} \dot{\mathcal{V}} &= -\mathcal{Y}^T\mathcal{Y} + 2\mathcal{Y}^T P_{ms}\left[g(\mathcal{Y} + \mathcal{X}_{\mathrm{ss}}(\rho), \rho) - A_{ms}(\rho)\mathcal{Y} + N(\rho)\mathcal{Z} - \frac{\partial \mathcal{X}_{\mathrm{ss}}}{\partial \rho}\dot{\rho}\right] \\ &\quad + \mathcal{Y}^T\left[\frac{d}{dt}P_{ms}(\rho)\right]\mathcal{Y} \\ &\quad - \frac{1}{\varepsilon}\,\mathcal{Z}^T\mathcal{Z} - \mathcal{Z}^T\left\{\frac{\partial \phi}{\partial \mathcal{X}}[g(\mathcal{Y} + \mathcal{X}_{\mathrm{ss}}(\rho), \rho) + N(\rho)\mathcal{Z}] + \frac{\partial \phi}{\partial \rho}\dot{\rho}\right\} \\ &\le -\|\mathcal{Y}\|_2^2 - \frac{1}{\varepsilon}\,\|\mathcal{Z}\|_2^2 + c_1\|\mathcal{Y}\|_2^3 + c_2\|\mathcal{Y}\|_2\,\|\mathcal{Z}\|_2 + c_3\|\mathcal{Y}\|_2\|\dot{\rho}\|_2 \\ &\quad + c_4\|\mathcal{Y}\|_2^2\|\dot{\rho}\|_2 + c_5\|\mathcal{Z}\|_2^2 + c_6\|\mathcal{Z}\|_2\|\dot{\rho}\|_2 \end{aligned}$$

in some neighborhood of the origin, for some positive constants c_i. Limiting our analysis to a neighborhood where $\|\mathcal{Y}\|_2 \le c_7 \le 1/(4c_1)$, we arrive at the inequality

$$\begin{aligned} \dot{\mathcal{V}} &\le -\frac{1}{2}\|\mathcal{Y}\|_2^2 - \frac{1}{2\varepsilon}\,\|\mathcal{Z}\|_2^2 + (c_3\|\mathcal{Y}\|_2 + c_4c_7\|\mathcal{Y}\|_2 + c_6\|\mathcal{Z}\|_2)\|\dot{\rho}\|_2 \\ &\quad - \begin{bmatrix} \|\mathcal{Y}\|_2 \\ \|\mathcal{Z}\|_2 \end{bmatrix}^T \begin{bmatrix} 1/4 & -\,c_2/2 \\ -\,c_2/2 & 1/(2\varepsilon) - c_5 \end{bmatrix} \begin{bmatrix} \|\mathcal{Y}\|_2 \\ \|\mathcal{Z}\|_2 \end{bmatrix} \end{aligned}$$

Choosing ε^* small enough that the 2×2 matrix is positive definite for all $0 < \varepsilon < \varepsilon^*$, we end up with

$$\dot{\mathcal{V}} \leq -2\alpha\mathcal{V} + 2\beta\sqrt{\mathcal{V}}\|\dot{\rho}\|_2$$

for some positive constants α and β. Hence, $W = \sqrt{\mathcal{V}}$ satisfies the inequality

$$D^+W \leq -\alpha W + \beta\|\dot{\rho}\|_2$$

Applying the comparison lemma concludes the proof.

C.21 Proof of Theorem 13.1

The proof uses the notions of Lie brackets and involutive distributions, introduced in Section 14.3, as well as the notion of complete integrability. A nonsingular distribution Δ on D, generated by $f_1, \ldots, f_k$, is completely integrable if for each $x_0 \in D$, there exists a neighborhood N of x_0 and $n-k$ real-valued smooth functions $h_1(x), \ldots, h_{n-k}(x)$ such that

$$\frac{\partial h_j}{\partial x} f_i(x) = 0, \quad \forall\ 1 \leq i \leq k \text{ and } 1 \leq j \leq n-k$$

and the row vectors $dh_1(x), \ldots, d_{n-k}(x)$ are linearly independent for all $x \in D$, where

$$dh(x) = \frac{\partial h}{\partial x} = \left[\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right]$$

is called the *differential* of h. A key result from differential geometry is **Frobenius theorem**,[14] which states that *a nonsingular distribution is completely integrable if and only if it is involutive.*

We start by stating and proving two preliminary lemmas.

Lemma C.8 *For all $x \in D$ and all integers k and j such that $k \geq 0$ and $0 \leq j \leq \rho - k - 1$, we have*

$$L_{ad_f^j g} L_f^k h(x) = \begin{cases} 0, & 0 \leq j+k < \rho - 1 \\ (-1)^j L_g L_f^{\rho-1} h(x) \neq 0, & j+k = \rho - 1 \end{cases} \tag{C.92}$$

◇

Proof: Prove it by induction of j. For $j = 0$, (C.92) holds by the definition of relative degree. Assume now that (C.92) holds for some j and prove it for $j+1$. Recall from the Jacobi identity (Exercise 13.8) that

$$L_{[f,\beta]}\lambda(x) = L_f L_\beta \lambda(x) - L_\beta L_f \lambda(x)$$

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[14] See [88] for the proof of Frobenius theorem.

for any real-valued function λ and any vector fields f and β. Taking $\lambda = L_f^k h$ and $\beta = ad_f^j g$, we obtain

$$L_{ad_f^{j+1} g} L_f^k h(x) = L_{[f, ad_f^j g]} L_f^k h(x) = L_f L_{ad_f^j g} L_f^k h(x) - L_{ad_f^j g} L_f^{k+1} h(x)$$

We note that the first term on the right-hand side vanishes since

$$j + k + 1 \leq \rho - 1 \Rightarrow j + k < \rho - 1 \Rightarrow L_f L_{ad_f^j g} L_f^k h(x) = 0$$

Moreover,

$$L_{ad_f^j g} L_f^{k+1} h(x) = \begin{cases} 0, & 0 \leq j + k + 1 < \rho - 1 \\ (-1)^j L_g L_f^{\rho-1} h(x) \neq 0, & j + k + 1 = \rho - 1 \end{cases}$$

by the assumption that (C.92) holds for j. Thus,

$$L_{ad_f^{j+1} g} L_f^k h(x) = \begin{cases} 0, & 0 \leq j + k + 1 < \rho - 1 \\ (-1)^{j+1} L_g L_f^{\rho-1} h(x) \neq 0, & j + k + 1 = \rho - 1 \end{cases}$$

which completes the proof of the lemma. □

Lemma C.9 *For all $x \in D$,*

- *the row vectors $dh(x), dL_f h(x), \ldots, dL_f^{\rho-1} h(x)$ are linearly independent;*
- *the column vectors $g(x), ad_f g(x), \ldots, ad_f^{\rho-1} g(x)$ are linearly independent.*

◇

Proof: We have

$$\begin{bmatrix} dh(x) \\ \vdots \\ dL_f^{\rho-1} h(x) \end{bmatrix} \begin{bmatrix} g(x) & \ldots & ad_f^{\rho-1} g(x) \end{bmatrix} =$$

$$\begin{bmatrix} L_g h(x) & L_{ad_f g} h(x) & \ldots & \ldots & L_{ad_f^{\rho-1} g} h(x) \\ L_g L_f h(x) & & & L_{ad_f^{\rho-2} g} L_f h(x) & * \\ \vdots & & & & \vdots \\ L_g L_f^{\rho-1} h(x) & * & \ldots & & * \end{bmatrix}$$

From the previous lemma, the right-hand side matrix takes the form

$$\begin{bmatrix} 0 & \dots & \dots & 0 & \diamond \\ 0 & & & \diamond & * \\ \vdots & & & & \vdots \\ 0 & \diamond & & & * \\ \diamond & * & \dots & & * \end{bmatrix}$$

where $\diamond$ denotes a nonzero element. Thus, the matrix is nonsingular, which proves the lemma, for if any of the two matrices on the left-hand side has rank less than ρ, their product must be singular. □

Lemma C.9 shows that $\rho \leq n$. We are now ready to prove Theorem 13.1. The proof for the case $\rho = n$ follows from Lemma C.9, whose first statement shows that $[\partial T/\partial x]$ is nonsingular. Consider the case $\rho < n$. The distribution $\Delta = \text{span}\{g\}$ is nonsingular, involutive, and has dimension one.[15] By Frobenius theorem, Δ is completely integrable. Hence, for every $x_0 \in D$, there exist a neighborhood N_1 of x_0 and $n-1$ smooth functions $\phi_1(x), \dots, \phi_{n-1}(x)$, with linearly independent differentials such that

$$L_g\phi_i(x) = 0, \quad \text{for } 1 \leq i \leq n-1, \ \forall \ x \in N_1$$

Because

$$L_gL_f^ih(x) = 0, \quad \text{for } 0 \leq i \leq \rho - 2$$

and $dh(x), \dots, dL_f^{\rho-2}h(x)$ are linearly independent, we can use $h, \dots, L_f^{\rho-2}h$ as part of these $n-1$ functions. In particular, we take them as $\phi_{n-\rho+1}, \dots, \phi_{n-1}$. Since $L_gL_f^{\rho-1}h(x) \neq 0$, the row vector $dL_f^{\rho-1}h(x_0)$ is linearly independent of the row vectors $d\phi_1(x_0), \dots, d\phi_{n-1}(x_0)$. Therefore,

$$\text{rank}\left[\frac{\partial T}{\partial x}(x_0)\right] = n \ \Rightarrow \ \frac{\partial T}{\partial x}(x_0) \text{ is nonsingular}$$

and there is a neighborhood N_2 of x_0 such that $T(x)$, restricted to N_2, is a diffeomorphism on N_2. Taking $N = N_1 \cap N_2$ completes the proof of the theorem. □

C.22 Proof of Theorem 13.2

The system

$$\dot{x} = f(x) + g(x)u$$

---

[15] Note that any nonsingular distribution of dimension one is automatically involutive.

is feedback linearizable if and only if a sufficiently smooth function $h(x)$ exists such that the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree n in $D_0 \subset D$; that is, $h(x)$ satisfies

$$L_g L_f^i h(x) = 0, \text{ for } 0 \le i \le n-2 \text{ and } L_g L_f^{n-1} h(x) \ne 0, \quad \forall\, x \in D_0 \tag{C.93}$$

Thus, to prove the theorem, we need to show that the existence of $h(x)$ satisfying (C.93) is equivalent to conditions 1 and 2.

Necessity: Suppose there is $h(x)$ satisfying (C.93). Lemma C.9 shows that rank $\mathcal{G} = n$. Then, $\mathcal{D}$ is nonsingular and has dimension $n-1$. From (C.92), with $k = 0$ and $\rho = n$, we have

$$L_g h(x) = L_{ad_f g} h(x) = \cdots = L_{ad_f^{n-2} g} h(x) = 0$$

which can be written as

$$dh(x)[g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)] = 0$$

This equation implies that $\mathcal{D}$ is completely integrable and it follows from Frobenius theorem that $\mathcal{D}$ is involutive.

Sufficiency: Suppose conditions 1 and 2 are satisfied. Then, $\mathcal{D}$ is nonsingular and has dimension $n-1$. By Frobenius theorem, there exists $h(x)$ satisfying

$$L_g h(x) = L_{ad_f g} h(x) = \cdots = L_{ad_f^{n-2} g} h(x) = 0$$

Using the Jacobi identity (Exercise 13.8), it can be verified that

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0$$

Furthermore,

$$dh(x)\mathcal{G}(x) = dh(x)[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)] = [0, \ldots, 0, L_{ad_f^{n-1} g} h(x)]$$

Since rank $\mathcal{G} = n$ and $dh(x) \ne 0$, it must be true that $L_{ad_f^{n-1} g} h(x) \ne 0$. Using the Jacobi identity, it can be verified that $L_g L_f^{n-1} h(x) \ne 0$, which completes the proof of the theorem.

C.23 Proof of Theorem 14.6

For the purpose of analysis, the observer dynamics are replaced by the equivalent dynamics of the scaled estimation error

$$\eta_{ij} = \frac{x_{ij} - \hat{x}_{ij}}{\varepsilon^{\rho_i - j}}$$

for $1 \le i \le m$ and $1 \le j \le \rho_i$. Hence, we have $\hat{x} = x - D(\varepsilon)\eta$, where

$$\begin{aligned} \eta &= [\eta_{11}, \ldots, \eta_{1\rho_1}, \ldots, \eta_{m1}, \ldots, \eta_{m\rho_m}]^T, \\ D(\varepsilon) &= \text{block diag}[D_1, \ldots, D_m] \\ D_i &= \text{diag}[\varepsilon^{\rho_i - 1}, \ldots, 1]_{\rho_i \times \rho_i} \end{aligned}$$

The closed-loop system can be represented by

$$\begin{aligned} \dot{x} &= Ax + B\phi(x, z, \gamma(\vartheta, x - D(\varepsilon)\eta, \zeta)) \\ \dot{z} &= \psi(x, z, \gamma(\vartheta, x - D(\varepsilon)\eta, \zeta)) \\ \dot{\vartheta} &= \Gamma(\vartheta, x - D(\varepsilon)\eta, \zeta) \\ \varepsilon\dot{\eta} &= A_0\eta + \varepsilon B\delta(x, z, \vartheta, D(\varepsilon)\eta) \end{aligned}$$

where

$$\delta(x, z, \vartheta, D(\varepsilon)\eta) = \phi(x, z, \gamma(\vartheta, \hat{x}, \zeta)) - \phi_0(\hat{x}, \zeta, \gamma(\vartheta, \hat{x}, \zeta))$$

and $(1/\varepsilon)A_0 = D^{-1}(\varepsilon)(A - HC)D(\varepsilon)$ is a $\rho \times \rho$ Hurwitz matrix. For convenience, we rewrite the system in the compact singularly perturbed form

$$\dot{\mathcal{X}} = F(\mathcal{X}, D(\varepsilon)\eta) \tag{C.94}$$

$$\varepsilon\dot{\eta} = A_0\eta + \varepsilon B\Delta(\mathcal{X}, D(\varepsilon)\eta) \tag{C.95}$$

where $F(\mathcal{X}, 0) = f(\mathcal{X})$. The initial states are $\mathcal{X}(0) = (x(0), z(0), \vartheta(0)) \in \mathcal{S}$ and $\hat{x}(0) \in \mathcal{Q}$. Thus, we have $\eta(0) = D^{-1}(\varepsilon)[x(0) - \hat{x}(0)]$. Setting $\varepsilon = 0$ in (C.95) yields $\eta = 0$ and the reduced system

$$\dot{\mathcal{X}} = f(\mathcal{X}) \tag{C.96}$$

is nothing, but the closed-loop system under state feedback. The boundary-layer model, obtained by applying the change of time variable $\tau = t/\varepsilon$ then setting $\varepsilon = 0$, is given by

$$\frac{d\eta}{d\tau} = A_0\eta$$

Since the origin of (C.96) is asymptotically stable and $\mathcal{R}$ is its region of attraction, by (the converse Lyapunov) Theorem 4.17, there is a smooth, positive definite function $V(\mathcal{X})$ and a continuous, positive definite function $U(\mathcal{X})$, both defined for all $\mathcal{X} \in \mathcal{R}$, such that

$$V(\mathcal{X}) \to \infty \text{ as } \mathcal{X} \to \partial\mathcal{R}$$

$$\frac{\partial V}{\partial \mathcal{X}} f(\mathcal{X}) \le -U(\mathcal{X}), \quad \forall\ \mathcal{X} \in \mathcal{R}$$

and for any $c > 0$, $\{V(\mathcal{X}) \le c\}$ is a compact subset of $\mathcal{R}$. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$. Choose positive constants b and c such that $c > b > \max_{\mathcal{X} \in \mathcal{S}} V(\mathcal{X})$. Then

$$\mathcal{S} \subset \Omega_b = \{V(\mathcal{X}) \le b\} \subset \Omega_c = \{V(\mathcal{X}) \le c\} \subset \mathcal{R}$$

For the boundary-layer system, the Lyapunov function $W(\eta) = \eta^T P_0 \eta$, where P_0 is the positive definite solution of the Lyapunov equation $P_0 A_0 + A_0^T P_0 = -I$, satisfies

$$\lambda_{\min}(P_0)\|\eta\|^2 \le W(\eta) \le \lambda_{\max}(P_0)\|\eta\|^2$$

$$\frac{\partial W}{\partial \eta} A_0 \eta \le -\|\eta\|^2$$

where throughout the proof we use $\|\cdot\| = \|\cdot\|_2$. Due to equivalence of norms, it is sufficient to prove (14.114) and (14.115) in the 2-norm. Let $\Sigma = \{W(\eta) \le \varrho\varepsilon^2\}$ and $\Lambda = \Omega_c \times \Sigma$. Due to the global boundedness of F and Δ in $\hat{x}$, for all $\mathcal{X} \in \Omega_c$ and $\eta \in R^\rho$, we have

$$\|F(\mathcal{X}, D(\varepsilon)\eta)\| \le k_1, \qquad \|\Delta(\mathcal{X}, D(\varepsilon)\eta)\| \le k_2$$

where k_1 and k_2 are positive constants independent of ε. Moreover, for any $0 < \tilde{\varepsilon} < 1$, there is L_1, independent of ε, such that for all $(\mathcal{X}, \eta) \in \Lambda$ and every $0 < \varepsilon \le \tilde{\varepsilon}$, we have

$$\|F(\mathcal{X}, D(\varepsilon)\eta) - F(\mathcal{X}, 0)\| \le L_1\|\eta\|$$

We will always consider $\varepsilon \le \tilde{\varepsilon}$. We start by showing that there exist positive constants ϱ and ε_1 (dependent on ϱ) such that the compact set Λ is positively invariant for every $0 < \varepsilon \le \varepsilon_1$. This can be done by verifying that

$$\dot{V} \le -U(\mathcal{X}) + \varepsilon k_3$$

and

$$\dot{W} \le -\frac{1}{\varepsilon}\|\eta\|^2 + 2\|\eta\|\|P_0\|\|B\|k_2 \le -\frac{1}{\varepsilon}\|\eta\|^2 + 2\|\eta\|\|P_0\|k_2$$

for all $(\mathcal{X}, \eta) \in \Lambda$, where $k_3 = L_1 L_2 \sqrt{\varrho/\lambda_{\min}(P_0)}$, $\|P_0\| = \lambda_{\max}(P_0)$, $\|B\| = 1$, and L_2 is an upper bound for $\|\partial V/\partial \mathcal{X}\|$ over Ω_c. Taking $\varrho = 16k_2^2\|P_0\|^3$ and $\varepsilon_1 = \beta/k_3$, where $\beta = \min_{\mathcal{X}\in\partial\Omega_c} U(\mathcal{X})$, it can be shown that, for every $0 < \varepsilon \le \varepsilon_1$, we have $\dot{V} \le 0$ for all $(\mathcal{X}, \eta) \in \{V(\mathcal{X}) = c\} \times \Sigma$ and $\dot{W} \le 0$ for all $(\mathcal{X}, \eta) \in \Omega_c \times \{W(\eta) = \varrho\varepsilon^2\}$. Hence, Λ is positively invariant.

Now we consider the initial state $(\mathcal{X}(0), \hat{x}(0)) \in \mathcal{S} \times \mathcal{Q}$. It can be verified that the corresponding initial error $\eta(0)$ satisfies $\|\eta(0)\| \le k/\varepsilon^{(\rho_{\max}-1)}$ for some nonnegative constant k dependent on $\mathcal{S}$ and $\mathcal{Q}$, where $\rho_{\max} = \max\{\rho_1, \dots, \rho_m\}$. Since $\mathcal{X}(0)$ is in the interior of Ω_c, it can be shown that

$$\|\mathcal{X}(t) - \mathcal{X}(0)\| \le k_1 t \tag{C.97}$$

as long as $\mathcal{X}(t) \in \Omega_c$. Thus, there exists a finite time T_0, independent of ε, such that $\mathcal{X}(t) \in \Omega_c$ for all $t \in [0, T_0]$. During this time interval, we have

$$\dot{W} \le -\frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta\|^2 + 2k_2\|P_0\|\|\eta\| \le -\frac{1}{2\varepsilon}\|\eta\|^2, \quad \text{for } W(\eta) \ge \varrho\varepsilon^2$$

since

$$W(\eta) \geq \varrho\varepsilon^2 \;\Rightarrow\; \|P_0\|\|\eta\|^2 \geq 16k_2^2\|P_0\|^3\varepsilon^2 \;\Leftrightarrow\; \|\eta\| \geq 4k_2\|P_0\|\varepsilon$$

Therefore,

$$W(\eta(t)) \leq \frac{\sigma_2}{\varepsilon^{2(\rho_{\max}-1)}} \exp\left(-\sigma_1 t/\varepsilon\right) \tag{C.98}$$

where $\sigma_1 = 1/(2\|P_0\|)$ and $\sigma_2 = k^2\|P_0\|$. Choose $\varepsilon_2 > 0$ small enough that

$$T(\varepsilon) \overset{\text{def}}{=} \frac{\varepsilon}{\sigma_1} \ln\left(\frac{\sigma_2}{\varrho\varepsilon^{2\rho_{\max}}}\right) \leq \tfrac{1}{2}T_0$$

for all $0 < \varepsilon \leq \varepsilon_2$. We note that ε_2 exists, since $T(\varepsilon)$ tends to zero as ε tends to zero. It follows that $W(\eta(T(\varepsilon))) \leq \varrho\varepsilon^2$ for every $0 < \varepsilon \leq \varepsilon_2$. Taking $\varepsilon_1^* = \min\{\tilde{\varepsilon}, \varepsilon_1, \varepsilon_2\}$ guarantees that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the trajectory $(\mathcal{X}(t), \eta(t))$ enters Λ during the interval $[0, T(\varepsilon)]$ and remains there for all $t \geq T(\varepsilon)$. Consequently, the trajectory is bounded for all $t \geq T(\varepsilon)$. On the other hand, for $t \in [0, T(\varepsilon)]$, the trajectory is bounded by virtue of inequalities (C.97) and (C.98).

Next, we show (14.114). We know that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the solutions are inside the set Λ for all $t \geq T(\varepsilon)$, where Λ is $O(\varepsilon)$ in the direction of the variable η. Thus, we can find $\varepsilon_3 = \varepsilon_3(\mu) \leq \varepsilon_1^*$ such that, for every $0 < \varepsilon \leq \varepsilon_3$, we have

$$\|\eta(t)\| \leq \mu/2, \qquad \forall\, t \geq T(\varepsilon_3) = \bar{T}(\mu) \tag{C.99}$$

Using the fact that $\dot{V} \leq -U(x) + \varepsilon k_3$ for all $(\mathcal{X}, \eta) \in \Lambda$, we conclude that

$$\dot{V} \leq -\tfrac{1}{2}U(x), \qquad \text{for } \mathcal{X} \notin \{U(\mathcal{X}) \leq 2k_3\varepsilon \overset{\text{def}}{=} \nu(\varepsilon)\} \tag{C.100}$$

Because $U(\mathcal{X})$ is positive definite and continuous, the set $\{U(\mathcal{X}) \leq \nu(\varepsilon)\}$ is a compact set for sufficiently small ε. Let $c_0(\varepsilon) = \max_{U(\mathcal{X}) \leq \nu(\varepsilon)}\{V(\mathcal{X})\}$; $c_0(\varepsilon)$ is nondecreasing and $\lim_{\varepsilon\to 0} c_0(\varepsilon) = 0$. Consider the compact set $\{V(\mathcal{X}) \leq c_0(\varepsilon)\}$. We have $\{U(\mathcal{X}) \leq \nu(\varepsilon)\} \subset \{V(\mathcal{X}) \leq c_0(\varepsilon)\}$. Choose $\varepsilon_4 = \varepsilon_4(\mu) \leq \varepsilon_1^*$ small enough such that, for all $\varepsilon \leq \varepsilon_4$, the set $\{U(\mathcal{X}) \leq \nu(\varepsilon)\}$ is compact, the set $\{V(\mathcal{X}) \leq c_0(\varepsilon)\}$ is in the interior of Ω_c, and

$$\{V(\mathcal{X}) \leq c_0(\varepsilon)\} \subset \{\|\mathcal{X}\| \leq \mu/2\} \tag{C.101}$$

Then, for all $\mathcal{X} \in \Omega_c$, but $\mathcal{X} \notin \{V(\mathcal{X}) \leq c_0(\varepsilon)\}$, we have an inequality similar to (C.100). Therefore, we conclude that the set $\{V(\mathcal{X}) \leq c_0(\varepsilon)\} \times \Sigma$ is positively invariant and every trajectory in $\Omega_c \times \Sigma$ reaches $\{V(\mathcal{X}) \leq c_0(\varepsilon)\} \times \Sigma$ in finite time. In other words, given (C.101), there exists a finite time $\tilde{T} = \tilde{T}(\mu)$ such that for every $0 < \varepsilon \leq \varepsilon_4$

$$\|\mathcal{X}(t)\| \leq \mu/2. \quad \forall\, t \geq \tilde{T} \tag{C.102}$$

Take $\varepsilon_2^* = \varepsilon_2^*(\mu) = \min\{\varepsilon_3, \varepsilon_4\}$ and $T_2 = T_2(\mu) = \max\{\bar{T}, \tilde{T}\}$. Then, (14.114) follows from (C.99), (C.102), $\hat{x} = x - D(\varepsilon)\eta$, and $\|D(\varepsilon)\| = 1$.

To show (14.115), we divide the interval $[0, \infty)$ into three intervals $[0, T(\varepsilon)]$, $[T(\varepsilon), T_3]$, and $[T_3, \infty)$ and show (14.115) for each interval. From the ultimate boundedness of $\mathcal{X}(t)$, shown in (14.114), and the asymptotic stability of the origin of (C.96), we conclude that there exists a finite time $T_3 \geq T(\varepsilon)$, independent of ε, such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, we have

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \geq T_3 \tag{C.103}$$

From (C.97), we know that

$$\|\mathcal{X}(t) - \mathcal{X}(0)\| \leq k_1 t$$

during the interval $[0, T(\varepsilon)]$. Similarly, it can be shown that

$$\|\mathcal{X}_r(t) - \mathcal{X}(0)\| \leq k_1 t$$

during the same interval. Hence,

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq 2k_1 T(\varepsilon), \quad \forall\, t \in [0, T(\varepsilon)]$$

Since $T(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists $0 < \varepsilon_5 \leq \varepsilon_2^*$ such that, for every $0 < \varepsilon \leq \varepsilon_5$, we have

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \in [0, T(\varepsilon)] \tag{C.104}$$

Over the interval $[T(\varepsilon), T_3]$, the solution $\mathcal{X}(t)$ satisfies

$$\dot{\mathcal{X}} = F(\mathcal{X}, D(\varepsilon)\eta(t)), \quad \text{with } \|\mathcal{X}(T(\varepsilon)) - \mathcal{X}_r(T(\varepsilon))\| \leq \delta_1(\varepsilon)$$

where $D(\varepsilon)\eta$ is $O(\varepsilon)$ and $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, by Theorem 3.5, we conclude that there exists $0 < \varepsilon_6 \leq \varepsilon_2^*$ such that, for every $0 < \varepsilon \leq \varepsilon_6$, we have

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \in [T(\varepsilon), T_3] \tag{C.105}$$

Take $\varepsilon_3^* = \min\{\varepsilon_5, \varepsilon_6\}$. Then, (14.115) follows from (C.103) through (C.105).

Finally, assuming that the origin of (C.96) is exponentially stable, it follows from (the converse Lyapunov) Theorem 4.14 that there exists a continuously differentiable Lyapunov function $V_1(\mathcal{X})$ which satisfies the inequalities

$$b_1\|\mathcal{X}\|^2 \leq V_1(\mathcal{X}) \leq b_2\|\mathcal{X}\|^2, \quad \frac{\partial V_1}{\partial \mathcal{X}} F(\mathcal{X}, 0) \leq -b_3\|\mathcal{X}\|^2, \quad \left\| \frac{\partial V_1}{\partial \mathcal{X}} \right\| \leq b_4\|\mathcal{X}\|$$

over the ball $B_r \subset \mathcal{R}$ for some positive constants r, b_1, b_2, b_3, and b_4. Using the local Lipschitz property of F and Δ and the fact that $F(0,0) = 0$ and $\Delta(0,0) = 0$, it can be shown that the composite Lyapunov function $V_2(\mathcal{X}, \eta) = V_1(\mathcal{X}) + W(\eta)$ satisfies

$$\dot{V}_2 \leq -\mathcal{Y}^T Q \mathcal{Y}$$

where

$$Q = \begin{bmatrix} b_3 & -\beta_1 \\ -\beta_1 & (1/\varepsilon) - \beta_2 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} \|\mathcal{X}\| \\ \|\eta\| \end{bmatrix}$$

for some nonnegative constants β_1 and β_2. The matrix Q will be positive definite for sufficiently small ε. Hence, there is a neighborhood $\mathcal{N}$ of the origin, independent of ε, and $\varepsilon_7 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_7$, the origin is exponentially stable and every trajectory in $\mathcal{N}$ converges to the origin as $t \to \infty$. By (14.114), there exists $\varepsilon_8 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_8$, the solutions starting in $\mathcal{S} \times \mathcal{Q}$ enter $\mathcal{N}$ in finite time. Hence, for every $0 < \varepsilon \leq \varepsilon_4^* = \min\{\varepsilon_7, \varepsilon_8\}$, the origin is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of the region of attraction.

Notes and References

The main references used in the preparation of this text are Hirsch and Smale [81], Hale [75], and Miller and Michel [135] for the theory of ordinary differential equations; Hahn [72], Krasovskii [107], and Rouche, Habets, and Laloy [154] for stability theory; and the texts by Vidyasagar [201](first edition) and Hsu and Meyer [85]. References for the various topics are listed under the respective chapters. The appendices on mathematical review and contraction mapping have been patterned after similar summaries in Bertsekas [27] and Luenberger [121]. For a complete coverage of topics in these appendices, the reader may consult any text on mathematical analysis. We have used Apostol [10]. Other standard texts are Rudin [157] and Royden [156].

Chapter 1. The tunnel diode circuit and negative resistance oscillator are taken from Chua, Desoer, and Kuh [39]. The presentation of the mass-spring system is based on Mickens [134] and Southward [184]. The Hopfield neural network description is based on Hopfield [82] and Michel, Farrel and Porod [131]. The adaptive control example is based on Sastry and Bodson [168]. An interesting book to read about nonlinear phenomena, including chaos, is Strogatz [187].

Chapter 2. The classical material on second-order systems in Sections 2.1 to 2.4 can be found in almost any text on nonlinear systems. Our presentation has followed closely the lucid presentation of Chua, Desoer, and Kuh [39]. Section 2.5 is based on Parker and Chua [149]. Section 2.6 is based on Hirsch and Smale [81, Chapters 10 and 11], Guckenheimer and Holmes [70, Section 1.8], and Strogatz [187]. The bifurcation material of Section 2.7 is based on [70] and [187].

Chapter 3. The material in Sections 3.1–3.3 is standard and can be found in one form or the other in any graduate text on ordinary differential equations. Section 3.1 comes very close to the presentation in Vidyasagar [201], while Sections 3.2 and 3.3 are based on Hirsch and Smale [81] and Coppel [43]. The comparison principle is based on Hale [75], Miller and Michel [135], and Yoshizawa [213].

Chapter 4. Hahn [72], Krasovskii [107], and Rouche, Habets, and Laloy [154] are authoritative references on Lyapunov stability. The presentation style of Section 4.1 is influenced by Hirsch and Smale [81]. The proof of Theorem 4.1 is taken from Hirsch and Smale [81, Section 9.3], while that of Theorem 4.3 is based on Hahn [72, Section 25], Hale [75], and Miller and Michel [135]. The invariance principle presentation of Section 4.2 follows the original work by LaSalle [112]. The proof of Lemma 4.1 is based on Rouche, Habets, and Laloy [154, Appendix III]. The application of the invariance principle to neural networks is standard and can be found in Hopfield [82]. Our presentation is influenced by Salam [163]. See also Cohen and Grossberg [42] for a generalization of Example 4.11. The material on linear time-invariant systems in Section 4.3 is taken from Chen [35]. The proof of Theorem 4.6 is taken from Kailath [94]. The proof of Theorem 4.7 on linearization is guided by Rouche and Mawhin [155, Sections 1.6 and 1.7], which includes a careful treatment of the case when the linearization has at least one eigenvalue in the right-half plane with other eigenvalues on the imaginary axis. The proof of Lemma 4.3 is taken from Hahn [72, Section 24B]. The statement and proof of Lemma 4.4 is guided by Hahn [72, Section 24E] and Sontag [181, Lemma 6.1]. The proof of Lemma 4.5 is taken from Hahn [72, Section 35] (for the local part) and Lin, Sontag, and Wang [118] (for the global part). The proofs of Theorems 4.8 and 4.9 combine ideas from Hahn [72, Section 25] and Rouche, Habets, and Laloy [154, Section I.6]. Section 4.6 is based on Vidyasagar [201]. The proofs of the converse Lyapunov theorems are based on Krasovskii [107, Theorem 11.1] for Theorem 4.14, Miller and Michel [135, Section 5.13] and Hahn [72, Section 49], with some insight from Hoppensteadt [83], for Theorem 4.16, and Kurzweil [109] and Lin, Sontag, and Wang [118] for Theorem 4.17. The proof of Theorem 4.18 is guided by Miller and Michel [135, Theorem 9.14] and Corless and Leitmann [45]. The concept of input-to-state stability was introduced by Sontag [181], who proved a few basic results. (See [182].) Our presentation benefited from a nice exposition in Krstic, Kanellakopolous, and Kokotovic [108].

Chapter 5. The treatment of $\mathcal{L}$-stability in Sections 5.1 and 5.2 is based on Desoer and Vidyasagar [53] and Vidyasagar [201]. Section 5.3 on the $\mathcal{L}_2$-gain is based on van der Schaft [199], with help from the papers by Willems [209] and Hill and Moylan [77], [79] on dissipative systems. The proof of Theorem 5.4 is based on [53, Section 2.6], [200, Section 3.1.2], and [220, Section 4.3]. The small-gain theorem presentation is based on Desoer and Vidyasagar [53] and the tutorial article by Teel, Georgiou, Praly, and Sontag [192]. Example 5.14 is taken from [192].

Chapter 6. The passivity approach is based on several references, including Hill and Moylan [77], [78], Sepulchre, Jankovic, and Kokotovic [172], Byrnes, Isidori and Willems [31], Krstic, Kanellakopoulos, and Kokotovic [108], Teel, Georgiou, Praly, and Sontag [192], and Vidyasagar [201]. An expanded treatment of positive real transfer functions and the positive real lemma is given in Anderson and Vongpanitlerd [6]. The proof of Lemma 6.1 is based on Tao and Ioannou [190] and Wen [206]. The proofs of Lemmas 6.2 and 6.3 are based on Anderson [4].

Chapter 7. Absolute stability has a long history in the control theory literature. For a historical perspective, see Hsu and Meyer [85, Sections 5.9 and 9.5]. Our presentation of the circle and Popov criteria is based on Hsu and Meyer [85, Section 9.5 and Chapter 10], Vidyasagar [201], Siljak [174, Sections 8.6 through 8.9 and Appendix H], and Moore and Anderson [138]. A comprehensive coverage of absolute stability can be found in Narendra and Taylor [140]. Detailed treatment of the describing function method can be found in Atherton [18] and Hsu and Meyer [85, Chapters 6 and 7]. Our presentation in Section 7.2 follows Mees [128, Chapter 5]. The proof of Lemma 7.1 is taken from Mees and Bergen [130]. The proof of Theorem 7.4 is based on Mees and Bergen [130] and Bergen and Franks [25]. The presentation of the error analysis has gained a lot from a lucid presentation in Siljak [174, Appendix G], from which Example 7.14 is taken.

Chapter 8. Section 8.1 is based mostly on Carr [34], with help from Guckenheimer and Holmes [70]. The proofs of Theorems 8.1 and 8.3 are taken from Carr [34, Chapter 2]. The proof of Theorem 8.2 by using Lyapunov analysis is simpler than the proof given in Carr [34]. Corollary 8.1 was suggested to the author by Miroslav Krstic. The proof of Lemma 8.1 is based on Hahn [72, Section 33]. Example 8.10 is taken from Willems [210]. There is a vast literature on estimating the region of attraction. Some methods are described or surveyed in [28], [37], [65], [80], [133], and [143]. The proof of Lemma 8.2 is taken from Popov [152, page 211]. The proof of Theorem 8.5 is based on Sastry and Bodson [168, Theorem 1.5.2]. The discussion on the stability of periodic solutions in Section 8.4 is based on Hahn [72, Section 81], Miller and Michel [135, Section 6.4], and Hale [75, Section VI.2]. A natural continuation of this discussion is the Poincaré map method, which is described in Hirsch and Smale [81, Chapter 13] and Guckenheimer and Holmes [70, Section 1.5]; see also the second edition of this text.

Chapter 9. The material of Sections 9.1 and 9.2 is based on a vast literature on robustness analysis in control theory. We can say, however, that the basic references are Hahn [72, Section 56] and Krasovskii [107, Sections 19 and 24]. Similar results are given in Coppel [43, Section III.3] for the case when the nominal system is linear, but the derivation is not based on Lyapunov theory; instead, it uses properties of fundamental matrices. The case of nonvanishing perturbations is referred to in Hahn [72] and Krasovskii [107] as the case of "persistent disturbance." The results on nonvanishing perturbations are also related to the concept of total stability. (See Hahn [72, Section 56].) The comparison method of Section 9.3 is based on sporadic use of the comparison lemma in the control literature. Section 9.5, on the stability of interconnected systems, is based mostly on the tutorial paper Araki [11], with help from the research monographs Siljak [175] and Michel and Miller [132].

The neural network example is taken from Michel, Farrel, and Porod [131]. The treatment of slowly varying systems is based on Desoer and Vidyasagar [53, Section IV.8], Vidyasagar [201], Kokotovic, Khalil, and O'Reilly [105, Section 5.2], and Hoppensteadt [83]. Lemma 9.8 is a specialization of Lemma 2 of Hoppensteadt [83] to the case of exponential stability.

Chapter 10. The perturbation method of Section 10.1 is classical and can be found in many references. A detailed treatment can be found in Kevorkian and Cole [98] and Nayfeh [141]. The asymptotic results of Theorems 10.1 and 10.2 are adapted from Hoppensteadt's work on singular perturbations. (See Hoppensteadt [84].) Section 10.3 is based on Halanay [75, Section 3.4]. The presentation of the averaging method in Section 10.4 is based on Sanders and Verhulst [166], Hale [75, Section V.3], Halanay [73, Section 3.5], and Guckenheimer and Holmes [70, Sections 4.1 and 4.2]. The vibrating pendulum example is taken from Tikhonov, Vasileva, and Volosov [194]. The application of averaging to weakly nonlinear oscillators in Section 10.5 is based on Hale [75, pp. 183–186]. The presentation of general averaging in Section 10.6 is based on Sanders and Verhulst [166], Hale [75, Section V.3], and Sastry and Bodson [168, Section 4.2].

Chapter 11. The presentation of the singular perturbation method follows very closely Kokotovic, Khalil, and O'Reilly [105]. The proofs of Theorems 11.1 and 11.2 adapt ideas from Hoppensteadt [83]. We have not included the construction of higher order approximations. For that, the reader may consult Hoppensteadt [84], Butuzov, Vasileva, and Fedoryuk [29], or O'Malley[145]. The articles [83], [84], and [29] appear in Kokotovic and Khalil [104]. Example 11.11 is taken from Tikhonov, Vasileva, and Volosov [194]. For extensions of this example, see Grasman [68]. The proofs of Theorems 11.3 and 11.4 are based on Saberi and Khalil [160].

Chapter 12. The design via linearization approach of Section 12.2 is standard and can be found in almost any book on nonlinear control. The use of integral control is also standard, but the results given in Section 12.4 can be traced back to Huang and Rugh [86] and Isidori and Byrnes [91]. The gain scheduling presentation of Section 12.5 is based Lawrence and Rugh [114] and Kaminer, Pascoal, Khargonekar, and Coleman [96], with help from Astrom and Wittenmark [15] and Rugh and Shamma [159].

Chapter 13. The feedback linearization chapter is based mainly on Isidori [88]. The lucid introduction of Section 13.1 is based on Spong and Vidyasagar [185, Chapter 10]. The local stabilization and tracking results are based on Isidori [88, Chapter 4]. The global stabilization material is based on many references, as the global stabilization problem has caught the attention of several researchers. The papers [30], [46], [119], [161], [188], and [196] and the books [88], [108], [124], and [172] will guide the reader to the basic results on global stabilization, not only for the problem formulation presented here, but also for more general problem formulations, including problems treated in the backstepping Section 14.3. Examples 13.16, 13.17, and 13.19 are taken from [30], [188], and [62], respectively.

Chapter 14. The sliding mode control section is based on the books by Utkin [198] and Slotine and Li [180], and the tutorial papers by DeCarlo, Zak, and Matthews [52] and Young, Utkin, and Ozguner [215]. Analysis of the continuous approximation of sliding mode controllers is based on [55]. The Lyapunov redesign section is guided by Corless and Leitmann [45], Barmish, Corless, and Leitmann [19], and Spong and Vidyasagar [185, Chapter 10]. See also Corless [44] for a survey of the use of Lyapunov redesign in the control of uncertain nonlinear systems. The robust stabilization design of Section 14.2.1 is also known as min–max control. The nonlinear damping of Section 14.2.2 is based on Krstic, Kanellakopolous, and Kokotovic [108, Section 2.5]. The backstepping section is based mainly on the same reference [108], with help from Qu [153] and Slotine and Hedrick [179]. The book [108] contains a comprehensive treatment of the backstepping design procedure, including its application to adaptive control of nonlinear systems. Backstepping is a recursive method that helps reduce the complexity of nonlinear systems. Other recursive methods include forwarding and interlacing [153], [172] and composite control of singularly perturbed systems [105]. The material on passivity-based control follows Sepulchre, Jankovic, and Kokotovic [172] and van der Schaft [199]. Extensive application of the method to physical systems can be found in [120] and [146]. The section on high-gain observers is based on Esfandiari and Khalil [56] and Atassi and Khalil [16].

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Symbols[1]

$\equiv$	identically equal
$\approx$	approximately equal
$\overset{\text{def}}{=}$	defined as
$<$ $(>)$	less (greater) than
$\leq$ $(\geq)$	less (greater) than or equal to
$\ll$ $(\gg)$	much less (greater) than
$\forall$	for all
$\in$	belongs to
$\subset$	subset of
$\rightarrow$	tends to
$\Rightarrow$	implies
$\Leftrightarrow$	equivalent to, if and only if
$\sum$	summation
$\prod$	product
$\|a\|$	the absolute value of a scalar a
$\\|x\\|$	the norm of a vector x (647)
$\\|x\\|_p$	the p-norm of a vector x (647)
$\\|A\\|_p$	the induced p-norm of a matrix A (648)
max	maximum
min	minimum
sup	supremum, the least upper bound
inf	infimum, the greatest lower bound
R^n	the n-dimensional Euclidean space (647)
B_r	the ball $\{x \in R^n \mid \\|x\\| \leq r\}$
$\overline{M}$	the closure of a set M
∂M	the boundary of a set M
$\mathrm{dist}(p, M)$	the distance from a point p to a set M (127)
$f : S_1 \rightarrow S_2$	a function f mapping a set S_1 into a set S_2 (649)

---

[1] The page where the symbol is defined is given in parentheses.

$f_2 \circ f_1$	the composition of two functions (650)
$f^{-1}(\cdot)$	the inverse of a function f (650)
$f'(\cdot)$	the first derivative of a real-valued function f (650)
$D^+ f(\cdot)$	the upper right-hand derivative (659)
∇f	the gradient vector (650)
$\frac{\partial f}{\partial x}$	the Jacobian matrix (651)
$\dot{y}$	the first derivative of y with respect to time
$\ddot{y}$	the second derivative of y with respect to time
$y^{(i)}$	the ith derivative of y with respect to time
$L_f h$	the Lie derivative of h with respect to the vector field f (510)
$[f, g]$	the Lie bracket of the vector fields f and g (523)
$ad_f^k g$	$[f, ad_f^{k-1} g]$ (523)
$\text{diag}[a_1, \ldots, a_n]$	a diagonal matrix with diagonal elements a_1 to a_n
$\text{block diag}[A_1, \ldots, A_n]$	a block diagonal matrix with diagonal blocks A_1 to A_n
A^T (x^T)	the transpose of a matrix A (a vector x)
$\lambda_{\max}(P)$ $(\lambda_{\min}(P))$	the maximum (minimum) eigenvalue of a symmetric matrix P
$P > 0$	a positive definite matrix P (117)
$P \geq 0$	a positive semidefinite matrix P (117)
$\text{Re}[z]$ or $\text{Re}\ z$	the real part of a complex variable z
$\text{Im}[z]$ or $\text{Im}\ z$	the imaginary part of a complex variable z
$\bar{z}$ or z^*	the conjugate of a complex variable z
Z^*	the conjugate of a complex matrix Z
$\text{sat}(\cdot)$	the saturation function (19)
$\text{sgn}(\cdot)$	the signum function (19)
$O(\cdot)$	order of magnitude notation (383)
$\Diamond$	designation of the end of theorems, lemmas and corollaries
$\triangle$	designation of the end of examples
$\Box$	designation of the end of proofs
[xx]	see reference number xx in the bibliography

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[1] 2.1 (68) means Corollary 2.1 appears on page 68. The Lemma and Theorem Indexes are written similarly.